Cyclically Indecomposable Triple Systems that are Decomposable

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February 15, 2005

Abstract

In this paper we investigate exhaustively the cyclically indecomposable triple systems $TS_\lambda(v)$ for $\lambda = 2$, $v \leq 33$ and $\lambda = 3, v \leq 21$ and we identify the decomposable ones. We also construct, by using Skolem-type and Rosa-type sequences, cyclically indecomposable twofold triple systems $TS_2(v)$ for all admissible orders. Further, we investigate exhaustively all cyclic $TS_2(v)$ that are constructed by Skolem-type and Rosa-type sequences up to $v \leq 45$ for indecomposability.

1 Introduction

A λ-fold triple system of order v, denoted $TS_\lambda(v)$ is a collection $\mathcal{B}$ of 3-subsets (called triples or blocks) from a v-set V, such that any given pair of elements in V lies in exactly λ triples. A one-fold triple system is called a Steiner triple system $STS(v)$.

A $TS_\lambda(v)$ is *simple* if it contains no repeated triples. A $TS_\lambda(v)$ is *cyclic*, $CTS_\lambda(v)$ if its automorphism group contains a v-cycle. A $TS_\lambda(v)$ is called *indecomposable* if its block set $\mathcal{B}$ cannot be partitioned into sets $\mathcal{B}_1, \mathcal{B}_2$ of blocks to form $TS_{\lambda_1}(v)$ and $TS_{\lambda_2}(v)$, where $\lambda_1 + \lambda_2 = \lambda$ with $\lambda_1, \lambda_2 \geq 1$.

The constructions of triple systems with the properties cyclic, simple and indecomposable, were studied by many researchers one property at a time; for example, cyclic triple systems for all λs were constructed by Colbourn and Colbourn [11], simple for $\lambda = 2$, by Stinson and Wallis [27]. Also, some of these properties were combined in studies; for example, indecomposable and simple for all λs studied Archdeacon and Dinitz [3], while Wang [28], constructed cyclic simple two-fold triple systems for all admissible orders and Zhang [29] constructed indecomposable simple $(v, 3, \lambda)$-BIBDs (for $v \geq 24\lambda - 5$).

When $v \equiv 0 \pmod 3$ a cyclic $CTS_2(v)$ must contain each block in the short orbit $\{0, v/3, 2v/3\} \pmod v$ twice. Provided that these are the only occurrences of repeated blocks, we will consider the $CTS_2(v)$ to be simple. In [22], the second and third authors constructed cyclic, simple and indecomposable two-fold triple systems for all admissible orders. They also introduced the notion of cyclically indecomposable triple systems. A $CTS_\lambda(v)$ is called *cyclically indecomposable* if its block set $\mathcal{B}$ cannot be partitioned into sets $\mathcal{B}_1, \mathcal{B}_2$ of blocks to form $CTS_{\lambda_1}(v)$ and $CTS_{\lambda_2}(v)$, where $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1, \lambda_2 \geq 1$. In [14] the first author computed the number of indecomposable non-isomorphic $BIBD(v, k, \lambda)$ for $k \leq 5, v \leq 13$ and $\lambda \leq 6$.

In this paper, we investigate the cyclically indecomposable triple systems. We construct two-fold cyclically indecomposable triple systems, $CTS_2(v)$, for all admissible orders. We also check exhaustively the triple systems that are cyclically indecomposable and we determine if they actually are decomposable (to non cyclic) or not. Furthermore, the structure of some non-cyclic decompositions is examined.

Up till now the only known examples for triple systems, that are cyclically indecomposable but decomposable, was $CTS_3(9)$. But we found many new examples the smallest for $\lambda = 3$ is this $CTS_3(15)$.

Example 1.1 *The Base blocks are:* $\{0, 1, 2\}$, $\{0, 1, 4\}$, $\{0, 2, 6\}$, $\{0, 2, 8\}$, $\{0, 3, 8\}$, $\{0, 3, 10\}$, $\{0, 4, 10\}$ (mod15). *This system can not be decomposed* *to two cyclic $CTS(15)$s but it can be decomposed to two non cyclic subsystems:*

1) $STS(15):$ $\{0,1,2\}$, $\{3,4,5\}$, $\{6,7,8\}$, $\{9,10,11\}$, $\{12,13,14\}$, $\{2,3,6\}$, $\{5,6,9\}$, $\{8,9,12\}$, $\{0,11,12\}$, $\{0,3,14\}$, $\{2,4,8\}$, $\{5,7,11\}$, $\{8,10,14\}$, $\{2,11,13\}$, $\{1,5,14\}$, $\{1,3,9\}$, $\{4,6,12\}$, $\{0,7,9\}$, $\{3,10,12\}$, $\{0,6,13\}$, $\{1,4,11\}$, $\{2,5,12\}$, $\{4,7,14\}$, $\{0,5,8\}$, $\{2,7,10\}$, $\{3,8,11\}$, $\{5,10,13\}$, $\{6,11,14\}$, $\{1,8,13\}$, $\{2,9,14\}$, $\{0,4,10\}$, $\{3,7,13\}$, $\{1,6,10\}$, $\{4,9,13\}$, $\{1,7,12\}$

2) $TS_2(15)$: $\{1,2,3\}$, $\{2,3,4\}$, $\{4,5,6\}$, $\{5,6,7\}$, $\{7,8,9\}$, $\{8,9,10\}$, $\{10,11,12\}$, $\{11,12,13\}$, $\{0,13,14\}$, $\{0,1,14\}$, $\{0,1,4\}$, $\{1,2,5\}$, $\{3,4,7\}$, $\{4,5,8\}$, $\{6,7,10\}$, $\{7,8,11\}$, $\{9,10,13\}$, $\{10,11,14\}$, $\{1,12,13\}$, $\{2,13,14\}$, $\{0,2,6\}$, $\{1,3,7\}$, $\{3,5,9\}$, $\{4,6,10\}$, $\{6,8,12\}$, $\{7,9,13\}$, $\{0,9,11\}$, $\{1,10,12\}$, $\{3,12,14\}$, $\{0,4,13\}$, $\{0,2,8\}$, $\{2,4,10\}$, $\{3,5,11\}$, $\{5,7,13\}$, $\{6,8,14\}$, $\{1,8,10\}$, $\{2,9,11\}$, $\{4,11,13\}$, $\{5,12,14\}$, $\{1,7,14\}$, $\{0,3,8\}$, $\{1,4,9\}$, $\{2,5,10\}$, $\{3,6,11\}$, $\{4,7,12\}$, $\{5,8,13\}$, $\{6,9,14\}$, $\{0,7,10\}$, $\{1,8,11\}$, $\{2,9,12\}$, $\{3,10,13\}$, $\{4,11,14\}$, $\{0,5,12\}$, $\{1,6,13\}$, $\{2,7,14\}$, $\{0,3,10\}$, $\{3,6,13\}$, $\{1,6,9\}$, $\{4,9,12\}$, $\{0,7,12\}$, $\{1,5,11\}$, $\{2,6,12\}$, $\{4,8,14\}$, $\{0,5,9\}$, $\{2,7,11\}$, $\{3,8,12\}$, $\{5,10,14\}$, $\{0,6,11\}$, $\{2,8,13\}$, $\{3,9,14\}$.

