COMPLETING PARTIAL LATIN SQUARES WITH TWO PRESCRIBED DIAGONALS

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Abstract. In the present paper we will prove that every partial latin square $L = (l_{ij})$ of odd order n with 2 cyclically generated diagonals ($l_{i+t,j+t} = l_{ij}+t$ if l_{ij} is not empty; with calculations modulo n) can be cyclically completed.

1. Introduction

A *partial latin square* L of order n is an $n \times n$ array in which each cell is either empty or contains a single element from an n-set S of symbols, such that each element occurs at most once in each row and at most once in each column. If every cell is filled, then L is a *latin square*. If not explicitly stated differently, we assume the elements of S to be the integers $0, 1, \ldots, n-1$ and also that the rows and columns are indexed by $0, 1, \ldots, n-1$. A *partial transversal* of a partial latin square of order n is a set of filled cells, at most one in each row, at most one in each column, and such that no two of the cells contain the same symbol. A partial transversal with n cells is called a *transversal*. We refer the reader to [4, 5] for undefined terms as well as a general overview of latin squares.

Completion of partial latin squares has been investigated in a number of papers. Best known is Evans' conjecture [6] that an $n \times n$ partial latin square which has $n-1$ cells occupied can always be completed to a latin square of order n. Based on work by Marica and Schönheim [10] and Lindner [9] this conjecture was proved to be true by Häggkvist [8] for $n \geq 1111$ and independently by Smetaniuk [12] and by Andersen and Hilton [2] for all n. We also like to mention a still unsolved conjecture stated by Daykin and Häggkvist [3] that says that if L is a partial $n \times n$ latin square where each row, column and symbol is used at most un times (where u is some constant, e.g. $u = \frac{1}{4}$), then L can be completed. Daykin and Häggkvist proved this for $n = 16k$ and $un = \sqrt{k}/32$ where $k \in \mathbb{N}$.

In connection with questions from design theory the following problem was posed by Alspach and Heinrich in 1990 [1]: Does there exist an $N(k)$ such that if k transversals of a partial latin square of order $n \geq N(k)$ are prescribed, the square can always be completed? For $k = 1$ one has $N(1) = 3$ since there exists an idempotent latin square for every order $n \neq 2$. A more specific version of their question was posed by Rees [11]: Does there exist an N such that if four cyclically generated transversals $l_{i+t,j+t} = l_{ij} + t \pmod{n}$ of a partial latin square of order $n \geq N$ are prescribed, the square can always be completed to one which contains a further five transversals? Grüttmüller [7] proved that if $N(k)$ exists, then $N(k) \geq 4k - 1$.

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$$\begin{array}{ccccc} 0 & * & * & 4 & * \\ * & 1 & * & * & 0 \\ 1 & * & 2 & * & * \\ * & 2 & * & 3 & * \\ * & * & 3 & * & 4 \end{array} \qquad\qquad \begin{array}{ccccc} 0 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 0 \\ 1 & 4 & 2 & 0 & 3 \\ 4 & 2 & 0 & 3 & 1 \\ 2 & 0 & 3 & 1 & 4 \end{array}$$

FIGURE 1. A partial latin square of order 5 with 2 prescribed diagonals and its unique completion

Figure 1 shows as an example a partial latin square with 2 cyclically generated diagonals (an asterisk indicates an empty cell) together with its unique completion. Notice that the remaining 3 diagonals in the completed latin square are also cyclically generated. Therefore, it seems natural to try a completion to a cyclically generated latin square. Such a cyclic completion is impossible if n is even. But for odd n it suggests the following question. Does there exist an odd constant $C(k)$ such that if k cyclically generated diagonals $l_{i+t,j+t} = l_{ij} + t \pmod n$ of a partial latin square of odd order $n \geq C(k)$ are prescribed, the square can always be cyclically completed? Among all these constants let $C(k)$ denote the smallest one. For example, a cyclically generated idempotent latin square $L = (l_{ij})$ can be constructed for all odd n by defining $l_{ij} = 2i - j \pmod n$. This implies that $C(1) = 1$.

In [7] Grüttmüller found lower bounds for $C(k)$ stating that $C(2) \geq 3$ (the trivial bound) and $C(k) \geq 3k - 1$ for $k \geq 3$. Completing all possible *partial latin squares of odd order n with k cyclically generated diagonals (briefly PLS(n, k))* with k in the range $2 \leq k \leq 7$ and $3k - 1 \leq n \leq 21$ he provided some evidence that the lower bounds for $C(k)$ mentioned above might be the best possible bounds.

In support of this we shall prove:

Theorem 1.1. *Every partial latin square of odd order n with 2 cyclically generated diagonals can be cyclically completed. This means $C(2) = 3$.*

Section 2 introduces some terminology and notation. In Section 3 we investigate direct constructions for latin squares of prime order or order nine which contain two prescribed diagonals and a special class of latin squares with a certain *cut-and-paste* property. These latin squares will be used in Section 4 as ingredients for recursive constructions. In Section 5 we combine the results obtained and prove the main result Theorem 1.1.

