An Improved Bound on the Cardinality of the Minimal Pairwise Balanced Designs on 18 Points with Maximum Block Size 4

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Abstract

The cardinality of the minimal pairwise balanced designs on v elements with largest block size k is denoted by $g^{(k)}(v)$. It is known that $31 \leq g^{(4)}(18) \leq 33$. In this paper we show that $g^{(4)}(18) \neq 31$.

Introduction

Let K be a set of positive integers. A *pairwise balanced design* PBD(v, K) (denoted by $\boldsymbol{P}$) of order v with block sizes from K is a pair $\boldsymbol{P} = (V, \mathcal{B})$, where V is a finite set (the *point set*) of cardinality v and $\mathcal{B}$ is a family of subsets (called *blocks*) of V which satisfy the following properties:

(i) every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$;

(ii) if $B \in \mathcal{B}$, then $|B| \in K$.

A *partial* PBD(v, K) is defined similarly, with the difference that $(V, \mathcal{B})$ satisfies instead of property (i) the property:

(i') every pair of distinct elements of V occurs in at most one block of $\mathcal{B}$.

The *dual* of a (partial) PBD with point set $V = \{0, 1, \ldots, v-1\}$ and block set $\mathcal{B} = \{B_0, B_1, \ldots, B_{m-1}\}$ is a pair $\boldsymbol{P^*} = (V^*, \mathcal{B}^*)$, where $V^* = \{0, 1, \ldots, m-1\}$ and $\mathcal{B}^* = \{B_0^*, B_1^*, \ldots, B_{v-1}^*\}$ is a family of subsets of V^* with the property that $x \in B_j^*$ if and only if $j \in B_x$. Clearly, the dual of a (partial) PBD is a partial PBD.

The cardinality of the minimal pairwise balanced designs on v elements with largest block size k is denoted by $g^{(k)}(v)$. The value $g^{(4)}(v)$ was investigated in [5, 8] and was determined for all v with the exception of 17 and 18. Lower and upper bounds for $v = 18$ were established by Stanton [6, 7] as $30 \leq g^{(4)}(18) \leq 33$. The lower bound was improved to $31 \leq g^{(4)}(18)$ by Grüttmüller, Roberts and Stanton [3]. The study of bounds on $g^{(k)}(v)$ for arbitrary k and different replication factors has been subject of numerous papers. The paper [2] includes a recent survey of known results.

In this paper, we prove that there does not exist a PBD$(18, \{2, 3, 4\})$ with exactly 31 blocks by showing that no partial design can be completed to be such a design. This has been achieved by case analysis and a combination of analytic and computational techniques. Most of the cases have been eliminated by an analytic approach using either a PBD or dual design. However only examples of the analytic arguments involved in either approach are included here, for the purpose of illustrating the methods of argument. Almost all cases have also been eliminated by exhaustive computer searches. The reason for the reliance on computer searches is that the analytic arguments became quite detailed and long as the number of subcases increased beyond 30, and thus many more pages of arguments of similar types but different detail are needed to achieve the same result as claimed here.

Volume $V(x\|_{\mathcal{B}})$	Possible point types				
6	$3^1 4^5$				
7	$2^2 4^5$	$2^1 3^2 4^4$	$3^4 4^3$		
8		$2^3 3^1 4^4$	$2^2 3^3 4^3$	$2^1 3^5 4^2$	$3^7 4^1$
9				$2^3 3^4 4^2$	$2^2 3^6 4^1$

Table 1: Possible point types in $\boldsymbol{P}$

1 Preliminaries

Let g_i be the number of blocks of size i for $i = 2, 3, 4$. Then counting pairs of points in two ways gives

$$g_2 + 3g_3 + 6g_4 = \binom{18}{2}.$$

Also, $g_2 + g_3 + g_4 = 31$, and there are three integer solutions to these two equations. It has been shown in [7, Cases 2, 6] that there is one solution, $(g_2, g_3, g_4) = (6, 1, 24)$, which cannot be realised as a PBD$(18, \{2, 3, 4\})$. The simple argument for this solution is included here as Lemma 1.1 for completeness. The other two solutions are $(g_2, g_3, g_4) = (3, 6, 22)$ and $(g_2, g_3, g_4) = (0, 11, 20)$. Illustration of the elimination of some of the subcases by analysis when $(g_2, g_3, g_4) = (0, 11, 20)$ and $(g_2, g_3, g_4) = (3, 6, 22)$ appear in Sections 2 and 3 respectively. All cases, except for some cases eliminated in Section 2, are eliminated by computation in Section 5.