In 1957, T. Skolem [26] , when studying Steiner triple systems, considered the possibility of distributing the numbers $1,2,\dots,2n$ in n pairs (a_r,b_r) such that $b_r - a_r = r$ for $r = 1,2,\dots,n$. For example, for $n = 4$, the pairs $(1,2),(5,7),(3,6)$, and $(4,8)$ will be such a partition. Later, this partition was written as a sequence; the previous partition can be written as $1,1,3,4,2,3,2,4$, which is now known as a Skolem sequence of order 4.

Formally, a *Skolem sequence of order* n is a sequence $S = (s_1, s_2, \dots, s_{2n})$ of $2n$ integers that satisfy the following conditions:

(1) For every $k \in \{1,2,\dots,n\}$ there exist exactly two elements s_i, s_j such that $s_i = s_j = k$.

(2) If $s_i = s_j = k, i < j$, then $j - i = k$.

An *extended Skolem sequence* of order n is a sequence $ES = (s_1, s_2, \dots, s_{2n+1})$ of $2n+1$ integers that satisfy conditions (1), (2), and:

(3) There is exactly one $i \in \{1,\dots,2n+1\}$ such that $s_i = 0$.

The $s_i = 0$ is also known as the hook $(*)$ of the sequence, if $s_{2n} = 0$, then the sequence is called a *hooked Skolem* sequence. If $s_{n+1} = 0$, for $n \equiv 0,3 (\mod 4)$ then the sequence is called a *Rosa* sequence and if $s_{n+1} = s_{2n+1} = 0$, for $n \equiv 1,2 (\mod 4)$ then the sequence is called a *hooked Rosa* sequence. It is known that the necessary conditions for the existence of (hooked) (extended) Skolem sequences are sufficient and also for (hooked) Rosa sequences.

Theorem 1.2 *[Skolem] [26] A Skolem sequence of order n exists if and only if $n \equiv 0, 1 (\mod 4)$.*

[O'Keefe] [19] A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 (\mod 4)$.

[Abrham & Kotzig][2] An extended Skolem sequence of order n exists for all n.

[Baker] [5] An extended Skolem sequence of order n exists for all positions i of the hook, if and only if i is odd and $n \equiv 0, 1 (\mod 4)$ or i is even and $n \equiv 2, 3 (\mod 4)$.

[Rosa] [24] A Rosa sequence of order n exists if and only if $n \equiv 0, 3 (\mod 4)$ and a hooked Rosa sequence of order n exists if and only if $n \equiv 1, 2 (\mod 4)$.

The existence of a (hooked) Skolem sequence of order n implies the existence of a cyclic $STS(6n + 1)$ [8,12], and the existence of a (hooked) Rosa sequence implies the existence of a cyclic $STS(6n + 3)$ [24].

For example, the extended Skolem sequence (or Rosa sequence) of order 4; $1, 1, 3, 4, 0, 3, 2, 4, 2$ gives rise to the pairs $(a_r, b_r), r = 1, \ldots, 4, \{(1, 2), (7, 9), (3, 6), (4, 8)\}$ which gives the base blocks $\{0, i, b_i + 4\}$ (or $\{0, a_i + 4, b_i + 4\}$), $i = 1, \ldots, 4(\{0, 1, 6\}, \{0, 2, 13\}, \{0, 3, 10\}, \{0, 4, 12\})(\mod 27)$. With the addition of the base block $\{0, 9, 18\})(\mod 27)$, we get the base blocks of an $STS(27)$.

An *m-fold Skolem sequence* of order n is a sequence $mS = (s_1, s_2, \ldots, s_{2mn})$ with the following condition:

$(1)'$ For every $k \in \{1, 2, \ldots, n\}$ there exist m disjoint pairs $(i, i+k), i, i+k \in \{1, \ldots, 2mn\}$ such that $s_i = s_{i+k} = k$.

An *m-fold extended Skolem sequence* of order n is a sequence $mES = (s_1, s_2, \ldots, s_{2mn+1})$ with property $(1)'$, as well as the condition $(2)'$ there exists exactly one $s_i = 0, 1 \leq i \leq 2mn + 1$. If $s_{2mn} = 0$, the extended sequence is called an m-fold hooked Skolem sequence.

In [4], [5], it is shown that the necessary conditions are sufficient for the existence of m-fold (hooked) (extended) Skolem sequences.

Theorem 1.3 *An m-fold Skolem sequence of order n exists if and only if*

(1) $n \equiv 0, 1 (\mod 4)$, or

(2) $n \equiv 2, 3(\mod 4)$ *and* m *even,*

and a hooked m*-fold Skolem sequence of order* n *exists if and only if* $n \equiv 2$ *or* $3(\mod 4)$ *and* m *is odd.*

Theorem 1.4 *Let* m, n, k *be positive integers. There exists an extended* m*-fold Skolem sequence of order* n *with* $s_k = 0$ *if and only if one of the following conditions hold:*

(1) $n \equiv 0$ *or* $1(\mod 4)$, *and* k *is odd;*

(2) $n \equiv 2$ *or* $3(\mod 4), m$ *is even and* k *is odd;*

(3) $n \equiv 2$ *or* $3(\mod 4), m$ *is odd and* k *is even.*

For example, $2, 3, 2, 2, 3, 2, 1, 1, 3, 1, 1, 3$ is a 2-fold Skolem sequence of order 3 and $2, 2, 2, 2, 2, 0, 2, 1, 1, 1, 1, 1, 1$ is a 3-fold extended Skolem sequence of order 2.
A sequence $2T = (t_1, t_2, ..., t_{4n+2})$ is a *two-fold Rosa* sequence of order n if:
i) for every $k \in \{1, 2, ..., n\}$ there exist 2 disjoint pairs $(i, i + k)$, where $i, i + k \in \{1, 2, ...4n + 2\}$, such that $t_i = t_{i+k} = k$.
ii) $t_{n+1} = t_{3n+2} = 0$.
In [8] it was shown that:

Theorem 1.5 *There exists a two-fold Rosa sequence of order* n *if and only if* $n \geq 2$.

A *Langford* sequence of order n and defect d is a sequence $L = (l_1, l_2, ..., l_{2n})$ of $2n$ integers satisfying the conditions:
1) for every $k \in \{d, d + 1, ..., d + n - 1\}$ there exist exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$,
2) if $l_i = l_j = k$,$i < j$, then $j - i = k$.
The extended Langford sequences are defined in a similar manner to that of the extended Skolem sequences. For more details about (extended) Langford sequences the reader may consult [6, 10].