2. PRELIMINARIES

We begin by introducing some terminology and notation. Clearly, a cyclically generated square L of order n is completely described by its first row $R = (l_{0,0}, l_{0,1}, \ldots, l_{0,n-1}) = (r_0, r_1, \ldots, r_{n-1})$ and it is a latin square if and only if all elements r_i as well as all differences $r_i - i \pmod n$ are mutually distinct. The latter condition ensures that the elements in every column are pairwise different. As mentioned before, it is easily checked that there is no cyclically generated latin square of even order n since $\sum_{i=0}^{n-1} i \equiv \frac{n}{2} \mod n$ but $\sum_{i=0}^{n-1}(r_i - i) \equiv 0 \mod n$. A *proper partial* *row* is a row $(r_0, r_1, \ldots, r_{n-1})$ where some of the r_i are empty and all nonempty r_i and the corresponding differences $r_i - i \pmod n$ are mutually distinct. Of course, a proper partial row with exactly k nonempty r_i corresponds to a PLS(n, k). We remark that most (but not all) computations are done in the ring $(\mathbb{Z}_n, +, \cdot)$. If $x \in \mathbb{Z}_n$ is relative prime to the ring order, then the multiplicative inverse exists and will be denoted by x^{-1}.

We will state most of the results in terms of proper (partial) rows and start here with a first observation on isomorphic rows:

Lemma 2.1. *Let n, m, a be three integers (n, m relatively prime) and let $R = (r_0, r_1, \ldots, r_{n-1})$ be a proper row. Then each of*
$R + a = (r_0 + a, r_1 + a, \ldots, r_i + a, \ldots, r_{n-1} + a)$,
$mR = (mr_{m^{-1}0}, mr_{m^{-1}1}, \ldots, mr_{m^{-1}i}, \ldots, mr_{m^{-1}(n-1)})$,
$R^T = (r_{j_0}, r_{j_1}, \ldots, r_{j_i}, \ldots, r_{j_{(n-1)}})$, *and*
$R^R = (-r_{j_{-1}} - 2, -r_{j_{-2}} - 2, \ldots, -r_{j_{-i-1}} - 2, \ldots, -r_{j_0} - 2)$
is also a proper row, where the indices j_i are uniquely determined by $r_{j_i} - j_i = i$ and all calculations are performed in the ring $(\mathbb{Z}_n, +, \cdot)$.

Proof. R is a proper row and, therefore, all the elements $r_i + a$ and differences $(r_i + a) - i$ are distinct. Thus, $R + a$ is a proper row. Furthermore, since m and m^{-1} are relatively prime to n we have $\{mr_{m^{-1}i} : i \in \mathbb{Z}_n\} = \{mr_i : i \in \mathbb{Z}_n\} = \{r_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$ and $\{mr_{m^{-1}i} - i : i \in \mathbb{Z}_n\} = \{m(r_{m^{-1}i} - m^{-1}i) : i \in \mathbb{Z}_n\} = \{m(r_i - i) : i \in \mathbb{Z}_n\} = \{r_i - i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$, implying that mR is a proper row. Similarly, $\{r_{j_i} : i \in \mathbb{Z}_n\} = \{r_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$ and $\{r_{j_i} - i : i \in \mathbb{Z}_n\} = \{j_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$, implying that R^T is a proper row. Moreover, R^R is a proper row since $\{-r_{j_{-i-1}} - 2 : i \in \mathbb{Z}_n\} = \{-r_i - 2 : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$ and $\{-r_{j_{-i-1}} - 2 - i : i \in \mathbb{Z}_n\} = \{-(-i - 1 + j_{-i-1}) - 2 - i : i \in \mathbb{Z}_n\} = \{-j_{-i-1} - 1 : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. $\square$

If R is proper row, then $R + (-r_0)$ is a proper row. Therefore, without loss of generality we can always assume that the first prescribed element in a row equals zero: $r_0 = 0$.

We remark that proper rows are equivalent to transversals in a special latin square as follows: $R = (r_0, r_1, \ldots, r_{n-1})$ is a proper row if and only if $T_R = \{(r_i - i \pmod n), i) : i \in \mathbb{Z}_n\}$ is a transversal in the latin square $L = (l_{ij}) = (i + j \pmod n))$. We will use this equivalence to illustrate statements and constructions, see for example Figure 2 for Lemma 2.1 and note that R^T can be obtained from R by transposing the corresponding transversal cells, while R^R can be obtained from R by reflecting the transversal cells in the main back diagonal.