Let $\mathcal{B}'$ be a subset of the block set $\mathcal{B}$. The *volume(frequency)* of a point x in $\mathcal{B}'$, denoted by $V(x|_{\mathcal{B}'})$, is the number of blocks in $\mathcal{B}'$ which contain x. Similarly, if X' is a subset of the point set, then $V(X'|_{\mathcal{B}'}) = \sum_{x \in X'} V(x|_{\mathcal{B}'})$. A point which occurs in exactly j blocks is called a *j-point*.

A point x has *point type* $P(x) = 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4}$ or $(\alpha_2, \alpha_3, \alpha_4)$ if x is contained in exactly α_2 blocks of size 2 (*doubles*), α_3 blocks of size 3 (*triples*) and α_4 blocks of size 4 (*quads*). Each point type must satisfy

$$\alpha_2 + 2\alpha_3 + 3\alpha_4 = v - 1 = 17. \tag{1}$$

With this equation and $\alpha_k \leq g_k$ it is easily checked that for a point x in a PBD with $(g_2, g_3, g_4) = (0, 11, 20)$ or $(3, 6, 22)$ then the only possible point types are exhibited in Table 1. Let a, b, c, d denote the number of points x with a volume $V(x|_{\mathcal{B}'}) = 6, 7, 8, 9$ respectively. Then

$$a + b + c + d = 18 \quad \text{and} \quad 6a + 7b + 8c + 9d = 2g_2 + 3g_3 + 4g_4. \tag{2}$$

It can be noted that in Sections 2 and 3 it is shown that $d = 0$.

The terms blocks and points are used when referring to a (partial) PBD and the terms elements and sets when referring to the dual design. In particular let $\boldsymbol{P}$ be a PBD$(18, \{2,3,4\})$ with 31 blocks, g_i blocks of size i and h_j j-points then the *dual design* $\boldsymbol{P^*} = (V^*, \mathcal{B}^*)$ is an antichain consisting of $|V^*| = 31$ elements in $|\mathcal{B}^*| = 18$ sets, with h_j sets of size j, and with g_i elements occurring in i sets in $\boldsymbol{P^*}$. Each pair of distinct points occurs together in exactly one block of $\boldsymbol{P}$, hence each pair of distinct sets in $\boldsymbol{P^*}$ has exactly one element in common: $|B_i^* \cap B_j^*| = 1$ for $i \neq j$. A set of size j is called a *j-set* and an element which occurs in exactly i sets is called an *i-element*, $i \geq 0$. If two distinct elements occur in the same set then it is said that they *are a pair*.

Lemma 1.1
There is no PBD$(18, \{2,3,4\})$ with the configuration $(g_2, g_3, g_4) = (6, 1, 24)$.
Proof Assume that the lemma is false. Then the volume of the quads is 96, and as there are 18 points in 24 blocks, there must be at least one point which occurs in six quads. However this point would then occur in pairs with 18 distinct elements, so this case is not possible.

2 Case $(g_2, g_3, g_4) = (0, 11, 20)$

As $g_2 = 0$ only point types with $\alpha_2 = 0$ are possible. In Table 1 the only such point types are a 6-point of type $3^1 4^5$, a 7-point of type $3^4 4^3$ and an 8-point of type $3^7 4^1$. As a refinement of equation (2), volumes of each point type in the triples and quads require that $a + 4b + 7c = 33$ and $5a + 3b + c = 80$. There are three feasible algebraic solutions: (a, b, c) =(15,1,2), $(14, 3, 1)$ or $(13, 5, 0)$. As $g_3 = 11$ there can be at most one point of type $3^7 4^1$, so $(a, b, c) = (15, 1, 2)$ is immediately eliminated.

2.1 Assume that $(a, b, c) = (14, 3, 1)$ (Using $\boldsymbol{P}$)

Let C be the point of type $3^7 4^1$ and A, B be two points of type $3^4 4^3$. Begin by assigning the points to triples. C occurs in seven triples. Each of A and B occur in four triples, once with C and once as a pair. This requires a minimum of 12 triples, but there are only 11 triples. So this case is not possible.