2 Constructing Simple Two-Fold Triple Systems

We will use the following constructions for $CTS_2(v)$ from Skolem sequences:

Construction 2.1 *(Rees, Shalaby, Sharary, [23]) Let* $2S = (s_1, s_2, \ldots, s_{4n})$ *be a two-fold Skolem sequence of order* n*. Then the set of triples* $\{\{0, r, b_r + n\}, \{0, r, d_r + n\} : r = 1, 2 \ldots, n\}$ *form the base blocks for a* $CTS_2(6n+1)$ *(where as usual* (a_r, b_r) *and* (c_r, d_r) *are the pairs of positions in* $2S$ *for which* $b_r - a_r = d_r - c_r = r, r = 1, 2, \ldots, n$*).*

Construction 2.2 *(Rees, Shalaby, Sharary, [23]) Let* $2T = (t_1, t_2, \ldots, t_{4n+2})$ *be a two-fold Rosa sequence of order* n*. (In particular,* $t_{n+1} = t_{3n+2} = 0$*). The set of triples* $\{\{0, r, b_r + n\}, \{0, r, d_r + n\} : r = 1, 2 \ldots, n\}$ *form the base blocks for a cyclic two-fold 3-GDD of type* 3^{2n+1} *(whose groups are given by* $\{0, 2n+1, 4n+2\}(\mod 6n+3)$*) which in turn gives rise to a* $CTS_2(6n+3)$*. (Again* (a_r, b_r) *and* (c_r, d_r) *are the pairs of positions in* $2T$ *for which* $b_r - a_r = d_r - c_r = r, r = 1, 2, \ldots, n$*).*

Construction 2.3 *(Rees, Shalaby, [22]) Let* $S = (s_1, s_2, \ldots, s_{2n})$ *be a Skolem sequence of order* n *and let* $\{(a_r, b_r) : r = 1, 2, \ldots, n\}$ *be the pairs of positions in* S *for which* $b_r - a_r = r$*. Then the set* $\{r, a_r + n, b_r + n\}$ *partitions the set* $\{1, 2, \ldots, 3n\}$ *into* n *triples* (a, b, c) *such that* $a + b \equiv c(\mod 3n+1)$*. Hence the set of triples* $\{\{0, r, b_r + n\} : r = 1, 2 \ldots, n\}$ *form the base blocks for a cyclic two-fold triple system* $CTS_2(3n+1)$*. For example,*

$$n = 1 \quad 11 \quad (1,2,3) \Rightarrow \{0,1,3\}(\mod 4)$$

$$n = 4 \quad 11342324 \quad (1,5,6)\ (2,9,11)\ (3,7,10)\ (4,8,12)$$
$$\Rightarrow \{0,1,6\}\ \{0,2,11\}\ \{0,3,10\}\ \{0,4,12\}\ (\mod 13)$$

Construction 2.4 *(Rees, Shalaby [22] Let* $T = (t_1, t_2, \ldots, t_{2n+1})$ *be a Rosa sequence of order* n*. (In particular,* $t_{n+1} = 0$*), and let* $\{(a_r, b_r)\}$ *be the set of positions in* T *for which* $b_r - a_r = r, r = 1, 2, \ldots, n$*. Then the set* $\{r, a_r + n + 1, b_r + n + 1\}$ *partitions the set* $\{1, 2, \ldots, 3n+2\} \backslash \{n+1, 2n+2\}$ *into* n *triples* (a, b, c) *such that* $a + b \equiv c(\mod 3n+3)$*. Hence the set of triples* $\{\{0, r, b_r + n + 1\} : r = 1, 2 \ldots, n\}$ *form the base blocks for a cyclic two-fold 3-GDD of type* 3^{n+1} *(whose groups are given by* $\{0, n+1, 2n+2\}(\mod 3n+3)$*) which in turn gives rise to a* $CTS_2(3n+3)$*. For example,*

$$n = 3 \quad 1130232 \quad (1,5,6)\ (2,9,11)\ (3,7,10)$$

$$\Rightarrow \{0,1,6\}\ \{0,2,11\}\ \{0,3,10\}\ (\textit{with 2 copies of } \{0,4,8\})\ (\mod 12)$$

Theorem 2.5 *(Rees, Shalaby [22])*

(i) The $CTS_2(v)$*s produced by Constructions 2.1 and 2.3 are simple.*

(ii) The GDDs produced by Constructions 2.2 and 2.4 are simple.

3 Cyclically Indecomposable Two-Fold Triple Systems

We will make use of the following results.

Lemma 3.1 *(Rees, Shalaby [22]) If $2S = (s_1, s_2, \ldots, s_{4n})$ is a two-fold Skolem sequence of order n and the pairs $(a_r, b_r), (c_r, d_r)$ contain among them a pair (x_r, y_r) where $x_r + y_r = 4n + 1$ then the corresponding $CTS_2(6n + 1)$ (arising out of Construction 2.1) is indecomposable.*

Lemma 3.2 *(Rees, Shalaby [22]) If $2T = (t_1, t_2, \ldots, t_{4n+2})$ is a two-fold Rosa sequence of order n and the pairs $(a_r, b_r), (c_r, d_r)$ contain among them a pair (x_r, y_r) where $x_r + y_r = 4n + 3$ then the corresponding $CTS_2(6n + 3)$ (arising out of Construction 2.2) is indecomposable.*

Lemma 3.3 *(Rees, Shalaby [22]) If $S = (s_1, s_2, \ldots, s_{2n})$ is a Skolem sequence of order n in which $s_{2n-1} = s_{2n} = 1$, then the corresponding $CTS_2(3n + 1)$ (arising out of Construction 2.3) is indecomposable.*

Lemma 3.4 *(Rees, Shalaby [22]) If $T = (t_1, t_2, \ldots, t_{2n+1})$ is a Rosa sequence of order n in which $t_{2n} = t_{2n+1} = 1$, then the corresponding $CTS_2(3n + 3)$ (arising out of Construction 2.4) is indecomposable.*

When discussing cyclic m-fold triple systems, there is a weaker notion of indecomposability that is sometimes useful to consider. We defined in the introduction a cyclic m-fold triple system to be cyclically indecomposable if it does not contain a cyclic m'-fold triple system for any $0 < m' < m$. In our context, where $m = 2$, a two-fold cyclic triple system $CTS_2(v)$ is cyclically indecomposable if it does not contain a cyclic $STS(v)$ as a subsystem (the complement of which would of course be a second cyclic $STS(v)$).

Thus, let $2S = (s_1, s_2, \ldots, s_{4n})$ be a two-fold Skolem sequence of order n and suppose that we can write $2S$ as a vector sum $2S = S_1 + S_2$ of sequences $S_1 = (s_1^1, s_2^1, \ldots, s_{4n}^1), S_2 = (s_1^2, s_2^2, \ldots, s_{4n}^2)$, (whence $2S = (s_1^1 + s_1^2, s_2^1 + s_2^2, \ldots, s_{4n}^1 + s_{4n}^2)$, each one of which satisfies the following two properties:

(1) For each $k \in \{1, 2, \ldots, n\}$ there are exactly two elements $s_i^\alpha, s_j^\alpha \in S_\alpha$ such that $s_i^\alpha = s_j^\alpha = k$ and $j - i = k$.