Let R be a row with n elements and define $u(R), \ell(R)$ to be two sets of indices as follows: $u(R) = \{i : (r_i \pmod n) - i \geq 0\}$ and $\ell(R) = \{i : (r_i \pmod n) - i < 0\}$. The set $u(R)$ can be viewed as the set of indices whose corresponding transversal cells are above or on the main back diagonal. Moreover, we define an operation cp(R, Q, w) ($w \in \{0, 1, -1\}$) which cuts out elements of row $R = (r_0, r_1, \ldots, r_{n-1})$ and pastes in elements of row $Q = (q_0, q_1, \ldots, q_{n-1})$ to form a new row $E = \text{cp}(R, Q, w) = (e_0, e_1, \ldots, e_{n-1})$ with

$$e_i = \begin{cases} r_i & \text{if } i \in u(R), \\ q_i + wn & \text{if } i \in \ell(R). \end{cases}$$

$$R \simeq \begin{array}{ccccc} \mathbf{0} & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \mathbf{4} & 0 \\ 2 & \mathbf{3} & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & \mathbf{2} \\ 4 & 0 & \mathbf{1} & 2 & 3 \end{array} \quad 2R \simeq \begin{array}{ccccc} \mathbf{0} & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \mathbf{4} & 0 \\ 2 & \mathbf{3} & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & \mathbf{2} \\ 4 & 0 & \mathbf{1} & 2 & 3 \end{array} \quad R^T \simeq \begin{array}{ccccc} \mathbf{0} & 1 & 2 & 3 & 4 \\ 1 & 2 & \mathbf{3} & 4 & 0 \\ 2 & 3 & 4 & 0 & \mathbf{1} \\ 3 & \mathbf{4} & 0 & 1 & 2 \\ 4 & 0 & 1 & \mathbf{2} & 3 \end{array} \quad R^R \simeq \begin{array}{ccccc} 0 & \mathbf{1} & 2 & 3 & 4 \\ 1 & 2 & 3 & \mathbf{4} & 0 \\ \mathbf{2} & 3 & 4 & 0 & 1 \\ 3 & 4 & \mathbf{0} & 1 & 2 \\ 4 & 0 & 1 & 2 & \mathbf{3} \end{array}$$

FIGURE 2. Rows $R = (0, 3, 1, 4, 2)$, $2R$, R^T, R^R illustrated as transversals

For example take $R = (0, 3, 1, 4, 2)$ to obtain $E = \text{cp}(R, R, 1) = (0, 3, 6, 4, 7)$. A row $E = (e_0, e_1, \ldots, e_{n-1})$ with all the e_i as well as all the $e_i - i \pmod n$ mutually distinct and the property that $0 \leq e_i - i < n$ will be called an *extended proper row*. An extended proper row E can also be viewed as a transversal $T_E = \{(e_i - i, i) : i \in \mathbb{Z}_n\}$ in the $n \times n$ square $L = (l_{ij}) = (i + j)$, see Figure 3. Squares of this kind occur later as subsquares in the structures to be considered.

3. Direct Constructions

In this section, we provide constructions for latin squares of prime order or order nine containing two prescribed diagonals. Furthermore, we describe a special class of latin squares and their properties.

Lemma 3.1. *Let n be a prime number. Then every proper PLS(n,2) L is cyclically completable.*

Proof. All calculations in this proof are done in the ring $(\mathbb{Z}_n, +, \cdot)$. Let $l_{0,0} = 0$ and $l_{0,j}$ be the two prescribed elements of the first row of L. Let $d = l_{0,j} - j$ and define a row R by $r_i = i(j^{-1}d + 1)$ for $i = 0, 1, \ldots, n-1$. Clearly all elements r_i and all differences $r_i - i = ij^{-1}d$ are distinct. Moreover, $r_0 = 0$ and $r_j = j(j^{-1}d + 1) = d + j = l_{0,j}$. Thus, R is the first row of a cyclic completion of L as desired. $\square$

Lemma 3.2. *Let L be a PLS$(9, 2)$. Then L is cyclically completable.*

Proof. Let R be the first row of L with prescribed elements $r_0 = l_{0,0} = 0$ and $r_j = l_{0,j}$ where $r_j, j, (r_j - j \pmod 9) \neq 0$. Obviously, R can be completed to one of the following 7 proper rows since every possible pair $(r_0 = 0, r_j)$ occurs among them: (0,2,4,6,8,1,3,5,7); (0,3,7,4,1,8,5,2,6); (0,4,3,8,2,7,1,6,5); (0,5,8,2,6,3,7,1,4); (0,6,1,5,7,2,4,8,3); (0,7,6,1,5,4,8,3,2); (0,8,5,7,3,6,2,4,1). $\square$

$$E \simeq \begin{array}{ccccc} \mathbf{0} & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \mathbf{4} & 5 \\ 2 & \mathbf{3} & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & \mathbf{7} \\ 4 & 5 & \mathbf{6} & 7 & 8 \end{array}$$

