Orthogonal double covers of complete graphs by fat caterpillars

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Abstract

An orthogonal double cover (ODC) of the complete graph K_n by some graph G is a collection of n spanning subgraphs of K_n, all isomorphic to G, such that any two of the subgraphs share exactly one edge and every edge of K_n is contained in exactly two of the subgraphs. A necessary condition for such an ODC to exist is that G has exactly $n-1$ edges. We show that for any given positive integer d, almost all caterpillars of diameter d admit an ODC of the corresponding complete graph.

Keywords: ODC, orthogonal double cover, graph decomposition, self-orthogonal factorization

MSC (2000): 05B40, 05C70

1 Introductory notes and definitions

An *orthogonal double cover (ODC)* of the complete graph K_n is a collection $\mathcal{G} = \{G_i : i = 1, 2, \ldots, n\}$ of spanning subgraphs of K_n, called *pages*, such that the following two conditions are satisfied:

1. *Double cover condition*
   Every edge of K_n belongs to the edge set of exactly two pages of $\mathcal{G}$.

2. *Orthogonality condition*
   Any two distinct pages of $\mathcal{G}$ share exactly one edge.

ODCs have been investigated for more than 25 years, and there is an extensive literature on the subject. For motivations to study ODCs and an overview of results and problems in the area we refer to the survey paper [1].

The above definition immediately implies that every page of $\mathcal{G}$ must have exactly $n-1$ edges. If all pages of $\mathcal{G}$ are isomorphic to some graph G, then $\mathcal{G}$ is said to be an ODC of K_n by G.

The problem usually studied in this context is: Given a class $\mathcal{C}$ of graphs, decide for which $G \in \mathcal{C}$ there is an ODC of the corresponding complete graph by G. As $e(G) = n-1$ is necessary for the existence of an ODC by G, it is very natural to ask the above question for $\mathcal{C}$ being the class of all trees. One can easily observe that there is no ODC of K_4 by P_4, the path of length three. For all other non-trivial trees on at most 14 vertices ODCs of the corresponding complete graphs exist [1] which supports the following conjecture.

Conjecture 1. [2] *If $T \neq P_4$ is a tree on $n \geq 2$ vertices, then there is an ODC of K_n by T.*

A complete proof of the above conjecture seems to be out of reach at the moment. It trivially holds for stars, and it was shown to be true for trees of diameter three [2] (see also [4]) and large classes of trees of diameter four (see [5] for details). In particular, Conjecture 1 is true for all caterpillars of diameter four. Studying ODCs by caterpillars of diameter $d \geq 5$, we continue this line of research. Our main result (Theorem 2) implies that, for fixed d, almost all caterpillars of diameter d admit an ODC of the corresponding complete graph. For the rest of the paper assume that $d \geq 5$.

By a $(q_1, q_2, \ldots, q_{d-1})$-caterpillar we mean a caterpillar of diameter d on $d - 1 + q_1 + q_2 + \cdots + q_{d-1}$ vertices in which the vertices with degree more than one, the *spine vertices* $s_1, s_2, \ldots, s_{d-1}$, in this order form a path and have degrees $d(s_1) = q_1 + 1$, $d(s_{d-1}) = q_{d-1} + 1$, $d(s_i) = q_i + 2$ $(i = 2, \ldots, d-2)$.

Theorem 2. *There exists a positive integer $k = k(d)$ such that all caterpillars R of diameter d and such that every spine vertex has at least k neighbors of degree one admit an ODC of the corresponding complete graph by R.*

2 Group-generated and surjective ODCs

Our proof of Theorem 2 is based on the fact that caterpillars with many pendant vetices have a large automorphism group which will allow us to make use of group-generated and surjective ODSs and apply a recursive construction.

Let Γ be an additive group of order n, and assume that $V(K_n) = \Gamma$. An ODC $\mathcal{G} = \{G_x : x \in \Gamma\}$ of K_n by some graph G is said to be *generated by* Γ if

$$E(G_x) = \{(u + x, v + x) : (u, v) \in E(G_0)\}$$

for all $x \in \Gamma$. For more information on group-generated ODCs we refer to Section 2.1 of [1].