(2) For each $1 \le i \le 2n$ exactly one of $s_i^\alpha, s_{4n-i+1}^\alpha$ is equal to 0.

Among the pairs $(a_r, b_r), (c_r, d_r)$ arising from $2S$ via Construction 2.1 we choose only those pairs that correspond to non-zero entries in S_1. Since there are $2n$ non-zero entries in S_1 we will therefore have a set of n difference triples, each of the form (r, a_r+n, b_r+n) or $(6n+1-r, d_r+n, c_r+n)$, over $\mathbb{Z}_{6n+1}$. Now because of Property (1), it follows that for each $k = 1, 2, \ldots, n$, exactly one of $k, 6n+1-k$ will appear as a difference among these n difference triples, and because of Property (2), the same will be true for each $k = n+1, n+2, \ldots, 3n$. Therefore, the set $\{\{0, r, y_r+n\} : r = 1, 2, \ldots, n\}$ of base blocks arising out of these n difference triples will generate a cyclic $STS(6n+1)$, whence the $CTS_2(6n+1)$ arising from the original sequence $2S$ is cyclically decomposable.

On the other hand, suppose that we have a two-fold Skolem sequence $2S = (s_1, s_2, \ldots, s_{4n})$ where the $CTS_2(6n+1)$ arising via Construction 2.1 is cyclically decomposable, that is, contains a cyclic $STS(6n+1)$ as a subsystem. Then among the $2n$ base blocks for the $CTS_2(6n+1)$ there are n of them which generate the cyclic $STS(6n+1)$; let these base blocks be $\{0, r, y_r+n\}$ for $r = 1, 2, \ldots, n$. Then for each $k = 1, 2, \ldots, 3n$, exactly one of $k, 6n+1-k$ will appear as a difference among the corresponding n difference triples, each of the form (r, a_r+n, b_r+n) or $(6n+1-r, d_r+n, c_r+n)$. Now construct a sequence $S_1 = (s_1^1, s_2^1, \ldots, s_{4n}^1)$ as follows. For each $k = 1, 2, \ldots, n$, if (k, a_k+n, b_k+n) is one of the foregoing n difference triples, then set $s_{a_k}^1 = s_{b_k}^1 = k$; otherwise $(6n+1-k, d_k+n, c_k+n)$ is one of the n difference triples and we set $s_{c_k}^1 = s_{d_k}^1 = k$. Set all remaining s_i^1 equal to 0. Now S_1 clearly satisfies Property (1) above. With regards to Property (2), suppose that $s_i^1 = k$ for some $1 \le i \le 2n$. Then the difference $i+n$ appears among the n difference triples, whence the difference $6n+1-(i+n) = 5n-i+1$ does not. Hence $s_{4n-i+1}^1 = 0$. On the other hand, if $s_i^1 = 0$, then the difference $i+n$ does not appear among the n difference triples and so the difference $6n+1-(i+n) = 5n-i+1$ must so appear, whence $s_{4n-i+1}^1 = k \in \{1, 2, \ldots, n\}$. Thus S_1 satisfies Property (2) above. Now let S_2 be the vector difference $S_2 = S - S_1$. Then we have $S = S_1 + S_2$ where each S_α satisfies Properties (1) and (2) above. The foregoing discussion now gives us the following.

Theorem 3.5 *Let $2S = (s_1, s_2, \ldots, s_{4n})$ be a two-fold Skolem sequence of order n. Then the two-fold cyclic triple system $CTS_2(6n+1)$, arising out of Construction 2.1, is cyclically indecomposable if and only if* $2S$ cannot *be written as a vector sum $2S = S_1 + S_2$, where each S_α satisfies Properties (1) and (2) above.*

Theorem 3.5 has an obvious analogue for $CTS_2(6n+3)$s:

Theorem 3.6 *Let $2T = (t_1, t_2, \ldots, t_{4n+2})$ be a two-fold Rosa sequence of order n. Then the two-fold cyclic triple system $CTS_2(6n+3)$, arising out* *of Construction 2.2 is cyclically indecomposable if and only if* $2T$ cannot *be written as a vector sum* $2T = T_1 + T_2$*, where each* T_α *satisfies the following two properties:*

(1)′ For each $k \in \{1, 2, \ldots, n\}$ *there are exactly two elements* $t_i^\alpha, t_j^\alpha \in T_\alpha$ *such that* $t_i^\alpha = t_j^\alpha = k$ *and* $j - i = k$*.*

(2)′ $t_{n+1}^\alpha = t_{3n+2}^\alpha = 0$ *and, for each* $i \in \{1, 2, \ldots, 2n+1\} \backslash \{n+1\}$ *exactly one of* $t_i^\alpha, t_{4n-i+3}^\alpha$ *is equal to 0.*

Thus, for example, the two-fold Skolem sequences of order $n > 2$ constructed in [[23], Theorem 2.2] all give rise to cyclically indecomposable two-fold triple systems of order $6n + 1$:

Theorem 3.7 *Let* $n > 2$ *and let* 0_n *be the largest odd integer not exceeding* n *and let* E_n *be the largest even integer not exceeding* n*. Then let* $2S = (E_n, E_n - 2, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n, E_n - 2, \ldots, 4, 2, E_n, 2, 4, \ldots, E_n - 2, 0_n, 0_n - 2, \ldots, 3, 1, 1, 3, \ldots, 0_n - 2, 0_n, 0_n, 0_n - 2, \ldots, 3, 1, 1, 3, \ldots, 0_n - 2, 0_n)$*. Then* $2S$ *yields (via Construction 2.1) a cyclically indecomposable two-fold cyclic triple system* $CTS_2(6n + 1)$*.*

Proof.
Suppose first that n is even, and that $2S = S_1 + S_2$. Without loss of generality, we may suppose that $s_1^1 = E_n = n$ and $s_{n+1}^1 = n$. But then we would have $s_{4n-1+1}^1 = s_{4n}^1 = 0$ and $s_{4n-(n+1)+1}^1 = s_{3n}^1 = 0$, whence S_1 will not contain $0_n = n - 1$, a contradiction. Hence $2S \neq S_1 + S_2$ and the corresponding $CTS_2(6n + 1)$ is cyclically indecomposable.