FIGURE 3. Extended proper row $E = (0, 3, 6, 4, 7)$ illustrated as transversal

$$
R(9) \simeq \begin{array}{ccccccccc}
\mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & \mathbf{2} & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\
2 & 3 & \mathbf{4} & 5 & 6 & 7 & 8 & 0 & 1 \\
3 & 4 & 5 & 6 & \mathbf{7} & 8 & 0 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 0 & \mathbf{1} & 2 & 3 \\
5 & 6 & 7 & \mathbf{8} & 0 & 1 & 2 & 3 & 4 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & \mathbf{5} \\
7 & 8 & 0 & 1 & 2 & \mathbf{3} & 4 & 5 & 6 \\
8 & 0 & 1 & 2 & 3 & 4 & 5 & \mathbf{6} & 7
\end{array}
\qquad
R(11) \simeq \begin{array}{ccccccccccc}
\mathbf{0} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & \mathbf{2} & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0 \\
2 & 3 & \mathbf{4} & 5 & 6 & 7 & 8 & 9 & 10 & 0 & 1 \\
3 & 4 & 5 & \mathbf{6} & 7 & 8 & 9 & 10 & 0 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & \mathbf{10} & 0 & 1 & 2 & 3 \\
5 & 6 & 7 & 8 & \mathbf{9} & 10 & 0 & 1 & 2 & 3 & 4 \\
6 & 7 & 8 & 9 & 10 & 0 & 1 & 2 & \mathbf{3} & 4 & 5 \\
7 & 8 & 9 & 10 & 0 & \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\
8 & 9 & 10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \mathbf{7} \\
9 & 10 & 0 & 1 & 2 & 3 & 4 & \mathbf{5} & 6 & 7 & 8 \\
10 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \mathbf{8} & 9
\end{array}
$$

FIGURE 4. Rows $R(9) = (0, 2, 4, 8, 7, 3, 1, 6, 5)$ and $R(11) = (0, 2, 4, 6, 9, 1, 10, 5, 3, 8, 7)$

In the following we define for every odd $n \geq 5$ the first row $R(n) = (r_0, r_1, \ldots, r_{n-1})$ of a special cyclically generated latin square of order n.

Construction 3.3. *If $n \equiv 1 \pmod 4$, then define*

$$
r_i = \begin{cases}
2i \pmod n & \text{for } i = 0, 1, \ldots, \frac{n-5}{2}, \\
2i + 2 \pmod n & \text{for } i = \frac{n-3}{2}, \frac{n+1}{2}, \ldots, n-4, \\
n - 2 \pmod n & \text{for } i = \frac{n-1}{2}, \\
n - 3 \pmod n & \text{for } i = n-2, \\
2i - 2 \pmod n & \text{for } i = \frac{n+3}{2}, \frac{n+7}{2}, \ldots, n-1
\end{cases}.
$$

If $n \equiv 3 \pmod 4$, then define

$$
r_i = \begin{cases}
2i \pmod n & \text{for } i = 0, 1, \ldots, \frac{n-5}{2}, \\
2i + 2 \pmod n & \text{for } i = \frac{n-1}{2}, \frac{n+3}{2}, \ldots, n-4, \\
n - 2 \pmod n & \text{for } i = \frac{n-3}{2}, \\
n - 3 \pmod n & \text{for } i = n-2, \\
2i - 2 \pmod n & \text{for } i = \frac{n+1}{2}, \frac{n+5}{2}, \ldots, n-1
\end{cases}.
$$

See for example $R(9)$ or $R(11)$ in Figure 4. We observe that all these rows have the following nice and for our constructions important property.

Property 3.4. *Let R be a proper row as constructed above, then $A = (a_i) = \mathrm{cp}(R, R, 1)$, $B = (b_i) = \mathrm{cp}(R, R^R, 1)$ and $C = (c_i) = \mathrm{cp}(R^R, R, 1)$ are extended proper rows and $\mathrm{cp}(A, A, -1) = R$, $\mathrm{cp}(A, C, -1) = R$, $\mathrm{cp}(B, A, -1) = R$ and $\mathrm{cp}(C, B, -1) = R^R$ are proper rows. Moreover, $a_0 = b_0 = 0$, $c_0 = 1$, and if $n \equiv 3 \pmod 4$ $b_{(n-1)/2} = n$ while if $n \equiv 1 \pmod 4$ $b_{(n+1)/2} = n$.*

4. RECURSIVE CONSTRUCTIONS

In this section, we present two constructions which build new proper rows from small ingredient rows. These constructions will then be used to establish that every $\mathrm{PLS}(n, 2)$ with prescribed elements $l_{0,0} = 0$ and $l_{0,j} = 2j + 1 \pmod n$ is cyclically completable.