Here we shall only need the following observation:

Theorem 3. [1, Thm. 2.12] *Let $\mathcal{G}$ be an ODC of K_n by some graph G which is generated by a group Γ. Furthermore, let V_i, $i = 1, 2, \dots, m-1$, be mutually disjoint sets of size n with $V_i \cap V(G) = \emptyset$ for $i = 1, 2, \dots, m-1$, and let G^* be obtained from G joining some $v_i \in V(G)$ to all $v \in V_i$ for $i = 1, 2, \dots, m-1$. Then there is an ODC of K_{mn} by G^* which is generated by $\Gamma \times \mathbb{Z}_m$.*

Note that the $v_i's$ in the above theorem are not necessarily distinct.

Consider an ODC $\mathcal{G} = \{G_i : i = 1, \dots, n\}$ of K_n by some G. A subset $U \subseteq V(G)$ is called *surjective* (with respect to $\mathcal{G}$) if there are isomorphisms $\varphi_i : G \mapsto G_i$, $i = 1, 2, \dots, n$, such that $\{\varphi_i(v) : i = 1, 2, \dots, n\} = V(K_n)$ for all $v \in U$. Clearly, if $\mathcal{G}$ is group-generated, then the whole vertex set $V(G)$ is surjective.

Surjective subsets will be useful for us because of the following lemma.

Lemma 4. [5] (see also [1, Lemma 2.10 and Remark 2.11]) *Let G be a graph of order $m+n$, and let G_1 and G_2 be subgraphs of G of orders m and n, respectively, such that the following conditions are satisfied:*

(1) $G_1 = G \setminus \{v_1, \dots, v_n\}$, where $v_1, \dots, v_n \in V(G)$ are vertices of degree one, all adjacent to the same vertex $v \in V(G) \setminus \{v_1, \dots, v_n\}$.

(2) There is an ODC $\mathcal{G}_1$ of K_m by G_1 such that $\{v\}$ is surjective with respect to $\mathcal{G}_1$.

(3) $G_2 = G \setminus U$, where $U \subseteq V(G)$ is an independent set of m vertices of degree one.

(4) There is an ODC $\mathcal{G}_2$ of K_n by G_2 such that the neighborhood $N(U)$ of U in G is surjective with respect to $\mathcal{G}_2$.

Then there is an ODC $\mathcal{G}$ of K_{m+n} by G. Moreover, if $U_1 \subseteq V(G_1)$ and $U_2 \subseteq V(G_2)$ are surjective with respect to $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, then $U_1 \cap U_2$ is surjective with respect to $\mathcal{G}$.

In the sequel, an ODC of some complete graph by a caterpillar R will be called surjective if the set of all spine vertices of R is surjective. As an immediate consequence of Lemma 4 we obtain:

Corollary 5. *Let R and S be caterpillars of diameter d such that S is a subgraph of R, and let s and r be the orders of S and R, respectively. Further let R' be a another caterpillar of diameter d such that $R = R' \setminus \{v_1, \dots, v_s\}$, where $v_1, \dots, v_s \in V(R')$ are pendant vertices, all adjacent to the same spine vertex. If there are surjective ODCs of K_s by S and of K_r by R, respectively, then there is a surjective ODC of K_{r+s} by R'.*

Iterated application of Corollary 5 gives:

Corollary 6. *Let S and T be caterpillars of diameter d and orders s and t, respectively, such that there are surjective ODCs of K_s by S and of K_t by T, respectively. Furthermore, assume that S and T are both subgraphs of the $(q_1, \dots, q_{d-1})$-caterpillar Q, and let k_i, $i = 1, \dots, d-1$, be linear combinations of s and t with non-negative integer coefficients. If there is a surjective ODC of the corresponding complete graph by Q, then there is a surjective ODC of the corresponding complete graph by the $(q_1 + k_1, \dots, q_{d-1} + k_{d-1})$-caterpillar.*

3 Proof of Theorem 2

Our strategy is as follows: We shall use Theorem 3 to generate group-generated, and hence surjective, ODCs of K_s and of K_t by caterpillars S and T, respectively, of diameter d. These ODCs will then be used together with Corollary 6 to show that almost all caterpillars of diameter d admit (surjective) ODCs. For the latter, it will be essential that s and t are relatively prime. We shall make use of the following fact a proof of which is an easy exercise.

Lemma 7. *Let s and t be positive integers. If s and t are relatively prime, then every integer greater than $st - s - t$ can be represented as a linear combination of s and t with non-negative integer coefficients.*

By Lemma 7 and Corollary 6, if we find S, T and Q as in Corollary 6 and such that s and t are relatively prime, then there are surjective ODCs by all $(r_1, \dots, r_{d-1})$-caterpillars with $r_i > q_i + st - s - t$ for $i = 1, 2, \dots, d-1$ and Theorem 2 is proved. It should be mentioned that if we already had S and T, then an appropriate Q can always be found just repeatedly applying Corollary 5 starting with $S = R$ and iterating until T is a subgraph of R'.