Now suppose that n is odd. If $n = 3$, then the corresponding sequence is $2S = (2, 2, 2, 2, 3, 1, 1, 3, 3, 1, 1, 3)$, which gives rise to an indecomposable $CTS_2(19)$ (apply Lemma 3.1 with $x_1 = 6$ and $y_1 = 7$) which is of course cyclically indecomposable. For $n \geq 5$, we suppose that $2S = S_1 + S_2$. Without loss of generality, we may suppose that $s_1^1 = E_n = n - 1$ and $s_n^1 = n - 1$. Now this forces $s_{4n-1+1}^1 = s_{4n}^1 = 0$ and $s_{4n-(n)+1}^1 = s_{3n+1}^1 = 0$, whence $s_{3n}^1 = 0$ and $s_{4n-1}^1 = 0$. But then $s_{4n-3n+1}^1 = s_{n+1}^1 = E_n - 2 = n - 3$ and $s_{4n-(4n-1)+1}^1 = s_2^1 = E_n - 2 = n - 3$. This means that S_1 must contain all four copies of the number $n - 3$, a contradiction. Hence $2S \neq S_1 + S_2$ and again the corresponding $CTS_2(6n + 1)$ is cyclically indecomposable. □

Remark 3.8 *With regards* $n = 1$ *and* 2 *in relation to Theorem 3.7, the only two-fold Skolem sequence of order 1 is* (1111)*, which gives rise to the cyclic* $CTS_2(7)$ *whose base blocks are* $\{0, 1, 3\}$ *and* $\{0, 1, 5\}$*, each of which generates a (cyclic)* $STS(7)$*. On the other hand, there are two two-fold Skolem* *sequences of order 2, namely* (11112222) *and* (11222211). *Now the* $CTS_2(13)$ *arising from* (11112222) *has as its base blocks* $\{0,1,4\},\{0,1,6\},\{0,2,9\}$ *and* $\{0,2,10\}$*, no pair of which generates an* $STS(13)$*; hence this* $CTS_2(13)$ *is cyclically indecomposable. But the sequence* (11222211) *can be written as* S_1+S_2*, where* $S_1=(11202000)$ *and* $S_2=(00020211)$ *whence the corresponding* $CTS_2(13)$ *(whose base blocks are* $\{0,1,4\},\{0,1,10\},\{0,2,7\}$*, and* $\{0,2,8\}$*) is cyclically decomposable into the two* $STS(13)$*s generated, respectively, by* $\{\{0,1,4\},\{0,2,7\}\}$ *and* $\{\{0,1,10\},\{0,2,8\}\}$.

In a similar fashion, the two-fold Rosa sequences of order $n\geq 3$ constructed in [9, Theorem 3.4] all give rise to cyclically indecomposable two-fold triple systems of order $6n+3$:

Theorem 3.9 *(i) Let* n *be even,* $n\geq 4$*, and let* $2T=(n-1,n-3,\ldots,3,\ 1,1,3,\ldots,n-3,n-1,0,n,n-2,\ldots,4,2,n,2,4,\ldots,n,n-2,\ldots,4,2,n,2,4,\ldots,n-2,0,n-1,n-3,\ldots,3,1,1,3,\ldots,n-3,n-1)$. *Then* $2T$ *yields (via Construction 2.2) a cyclically indecomposable two-fold cyclic triple system* $CTS_2(6n+3)$.

(ii) Let n *be odd,* $n\geq 3$*, and let* $2T=(11202232330311)$ *if* $n=3$*,* $2T=(31135023253411540425 24)$ *if* $n=5$*,* $2T=(5311357035232756411746064272 46)$ *if* $n=7$*; if* $n\equiv 1 \pmod 4$ *and* $n\geq 9$*, then take* $2T=(n-2,n-4,\ldots,1,1,3,\ldots,n-2,n,0,n-4,n-2,n-8,n-6,\ldots,[572325397],\ldots,n-4,n-6,n,n-2,n-1,n-3,\ldots,4,1,1,n,4,6,\ldots,n-1,0,n-1,n-3,\ldots,2,n,2,4,\ldots,n-1)$*, while if* $n\equiv 3 \pmod 4$ *and* $n\geq 11$*, then take* $2T$ *to be the foregoing sequence, with the subsequence* [572325397] *replaced by* [793523275]*. Then* $2T$ *yields (via Construction 2.2) a cyclically indecomposable two-fold cyclic triple system* $CTS_2(6n+3)$.

Proof.

(i) Suppose that $2T=T_1+T_2$. Without loss of generality, we may suppose that $t^1_{n+2}=t^1_{2n+2}=n$. But then $t^1_{4n-(n+2)+3}=t^1_{3n+1}=0$ and $t^1_{4n-(2n+2)+3}=t^1_{2n+1}=0$; this means that T_1 will not contain $n-2$, a contradiction. Hence $2T\neq T_1+T_2$ and the corresponding $CTS_2(6n+3)$ is cyclically indecomposable.

(ii) Let $n\geq 7$ and suppose that $2T=T_1+T_2$. Without loss of generality, we may suppose that $t^1_1=n-2$, whereupon $t^1_{4n+2}=0$ and so $t^1_{3n+3}=0$. Now $t^1_{3n+3}=0\Rightarrow t^1_{3n+1}=n-1\Rightarrow t^1_{2n+2}=n-1\Rightarrow t^1_{2n+1}=0\Rightarrow t^1_{n+3}=0\Rightarrow t^1_{4n-(n+3)+3}=t^1_{3n}=n-3\Rightarrow t^1_{2n+3}=n-3\Rightarrow t^1_{2n}=$

$0 \Rightarrow t^1_n = 0$. Thus, we have $t^1_{3n+3} = 0$ and $t^1_n = 0$, a contradiction. Hence $2T \neq T_1 + T_2$ and the corresponding $CTS_2(6n+3)$ is cyclically indecomposable.

We leave the verification for $n = 3$ and $n = 5$ as an exercise for the reader.

□

Remark 3.10 *With regards $n = 1$ and 2 in relation to Theorem 3.9, there is no two-fold Rosa sequence of order 1, while the only two-fold Rosa sequence of order 2 is (*1102222011*), and this sequence can be written as $T_1 + T_2$ where $T_1 = (1102020000)$ and $T_2 = (0000202011)$. The corresponding $CTS_2(15)$ (whose base blocks are $\{0,1,4\}, \{0,1,12\}, \{0,2,8\}, \{0,2,9\}$, and $\{0,5,10\}, \{0,5,10\}$) is therefore cyclically decomposable into the two $STS(15)$s generated, respectively, by $\{\{0,1,4\}, \{0,2,8\}, \{0,5,10\}\}$ and $\{\{0,1,12\}, \{0,2,9\}, \{0,5,10\}\}$.*