Construction 4.1. *Let Q be a proper row of length m and $E = \mathrm{cp}(Q, Q, 1) = (e_0, e_1, \dots, e_{m-1})$. Let $Q_0, Q_1, \dots, Q_{m-1}$ with $Q_i = (q_{i,0}, q_{i,1}, \dots, q_{i,n-1})$ are m proper rows of length n each for $i = 0, 1, \dots, m-1$. Then $R = (r_0, r_1, \dots, r_{mn-1})$ defined by*

$$r_i = q_{b,a}m + e_b \pmod{mn}$$

where the integers a, b are uniquely defined by $i = am + b$ with $0 \le b < m$ is a proper row of length mn.

Before proving the claim we present an example.

Example 4.2. *Let $m = 5$, $Q = (0, 3, 1, 4, 2)$, $E = (0, 3, 6, 4, 7)$, $n = 3$, $Q_0 = (0, 2, 1)$, $Q_1 = (1, 0, 2)$, $Q_2 = (2, 1, 0)$, $Q_3 = (1, 0, 2)$, $Q_4 = (0, 2, 1)$. Then $R = (0, 8, 1, 9, 7, 10, 3, 11, 4, 2, 5, 13, 6, 14, 12)$. See Figure 5 for a representation of R by a transversal. The second latin square is a rearrangement of the first latin square obtained by simultaneously permuting columns and rows which provides a better understanding of the foregoing construction which can also be stated as: Take an $m \times m$ square $L = (l_{ij}) = (i + j)$ with a transversal T_E and replace every cell (i, j) by a latin square $L^{ij} = (l^{ij}_{kl}) = ((i + j) + (k + l)m \pmod{mn})$ of order n (the set of symbols in L^{ij} is $\{i + j, i + j + m, \dots, i + j + (n-1)m \pmod{mn}\}$) to obtain a latin square of order mn. While doing so every transversal cell $(i, j) \in T_E$ will be replaced by n new transversal cells from T_{Q_j} in L^{ij}.*

Proof. We have to prove that in R all elements r_i as well as all the $r_i - i \pmod{mn}$ are distinct, i.e., $\{r_i : i \in \mathbb{Z}_{mn}\} = \{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\} = \mathbb{Z}_{mn}$. Obviously $\{r_i : i \in \mathbb{Z}_{mn}\} \subseteq \mathbb{Z}_{mn}$ and $\{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\} \subseteq \mathbb{Z}_{mn}$ and, therefore, it remains to prove that $\mathbb{Z}_{mn} \subseteq \{r_i : i \in \mathbb{Z}_{mn}\}$ and $\mathbb{Z}_{mn} \subseteq \{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\}$. Let $x \in \mathbb{Z}_{mn}$ with $x = \alpha m + \beta$ and $0 \le \beta < m$. There is exactly one q_b in Q with $q_b = \beta$. Thus, $e_b \equiv \beta \pmod{m}$ and $\{q_{b,a}m + e_b \pmod{mn} : a \in \mathbb{Z}_n\}$ contains all residues congruent to $\beta \pmod{m}$. This implies that there exists an a such that $x = q_{b,a}m + e_b \pmod{mn} = r_{am+b} \in \{r_i : i \in \mathbb{Z}_{mn}\}$. Moreover, there is exactly one index b such that $e_b - b = \beta$ and exactly one a such that $q_{b,a} - a \pmod{n} = \alpha$. Therefore, $x = (q_{b,a} - a)m + (e_b - b) \pmod{mn} = r_{am+b} - (am + b) \pmod{mn} \in \{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\}$. That completes the proof. □

Construction 4.3. *Let $Q = (q_0, q_1, \dots, q_{m-1})$ be a proper row of length m. Let $E_0, E_1, \dots, E_{m-1}$ with $E_i = (e_{i,0}, e_{i,1}, \dots, e_{i,n-1})$ are m extended proper rows of length n each with the additional property that $\mathrm{cp}(E_i, E_{i-1 \pmod{m}}, -1)$ is a proper row for $i = 0, 1, \dots, m-1$. Then $R = (r_0, r_1, \dots, r_{mn-1})$ defined by*

$$r_i = q_a n + e_{q_a,b} \pmod{mn}$$

where the integers a, b are uniquely defined by $i = an + b$ with $0 \le b < n$ is a proper row of length mn.

Again, we first provide an example.

Example 4.4. *Let $m = 3$, $Q = (0, 2, 1)$, $n = 5$, $E_0 = (0, 4, 3, 7, 6)$, $E_1 = (0, 4, 3, 5, 8)$, $E_2 = (1, 4, 2, 7, 6)$. Then $R = (0, 4, 3, 7, 6, 11, 14, 12, 2, 1, 5, 9, 8, 10, 13)$. See Figure 6 for a representation of R by T_R. In terms of transversal cells we may formulate the*