Lemma 8. *For every $d \geq 2$, there are group-generated ODCs of K_{2^d} and K_{3^d} by the $(2, 2^2 - 1, 2^3 - 1, \dots, 2^{d-2} - 1, 2^{d-1})$-caterpillar and the $(7, 2 \cdot 3^2 - 1, 2 \cdot 3^3 - 1, \dots, 2 \cdot 3^{d-2} - 1, 3^{d-1})$-caterpillar, respectively.*

Proof: The claim follows by induction on d and Theorem 3 with $m = 2$ and $m = 3$, respectively. $\square$

Proof of Theorem 2: Lemma 8 provides an appropriate choice for S and T with $s = 2^d$, $t = 3^d$, and such that S is a subgraph of T. Hence, Q can be chosen to be equal to T. Therefore, we have $q_i \leq 3^{d-1}$ for $i = 1, 2, \ldots, d-1$, and by

$$q_i + st - s - t \leq 3^{d-1} + 6^d - 2^d - 3^d < 6^d,$$

Theorem 2 holds with $k = 6^d$. $\square$

4 An improved bound for *k(d)*

In the previous section we did not make any effort to keep s and t small and ended up with $k(d)$ in Theorem 2 being exponential. Here we give some improvement based on group-generated ODCs of complete graphs by hamiltonian paths. For simplicity, assume that $d \geq 120$ in this section.

For $\ell \geq 2$, let P_ℓ denote the path of length $\ell - 1$.

Lemma 9. *Let $\ell \geq 3$, $m \geq 2$ and $j \geq 0$ be integers. If there is a group-generated ODC of K_ℓ by P_ℓ, then there is a group-generated ODC of the complete graph on $m^{k+1} \cdot \ell$ vertices by the $(q_1, q_2, \ldots, q_{\ell+j-1})$-caterpillar, where*

$$q_i = \begin{cases} 1 & \left| \; i = 1, \right. \\ 0 & \left| \; 2 \leq i \leq \ell - 2, \right. \\ (m-1)m^{i-\ell+1} \cdot \ell - 1 & \left| \; \ell - 1 \leq i \leq \ell + j - 2, \right. \\ (m-1)m^k \cdot \ell & \left| \; i = \ell + j - 1. \right. \end{cases}$$

Proof: The claim follows by induction on j and Theorem 3. $\square$

Let a^2 be the largest odd perfect square that is not divisible by 3 and not larger than $d + 1$. Group-generated ODCs of K_ℓ by P_ℓ have been constructed for all odd squares ℓ in [3]. Hence, there are such ODCs for $\ell = a^2$ and for $\ell = (a-2)^2$, and by Lemma 9 it follows that there are group-generated ODCs of K_s and K_t by S and T, respectively, where

$$s = 2^{d+1-a^2} \cdot a^2, \quad t = 3^{d+1-(a-2)^2} \cdot (a-2)^2,$$

S is the

$$(1, 0, 0, \ldots, 0, a^2 - 1, 2a^2 - 1, \ldots, 2^{d-a^2-1} \cdot a^2 - 1, 2^{d-a^2} \cdot a^2)\text{-caterpillar}$$

of diameter d, and T the

$$(1, 0, 0, \ldots, 0, 2(a-2)^2 - 1, 2 \cdot 3(a-2)^2 - 1, \ldots, \\ 2 \cdot 3^{d-(a-2)^2-1} \cdot (a-2)^2 - 1, 2 \cdot 3^{d-(a-2)^2} \cdot (a-2)^2)\text{-caterpillar}$$

of diameter d. By the choice of a, the numbers s and t are relatively prime. Furthermore, S is a subgraph of T, so we can choose $Q = T$ in Corollary 6. As the maximum degree in T is $2t/3$, Corollary 6 and Lemma 7 yield ODCs for all caterpillars of diameter d in which every spine vertex has at least $st - s - t/3$ neighbors of degree one.

By the choice of a, st is easily seen to be smaller than $d^2 \cdot (2^8 \cdot 3^{12})^{\sqrt{d+2}}$ which therefore can be chosen as $k(d)$ in Theorem 2 (if d is large enough).

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