4 Cyclically indecomposable triple systems that are decomposable

In this section, we will investigate exhaustively the decomposability of $CTS_\lambda(v)$ for $\lambda = 2, v \leq 33$ and $\lambda = 3, v \leq 21$. To do so, we need some more definition. Let $B = \{b_1, b_2, b_3\}$ be a block. A *translate* $B + i, i \in \mathbb{Z}_v$ of B is the block $B + i = \{b_1 + i, b_2 + i, b_3 + i\} \mod v$. In a CTS the set of distinct translates forms a block orbit. An arbitrarily fixed block in a block orbit is called a *base block* for this orbit. A base block B is *canonical* if it is lexicographically smallest in its block orbit and is said to be *short* if $B + i = B$ for some nonzero $i \in \mathbb{Z}_v$. To represent a CTS it suffices to list all its canonical base blocks. All blocks in one orbit provide the same (multi) set of differences $d(B) = \{\pm(b_2 - b_1), \pm(b_3 - b_1), \pm(b_3 - b_2)\}$ or, if B is a short block $d(B) = \{\pm(b_2 - b_1)\} = \{\pm(v/3)\}$. Given a block B and an integer w which is co-prime to v, we define $w \cdot B = \{wb_1, wb_2, wb_3\} \mod v$. Two CTS with block sets $\mathcal{B}_1, \mathcal{B}_2$ are *equivalent* if there exist $w, i \in \mathbb{Z}_v$ such that for each canonical base block $B_1 \in \mathcal{B}_1$ there is some canonical base block $B_2 \in \mathcal{B}_2$ with $w \cdot B_1 + i = B_2$. Non-isomorphic CTS are clearly inequivalent. Unfortunately, the converse is not true in general the smallest known counterexample being a $CTS_2(16)$, see Brand [7]. Under certain circumstances one can ensure that inequivalent CTS are also non-isomorphic, see Bays [6], Lambossy [17], Pálfy [20], Phelps [21] or Brand [8]. Although these conditions do not apply for all orders v considered here, we use the equivalence notation because this is computational less demanding as a complete isomorphism test. A CTS is said to be *canonical* if its representation by canonical base blocks is lexicographically smallest among the representation of all CTS in its equivalence class.

We start our investigations by determining a list with all inequivalent $CTS_\lambda(v)$, for $\lambda = 2$ or 3, $v \equiv 1, 3 \mod 6$. Note, that $CTS_2(v), v \equiv 0, 4 \mod 6$ and $CTS_3(v), v \equiv 5 \mod 6$ also exist, but are trivially indecomposable as there is no $STS(v)$ for $v \equiv 0, 4, 5 \mod 6$. The list is created by a backtrack-algorithm, a search technique which builds up partial solutions, exhaustively covering all possibilities in a systematic fashion. For more information on search techniques used in design theory see for example Colbourn [10], Gibbons [13] or Kreher and Stinson [16]. In our problem the search space for the backtrack consists of all canonical base blocks, and a *partial* $CTS_\lambda(v)$ representation is a collection of canonical base blocks with the additional property that every difference $d \in \mathbb{Z}_v \setminus \{0\}$ occurs at most λ times among the differences of the base blocks. A partial CTS with canonical base block representation $\mathcal{R}$ is said to be *proper* if $\mathcal{R}$ is lexicographically smallest among the partial CTS representations in its equivalence class. The task of our enumeration problem is to find all proper partial CTS representations where every difference d occurs exactly λ times among the differences of the base blocks. Using this approach we constructed all inequivalent, canonical $CTS_\lambda(v)$ for $\lambda = 2, v \leq 31$ and $\lambda = 3, v \leq 21$. The number $IECTS$ of inequivalent $CTS_\lambda(v)$ over $\mathbb{Z}_v$ is listed in Tables 1 and 2 and is the same as listed in [1, Table IV.10.79].

In a second step we try to (cyclically) decompose each of the constructed canonical CTS. Colbourn and Colbourn [12] proved that deciding whether a $TS_\lambda(v)$ ($\lambda = 3, 4$) is decomposable is NP-complete. Whereas deciding whether a $TS_2(v)$ and therefore a $CTS_2(v)$ is decomposable can be done by a polynomial time algorithm, see Kramer [15]. To do this we formulate our problem as a problem for (multi)graphs. The *(pair) block-intersection graph* of a CTS has the block set $\mathcal{B}$ as vertex set and there is an edge between blocks B_1 and B_2 ($B_1 \neq B_2$) labeled with the pair $\{i, j\}$ if $\{i, j\} \subseteq B_1 \cap B_2$. Note that multiple edges (with distinct labels) are possible if two blocks intersect in more than two elements. Moreover, the edges with label $\{i, j\}$ form for each pair $\{i, j\} \subset \mathbb{Z}_v$ ($i \neq j$) a λ-clique.

Theorem 4.1 *A $CTS_\lambda(v)$ is decomposable if and only if there is a $\lambda' \in \mathbb{N}$ with $1 \leq \lambda' \leq \lfloor \frac{\lambda}{2} \rfloor$ and a coloring (red, blue) of the vertices (i.e. blocks) of the block-intersection graph such that in the subgraph induced by the red vertices the edges with label $\{i, j\}$ form for each pair $\{i, j\} \subset \mathbb{Z}_v$ ($i \neq j$) a λ'-clique.*

In the case $\lambda = 2$ such a coloring exists if and only if the block-intersection

v	7	9	13	15	19	21	25	27	31
$IECTS$	2	0	9	9	201	175	19543	10841	2532755
$IDCTS$	0	0	6	5	161	109	18201	10320	2468671
$CIDCTS$	0	0	6	5	161	109	18201	10320	2468671

Table 1: Decomposability for $CTS_2(v)$ with $v \leq 31$

v	7	9	13	15	19	21
$IECTS$	3	4	47	421	13316	212968
$IDCTS$	1	1	24	355	8839	209825
$CIDCTS$	1	4	24	400	8840	202578

Table 2: Decomposability for $CTS_3(v)$ with $v \leq 21$

graph is bipartite which can efficiently be checked in linear time. For $\lambda \geq 3$ we used a backtrack algorithm described in [14] to obtain the number $IDCTS$ of indecomposable CTS. The results are summarized in Tables 1 and 2 and the actual decompositions are available from the authors upon request.

Similarly, in order to decide if a CTS represented by the set of canonical base blocks $\mathcal{R}$ is cyclically decomposable we define the *base block-difference graph* with vertex set $\mathcal{R}$ which has an edge between base blocks B_1 and B_2 labeled with d if either $B_1 \neq B_2$ and $d \in d(B_i) \cap d(B_j)$, or $B_1 = B_2$ and $r_d(B_1) > 1$, where $r_d(B)$ counts how often difference d is repeated in the multi set $d(B)$. Edges of the first kind are repeated $r_d(B_1) \cdot r_d(B_2)$ times, while loops are repeated $\binom{r_d(B_1)}{2}$ times. Here, the edges with label d form for each $d \in Z_v \setminus \{0\}$ a (possibly degenerated) λ-clique. Degenerated means that some vertices of the clique may collapse into one vertex generating multiple edges and loops.