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	3	4	5	6	7	8	9	10	11	12	13	**14**	0
2	3	4	5	6	7	8	9	10	11	12	**13**	14	0	1
3	4	5	6	**7**	8	9	10	11	12	13	14	0	1	2
4	5	6	7	8	9	10	**11**	12	13	14	0	1	2	3
5	6	7	8	9	**10**	11	12	13	14	0	1	2	3	4
6	7	8	**9**	10	11	12	13	14	0	1	2	3	4	5
7	**8**	9	10	11	12	13	14	0	1	2	3	4	5	6
8	9	10	11	12	13	14	0	1	**2**	3	4	5	6	7
9	10	11	12	13	14	0	1	2	3	4	5	**6**	7	8
10	11	12	13	14	0	1	2	3	4	**5**	6	7	8	9
11	12	13	14	0	1	2	3	**4**	5	6	7	8	9	10
12	13	14	0	1	2	**3**	4	5	6	7	8	9	10	11
13	14	0	1	2	3	4	5	6	7	8	9	10	11	**12**
14	0	**1**	2	3	4	5	6	7	8	9	10	11	12	13

0	5	10	1	6	11	2	7	12	3	8	13	4	9	14
5	**10**	0	6	11	1	7	12	2	8	13	3	9	14	4
10	0	**5**	11	1	6	12	2	7	13	3	8	14	4	9
1	6	11	2	7	12	3	8	13	4	9	**14**	5	10	0
6	11	1	7	12	2	8	13	3	**9**	14	4	10	0	5
11	1	6	12	2	7	13	3	8	14	**4**	9	0	5	10
2	7	12	3	8	**13**	4	9	14	5	10	0	6	11	1
7	12	2	**8**	13	3	9	14	4	10	0	5	11	1	6
12	2	7	13	**3**	8	14	4	9	0	5	10	1	6	11
3	8	13	4	9	14	5	10	0	6	11	1	**7**	12	2
8	13	3	9	14	4	10	0	5	11	1	6	12	**2**	7
13	3	8	14	4	9	0	5	10	1	6	11	2	7	**12**
4	9	14	5	10	0	6	**11**	1	7	12	2	8	13	3
9	14	4	10	0	5	11	1	**6**	12	2	7	13	3	8
14	4	9	0	5	10	**1**	6	11	2	7	12	3	8	13

FIGURE 5. Transversals representing row $R = (0, 8, 1, 9, 7, 10, 3, 11, 4, 2, 5, 13, 6, 14, 12)$

construction above as: Take a latin square $L = (l_{ij}) = (i + j \pmod m)$ and replace every cell (i, j) by an $n \times n$ square $L^{ij} = (l^{ij}_{kl}) = ((i + j)n + (k + l) \pmod{mn})$ of order n to obtain a latin square of order mn. While doing so every transversal cell $(i, j) \in T_Q$ will be replaced by n new transversal cells from $T_{E_{i+j \pmod m}}$ in L^{ij}.

Proof. It suffices to prove that $\{r_i : i \in \mathbb{Z}_{mn}\} = \{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\} = \mathbb{Z}_{mn}$. At first, we show $\mathbb{Z}_{mn} \subseteq \{r_i : i \in \mathbb{Z}_{mn}\}$. Let $x \in \mathbb{Z}_{mn}$ with $x = \alpha n + \beta$ and $0 \leq \beta < n$,

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	**3**	4	5	6	7	8	9	10	11	12	13	14	0
2	3	4	5	**6**	7	8	9	10	11	12	13	14	0	1
3	**4**	5	6	7	8	9	10	11	12	13	14	0	1	2
4	5	6	**7**	8	9	10	11	12	13	14	0	1	2	3
5	6	7	8	9	10	11	**12**	13	14	0	1	2	3	4
6	7	8	9	10	**11**	12	13	14	0	1	2	3	4	5
7	8	9	10	11	12	13	14	0	**1**	2	3	4	5	6
8	9	10	11	12	13	**14**	0	1	2	3	4	5	6	7
9	10	11	12	13	14	0	1	**2**	3	4	5	6	7	8
10	11	12	13	14	0	1	2	3	4	**5**	6	7	8	9
11	12	13	14	0	1	2	3	4	5	6	7	**8**	9	10
12	13	14	0	1	2	3	4	5	6	7	8	9	**10**	11
13	14	0	1	2	3	4	5	6	7	8	**9**	10	11	12
14	0	1	2	3	4	5	6	7	8	9	10	11	12	**13**

FIGURE 6. Transversal representing row $R = (0, 4, 3, 7, 6, 11, 14, 12, 2, 1, 5, 9, 8, 10, 13)$

then there is exactly one index a such that $q_a = \alpha$. Moreover, there is exactly one index b such that the b-th entry in row $\mathrm{cp}(E_{q_a}, E_{q_a - 1 \pmod m}, -1)$ equals β. Now, if $b \in u(E_{q_a})$, then $e_{q_a,b} = \beta$ and, therefore, $x = q_a n + e_{q_a,b} = r_{an+b} \in \{r_i : i \in \mathbb{Z}_{mn}\}$. If $b \in \ell(E_{q_a})$, then $e_{q_a - 1 \pmod m),b} - n = \beta$ and, therefore, $x = q_a n + e_{q_a-1 \pmod m),b} - n = (q_a - 1)n + e_{q_a - 1 \pmod m),b} \pmod{mn} = r_{a'n+b} \in \{r_i : i \in \mathbb{Z}_{mn}\}$ for some a'. Clearly $\{r_i : i \in \mathbb{Z}_{mn}\} \subseteq \mathbb{Z}_{mn}$. This implies that $\{r_i : i \in \mathbb{Z}_{mn}\} = \mathbb{Z}_{mn}$.

Secondly, we need to prove that $\{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\} = \mathbb{Z}_{mn}$. For an arbitrary $x \in \mathbb{Z}_{mn}$ with $x = \alpha n + \beta$ and $0 \leq \beta < n$ we have $x \in \{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\}$, since there is a uniquely determined pair (a, b) such that $q_a - a \pmod m = \alpha$ and $e_{q_a,b} - b = \beta$ with the property that $x = (q_a - a)n + e_{q_a,b} - b \pmod{mn} = r_{an+b} - (an+b) \pmod{mn} \in \{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\}$. Thus $\{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\} = \mathbb{Z}_{mn}$ follows since clearly $\{r_i - i \pmod{mn} : i \in \mathbb{Z}_{mn}\} \subseteq \mathbb{Z}_{mn}$. □

Our next result concerns the completion of $\mathrm{PLS}(n, 2)$ where n is a composite integer and the symbols prescribed in the first row are 0 and $2j+1 \pmod n$ at the j-th position.

Lemma 4.5. *Let $p > 3$ be a prime number and m an odd integer. Furthermore, let $l_{0,0} = 0$ and $l_{0,j} = 2j + 1 \pmod{mp}$ be the two prescribed elements of the first row of a PLS$(mp, 2)$ L. Then L is cyclically completable.*

Proof. We consider four cases depending on the residues of $j, 2j + 1$ and $j + 1$ modulo p. First of all assume that $j \not\equiv 0 \pmod p$, $2j + 1 \not\equiv 0 \pmod p$ and $j + 1 \not\equiv 0 \pmod p$. There exists a proper row U of size p with $u_0 = 0$ and $u_{j \pmod p} = 2j + 1 \pmod p$ by Lemma 3.1. Take as ingredients in Construction 4.3 the proper row $Q = (0, 2, 4, \dots, 2i, \dots, m - 4, m - 2)$ of length m and take m extended proper rows $E_i = \mathrm{cp}(U, U, 1)$. Thus, we obtain the first row R of a cyclic completion of L, since $r_0 = q_0 p + e_{q_0,0} \pmod{mp} = 0 \cdot p + e_{0,0} = 0$ and, moreover, with $j = ap + b$ $(0 \leq b < p)$

we have $r_j = q_a p + e_{q_a,b} \pmod{mp} = 2ap + e_{2a \pmod m, b} \pmod{mp} = 2ap + 2b + 1 \pmod{mp} = 2j+1 \pmod{mp}$ as prescribed in L.

Now assume that $j \equiv 0 \pmod p$. Use the row $U = R(p)$ from Construction 3.3 to define extended proper rows as follows: $E_{(2j/p) \pmod m} = \mathrm{cp}(U^R, U, 1)$, $E_{(2j/p-1) \pmod m} = \mathrm{cp}(U, U^R, 1)$ and $E_i = \mathrm{cp}(U, U, 1)$ for $i = 0, 1, \ldots, m-1$ with $i \not\equiv \frac{2j}{p}, \frac{2j}{p} - 1 \pmod m$. With these rows and $Q = (0, 2, 4, \ldots, 2i, \ldots, m-4, m-2)$, Construction 4.3 provides a proper row R. Here, $r_0 = q_0 p + e_{q_0,0} \pmod{mp} = 0 \cdot p + e_{0,0} = 0 + u_0 = 0$ and, moreover, with $j = ap$ and Property 3.4 $r_j = 2ap + e_{2a \pmod m, 0} \pmod{mp} = 2ap + 1 \pmod{mp} = 2j + 1 \pmod{mp}$. Hence, R is the first row of a cyclic completion of L.