Theorem 4.2 *A $CTS_\lambda(v)$ is cyclically decomposable if and only if there is a $\lambda' \in \mathbb{N}$ with $1 \leq \lambda' \leq \lfloor \frac{\lambda}{2} \rfloor$ and a coloring (red, blue) of the vertices (i.e. base blocks) of the base block-difference graph such that in the subgraph induced by the red vertices the edges with label d form for each $d \in \mathbb{Z}_v \setminus \{0\}$ a (degenerated) λ'-clique.*

Again, such a coloring exists for $\lambda = 2$ if and only if the block-intersection graph is bipartite which can efficiently be computed. For $\lambda \geq 3$ we used a variation of the backtrack algorithm described in [14] that is able to deal with loops to obtain the number $CIDCTS$ of cyclically indecomposable CTS. The results are displayed Tables 1 and 2.

v	7	9	13	15	19	21	25	27	31	33
$IECTSwRD$	2	0	7	8	116	118	11774	6257	1512940	1050764
$IDCTSwRD$	0	0	4	4	76	52	10432	5736	1448856	992656
$CIDCTSwRD$	0	0	4	4	76	52	10432	5736	1448856	992656

Table 3: Decomposability for $CTS_2(v)$ without repeated differences with $v \leq 33$

The following observation was helpful to speed up the computations in the case $\lambda = 2$ and to get an additional result when $v = 33$.

Lemma 4.3 *A $CTS_2(v)$ having a base block B those set of differences $d(B)$ contains a repeated difference d is indecomposable.*

Proof.
As already mentioned, we only need to consider $v \equiv 1, 3 \mod 6$. Suppose that $B = \{x, x+d, x+2d\}$, then $B + d = \{x+d, x+2d, x+3d\}$ contains a common pair $\{x+d, x+2d\}$ with B. Thus, if B is colored red, then $B + d$ must be colored blue, $B + 2d$ red, $B + 3d$ blue again, and so forth. So for all $i \in \mathbb{Z}_v$ the blocks $B + 2id$ need to be colored red and the blocks $B + (2i+1)d$ need to be colored blue, which is impossible as $2id$ and $(2i+1)d$ generate the same orbit for odd v. $\square$

In Table 3 we present the results where we only considered inequivalent $CTS_2(v)$ without repeated differences (wRD) in the canonical base blocks. In the case $v = 33$ we did not create all inequivalent CTS_2 just those without repeated differences so this value is missing in Table 1.

We remark that there is no cyclically indecomposable $CTS_2(v), v \leq 33$ that is decomposable. But it is worth to notice that some cyclically decomposable CTS_2 also admit a non-cyclic decomposition.

Example 4.4 *The $CTS_2(21)$ generated by the base blocks $\{0,1,3\}, \{0,1,9\}, \{0,2,5\}, \{0,4,10\}, \{0,4,12\}, \{0,5,15\}, \{0,7,14\}, \{0,7,14\}$ contains a cyclic sub-design with base blocks $\{0,1,3\}, \{0,4,12\}, \{0,5,15\}, \{0,7,14\}$, but also contains a non-cyclic triple system which can be obtained by developing the following blocks $+3 \mod 21$: $\{0,1,3\}, \{1,2,4\}, \{2,3,11\}, \{0,2,5\}, \{2,6,12\}, \{0,4,12\}, \{1,5,13\}, \{1,6,16\}, \{0,7,14\}, \{1,8,15\}, \{2,9,16\}$.*

Cyclically indecomposable $CTS_3(v)$ that are decomposable exist for $v =$ 9, 15, 19 or 21, but not for $v = 7$ or 13. Concerning the structure of the decompositions we observe that most sub STS are generated $+3 \mod v$. So the $STS(15)$ in Example 1.1 can be represented by the blocks $\{0,1,2\}, \{2,3,6\}$,

$\{2,4,8\},\{1,3,9\},\{1,4,11\},\{2,5,12\},\{0,4,10\}$, all remaining blocks are formed by adding 3 modulo 15. On the other hand there are decompositions which are not that easy to generate.

Example 4.5 *The $CTS_3(9)$ represented by base blocks $\{0,1,2\}$, $\{0,1,5\}$, $\{0,2,4\},\{0,3,6\},\{0,3,6\},\{0,3,6\}$ can be decomposed into a $STS(9)$ and a $TS_2(9)$ in the following way. For the $STS(9)$ take blocks $\{0,1,2\}$, $\{3,5,7\}$, $\{6,4,8\}$, $\{0,7,8\}$, $\{2,3,4\}$, $\{1,5,6\}$, $\{0,4,5\}$, $\{3,1,8\}$, $\{6,2,7\}$, $\{0,3,6\}$, $\{1,4,7\}$, $\{2,5,8\}$, which are not closed under addition with $+3 \mod 9$. But note that there is also a cyclic sub Steiner triple system of the $CTS_3(9)$ which is generated by developing the blocks $\{0,1,2\},\{2,3,7\},\{2,4,6\},\{0,3,6\}$, $\{1,4,7\}$, $\{2,5,8\}$ $+3 \mod 9$ (the last 3 blocks are short blocks).*

With the examples above in mind one might ask whether for all decomposable $CTS_3(v)$ there is a decomposition generated $+3 \mod v$. This is not the case as the unique cyclically indecomposable, but decomposable $CTS_3(19)$ shows.

Example 4.6 *The $CTS_3(19)$ represented by base blocks $\{0,1,2\}$, $\{0,1,8\}$, $\{0,2,4\}$, $\{0,3,6\}$, $\{0,3,11\}$, $\{0,4,10\}$, $\{0,4,13\}$, $\{0,5,10\}$, $\{0,5,12\}$ contains the following sub $STS(9)$: $\{0,1,2\}$, $\{0,17,18\}$, $\{2,3,4\}$, $\{4,5,6\}$, $\{6,7,8\}$, $\{8,9,10\}$, $\{10,11,12\}$, $\{12,13,14\}$, $\{14,15,16\}$, $\{5,16,17\}$, $\{1,3,5\}$, $\{1,16,18\}$, $\{5,7,9\}$, $\{9,11,13\}$, $\{13,15,17\}$, $\{1,4,17\}$, $\{4,7,10\}$, $\{5,8,11\}$, $\{10,13,16\}$, $\{11,14,17\}$, $\{0,3,11\}$, $\{0,8,16\}$, $\{1,9,12\}$, $\{2,5,13\}$, $\{2,10,18\}$, $\{3,6,14\}$, $\{4,12,15\}$, $\{6,9,17\}$, $\{7,15,18\}$, $\{1,10,14\}$, $\{2,6,12\}$, $\{2,8,17\}$, $\{2,11,15\}$, $\{3,7,13\}$, $\{3,9,18\}$, $\{3,12,16\}$, $\{4,8,14\}$, $\{5,14,18\}$, $\{8,12,18\}$, $\{0,4,13\}$, $\{0,6,10\}$, $\{0,9,15\}$, $\{1,7,11\}$, $\{1,6,15\}$, $\{2,7,16\}$, $\{5,10,15\}$, $\{6,11,16\}$, $\{7,12,17\}$, $\{0,5,12\}$, $\{0,7,14\}$, $\{1,8,13\}$, $\{2,9,14\}$, $\{3,8,15\}$, $\{3,10,17\}$, $\{4,9,16\}$, $\{4,11,18\}$, $\{6,13,18\}$.*

5 Cyclically indecomposable two-fold triple systems constructed from Skolem-type and Rosa-type sequences

We also investigated exhaustively all $CTS_2(v)$ that are constructed by Skolem-type and Rosa-type sequences up to $v \leq 45$ for indecomposability. All Skolem and Rosa sequences used are constructed by Churchill and Shalaby [9], the listings of the sequences are available from the authors upon request. The number of sequences considered are presented in the Appendix in Tables 8 and 9.