In the case $2j + 1 \equiv 0 \pmod p$ we use $U = R(p)$ from Construction 3.3 and $E_{((2j+1)/p) \pmod m} = \mathrm{cp}(U^R, U, 1)$, $E_{((2j+1)/p-1) \pmod m} = \mathrm{cp}(U, U^R, 1)$ and $E_i = \mathrm{cp}(U, U, 1)$ for $i = 0, 1, \ldots, m-1$ with $i \not\equiv \frac{2j+1}{p}, \frac{2j+1}{p} - 1 \pmod m$. With $Q = (0, 2, 4, \ldots, 2i, \ldots, m-4, m-2)$, Construction 4.3 provides a proper row R which is the desired row if $p \equiv 3 \pmod 4$ since $r_0 = 0 + u_0 = 0$ and $r_j = 2ap + e_{2a \pmod m, b} \pmod{mp} = 2ap + e_{((2j+1)/p-1) \pmod m, (p-1)/2} \pmod{mp} = 2ap + p \pmod{mp} = 2j + 1 \pmod{mp}$ with $j = ap + b$ $(0 \leq b < p)$ and $2j + 1 = 2ap + 2b + 1 = (2a + 1)p$. If $p \equiv 1 \pmod 4$, then R^T is the desired row since $r_0^T = r_0 = 0$ and $r_j^T = r_{j+1} = 2ap + e_{2a \pmod m, b+1} \pmod{mp} = 2ap + e_{((2j+1)/p-1) \pmod m, (p+1)/2} \pmod{mp} = 2ap + p \pmod{mp} = 2j + 1 \pmod{mp}$.

Finally assume $j + 1 \equiv 0 \pmod p$. In view of the foregoing (case $j \equiv 0 \pmod p$) we are able to find a proper row R' with $r'_0 = 0$ and $r'_{-(j+1) \pmod{mp}} = -2(j+1) + 1 \pmod{mp}$. Then $R = (-1) \cdot (R')^T$ is a proper row with $r_0 = 0$ and $r_j = -(r')^T_{-j \pmod{mp}} = -r'_{-(j+1) \pmod{mp}} = -(-2(j+1) + 1) \pmod{mp} = 2j + 1 \pmod{mp}$. □

Lemma 4.6. *Let $l_{0,0} = 0$ and $l_{0,j} = 2j + 1 \pmod{3^\alpha}$ be the two prescribed elements of the first row of a PLS(3^α,2) L. Then L is cyclically completable.*

Proof. If $\alpha = 1$, then there is no PLS(3, 2) with prescribed elements of the kind described above. If $\alpha = 2$, then the result follows from Lemma 3.2. If $\alpha \geq 3$, then we use a similar argument as in the proof of Lemma 4.5: instead of a prime p we consider $p = 9$ and the result follows immediately. □

5. Results and Problems

Now, we are ready to prove the main result of this paper.

Proof of Theorem 1.1. We use induction to prove the claim and assume that every PLS(m,2) with $m < n$ is cyclically completable. This is true by Lemma 3.1 and Lemma 3.2 for $m \leq 13$.

Let $l_{0,0} = 0$ and $l_{0,j}$ be the two prescribed elements of the first row of a PLS(n,2) L. We consider three cases depending on the factorization of n. Suppose first that there exists a prime factor p of n such that $j = ap$ and $l_{0,j} = \alpha p$. Let $n = mp$. Take a proper row Q of length m with $q_0 = 0$ and $q_a = \alpha$ (which exists by the induction hypothesis) and m extended proper rows E_i of length p defined by $e_{i,s} = 2s \pmod p$. The result follows from Construction 4.3.

Now, suppose that there exists a prime factor p of n such that $j = ap+b$ $(0 < b < p)$, $l_{0,j} = \alpha p + \beta$ $(0 < \beta < p)$ and $l_{0,j} - j \not\equiv 0 \pmod{p}$. Let $n = mp$. Take a proper row Q of size p with $q_0 = 0$ and $q_b = \beta$ (which exists by Lemma 3.1). Choose p proper rows Q_i of size m such that $q_{0,0} = 0$ and $q_{b,a} = \alpha$ (which exist since C(1)=1). Applying Construction 4.1, we get a proper row R with the prescribed elements, as desired.

Finally, suppose for all prime divisors p of n exactly one of the following conditions is true: $j \equiv 0 \pmod{p}$ or $l_{0,j} \equiv 0 \pmod{p}$ or $l_{0,j} - j \equiv 0 \pmod{p}$. Define $m = l_{0,j} - 2j$. Note that m is relatively prime to n, since otherwise for at least one prime divisor of n all three conditions above are simultaneously satisfied. Hence, m^{-1} exists in $(\mathbb{Z}_n, +, \cdot)$. Lemma 4.5 or Lemma 4.6 yields a proper row R' of length n with $r'_0 = 0$ and $r'_{m^{-1}j} = 2(m^{-1}j) + 1 \pmod{n}$. Now, $R = mR'$ is a proper row with $r_0 = 0$ and $r_j = mr'_{m^{-1}j} \pmod{n} = m(2(m^{-1}j)+1) \pmod{n} = 2j + (l_{0,j} - 2j) \pmod{n} = l_{0,j}$, as desired. $\square$

We hope that this result provides some ideas that might be helpful to solve the general problem.

Problem 5.1. *Is it possible to prove that every partial latin square of odd order n with k cyclically generated diagonals can be cyclically completed if $n \geq 3k - 1$?*

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