We form with Constructions 2.1 to 2.4 for each given sequence the corresponding $CTS_2(v)$. Following Lemma 4.3 we only need to do this for

n	1	2	3	4	5	6	7
v	7	13	19	25	31	37	43
No. of CTS from 2-Skolem seq.	1	3	12	186	3212	79238	2770026
Indecomposable	0	2	8	146	2992	74916	2692464
Cyclically indecomposable	0	2	8	146	2992	74916	2692464

Table 4: $CTS_2(v)$ with $v \leq 43$ constructed from two-fold Skolem sequences (Construction 2.1)

$CTS_2(v)$ without repeated differences in some base block. Two-fold Skolem and Rosa sequences which provide base blocks with repeated differences are characterized by Lemma 3.1 and 3.2. We generalize Lemma 3.3 and 3.4 to identify all Skolem and Rosa sequences which would give base blocks with repeated differences.

Lemma 5.1 *1. If $S = (s_1, s_2, \ldots, s_{2n})$ is a Skolem sequence of order n in which $s_i = 2n+1-i$ for some $n+1 \leq i \leq 2n$ or $s_i = s_{n+1-i} = n+1-2i$ for some $1 \leq i \leq n/2$, then the corresponding $CTS_2(3n+1)$ (arising out of Construction 2.3) is indecomposable.*

2. If $T = (t_1, t_2, \ldots, t_{2n+1})$ is a Rosa sequence of order n in which $t_i = 2n+2-i$ for some $n+2 \leq i \leq 2n+1$ or $t_i = t_{n+1-i} = n+1-2i$ for some $1 \leq i \leq n/2$, then the corresponding $CTS_2(3n+3)$ (arising out of Construction 2.4) is indecomposable.

Proof.
If $s_i = 2n+1-i$ for some $n+1 \leq i \leq 2n$, then Construction 2.3 provides the base block $\{0, 2n+1-i, i+n\}$ with difference set $\{\pm(2n+1-i), \pm(i+n) = \mp(2n+1-i), \pm(n+1-2i)\}$ that contains $d = 2n+1-i$ twice. If $s_i = s_{n+1-i} = n+1-2i$ for some $1 \leq i \leq n/2$, then the base block $\{0, n+1-2i, 2n+1-i\}$ providing differences $\{\pm(n+1-2i), \pm(2n+1-i), \pm(-n-i) = \pm(2n+1-i)\}$ is obtained from Construction 2.3. Again, difference $d = 2n+1-i$ is repeated. Similarly, Construction 2.4 provides repeated difference $d = 2n+2-i$ if $t_i = 2n+2-i$ for some $n+2 \leq i \leq 2n+1$ or $t_i = t_{n+1-i} = n+1-2i$ for some $1 \leq i \leq n/2$. It is a short exercise to check that other repeated differences can not occur. $\square$

The $CTS_2(v)$ obtained are treated as described in the previous section in order to decide (cyclically) decomposability. The results are presented in Tables 4 to 7.

n	2	3	4	5	6	7
v	15	21	27	33	39	45
No. of CTS from 2-Rosa seq.	1	8	50	912	22286	782374
Indecomposable	0	4	44	802	21258	764196
Cyclically indecomposable	0	4	44	802	21258	764196

Table 5: $CTS_2(v)$ with $v \leq 45$ constructed from two-fold Rosa sequences (Construction 2.2)

n	4	5	8	9	12	13
v	13	16	25	28	37	40
No. of CTS from Skolem seq.	6	10	504	2656	455936	3040560
Indecomposable	6	10	481	2656	452123	3040560
Cyclically indecomposable	6	10	481	2656	452123	3040560

Table 6: $CTS_2(v)$ with $v \leq 40$ constructed from Skolem sequences (Construction 2.3)

n	3	4	7	8	11	12
v	12	15	24	27	36	39
No. of CTS from Rosa seq.	2	2	44	260	33104	203712
Indecomposable	2	2	44	251	33104	202415
Cyclically indecomposable	2	2	44	251	33104	202415

Table 7: $CTS_2(v)$ with $v \leq 39$ constructed from Rosa sequences (Construction 2.4)

Order	Number of Skolem sequences	Number of Rosa sequences
1	1	0
2	0	0
3	0	2
4	6	2
5	10	0
6	0	0
7	0	44
8	504	260
9	2656	0
10	0	0
11	0	33104
12	455936	203712
13	3040560	0

Table 8: Number of Skolem and Rosa sequences of order $n \leq 13$

6 Appendix

We give listings of small orders of (2-fold) Skolem and Rosa sequences and present in Tables 8 and 9 the number of distinct sequences of small order.

Listings of small orders of Skolem sequences:
n= 4: 11423243; 11342324; 41134232; 23243114; 42324311; 34232411
n= 5: 1152423543; 1134532425; 4115423253; 5113453242; 4511435232; 2325341154; 2423543115; 3523245114; 5242354311; 3453242511
Listings of small orders of Rosa sequences:
n= 3: 1130232; 2320311
n=4: 113403242; 242304311
Listings of small orders of 2-fold Skolem sequences
n=1: 1111
n=2: 11112222; 11222211; 22221111
n=3: 111123233232; 113113232232; 112322323113; 112323323211; 311311232232; 311331132222; 311322223113; 311323223211; 222231133113; 232232113113; 232232311311; 232332321111
Listings of small orders of 2-fold Rosa sequences
n=2: 1102222011
n= 3: 23203112320311; 23203111130232; 23203311320211; 11302322320311; 11302321130232; 11303323220211; 11202311330232; 11202232330311

Order	Number of 2-fold Skolem sequences	Number of 2-fold Rosa sequences
1	1	0
2	3	1
3	12	8
4	186	50
5	3212	912
6	79238	22286
7	2770026	782374
8	127860956	36649766
9	> 5000000000	

Table 9: Number of 2-fold Skolem and 2-fold Rosa sequences of order $n \leq 9$

7 Conclusion

In this paper we investigated $CTS_\lambda(v)$ for the properties of being indecomposable or cyclically indecomposable. On first inspection it seems that for $\lambda = 2$ all cyclically indecomposable CTS are also indecomposable. So it would be of interest to either find a $CTS_2(v)$ which is cyclically indecomposable but decomposable or to prove that this is impossible. For $\lambda = 3$ we are interested in the spectrum of those integers v for which there exists a cyclically indecomposable but decomposable $CTS_3(v)$.

8 Acknowledgement

Some of this research was conducted while the first author was visiting the Memorial University of Newfoundland; this author wishes to thank the university for its hospitality.

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