2.2 Assume that $(a, b, c) = (13, 5, 0)$ (Using $\boldsymbol{P}^*$)

Let $\mathcal{A} \times \mathcal{B}$ denote the Cartesian product of sets $\mathcal{A}$ and $\mathcal{B}$. Say that a pair of distinct sets $(A, B) \in \mathcal{A} \times \mathcal{B}$ is *covered by* an element a if $A \cap B = \{a\}$. Alternatively it is said that *a covers the pair* $(A, B) \in \mathcal{A} \times \mathcal{B}$. If $a \in X$ then it can be said that X *covers* (A, B).

Let $X \subseteq V^*$ and let $\mathcal{C} \subseteq \mathcal{B}^*$. An element $x \in X$ is called an i-element with respect to $\mathcal{C}$ if x occurs in exactly i sets of $\mathcal{C}$. Say that $X|_\mathcal{C}$ has *frequency type* $0^a1^b2^c3^d4^e$ if X consists of a 0-elements, b 1-elements, c 2-elements, d 3-elements and e 4-elements with respect to $\mathcal{C}$. Let $\mathcal{S} \in \mathcal{C}$ be a set. $\mathcal{S}$ has *set type* $1^a2^b3^c4^d$ with respect to $X|_\mathcal{C}$ if a, b, c, d elements of $\mathcal{S}$ are $1, 2, 3, 4$-elements (respectively) of X with respect to $\mathcal{C}$.

Considering the dual $\boldsymbol{P^*}$, there are five 7-sets and thirteen 6-sets. The 7-sets each contain four 3-elements and three 4-elements. The 6-sets each contain one 3-element and five 4-elements. Let $\mathcal{P}$ denote the five 7-sets, let $a, \ldots, k$ denote the eleven 3-elements and let $A, \ldots, T$ denote the twenty 4-elements. At most one of the eleven 3-elements occurs three times in $\mathcal{P}$ as $V(\{a, \ldots, k\}|_\mathcal{P}) = 20$ and there are 10 pairs in $\mathcal{P} \times \mathcal{P}$. Remember that any two distinct sets have exactly one element in common, thus any pair of distinct sets from $\mathcal{P} \times \mathcal{P}$ is covered by exactly one element of V^*, and an element of V^* which occurs in exactly i sets in $\mathcal{P}$ covers $i(i-1)/2$ such pairs. Therefore the only possible frequency types for $\{a, \ldots, k\}|_\mathcal{P}$ are $1^32^73^1$, 0^12^{10} or 1^22^9. Each of these cases require a separate argument. A part of one argument is included here for illustration.

2.2.1. Assume that $\{a, \ldots, k\}|_\mathcal{P}$ has frequency type $1^32^73^1$.

As the 3-elements cover all of the pairs in $\mathcal{P} \times \mathcal{P}$, it can be assumed that $\mathcal{P} = \{abciABC, adejDEF, afgkGHI, bdfhJKL, ceghMNO\}$. Partition the remaining 13 sets of $\boldsymbol{P^*}$ into $\mathcal{Q}$, the seven 6-sets each containing one of b, c, d, e, f, g or h, and $\mathcal{R}$ the six 6-sets containing one of i, j or k. Let $X = \{A, \ldots, O\}, Y = \{P, \ldots, T\}$ and $Z = X \cup Y$. Table 2 shows the set types that might occur in $\mathcal{Q}$ with respect to $Z|_\mathcal{Q}$ given that each set in $\mathcal{Q}$ must have six pairs in $\mathcal{Q} \times \mathcal{Q}$ covered by elements of Z.

Number of set types	Possible set types
r	1^34^2
s	$1^22^13^14^1$
t	$1^12^34^1$
u	1^23^3
v	$1^12^23^2$
w	2^43^1

Table 2: Set types possible for $\mathcal{Q}$ with respect to $Z|_\mathcal{Q}$

It follows that if the collection $Z|_{\mathcal{Q}}$ has frequency type $1^m2^n3^o4^p$, then

$$\begin{aligned} r+s+t+u+v+w &= 7 \\ 3r+2s+t+2u+v &= m \\ s+3t+2v+4w &= 2n \\ s+3u+2v+w &= 3o \\ 2r+s+t &= 4p \end{aligned} \tag{3}$$

Unless otherwise stated the elements of Z are now considered in the context of their occurrences in $\mathcal{Q}$ or $\mathcal{R}$ alone. It will be useful to note that all elements of X are 3-elements in $\mathcal{Q} \cup \mathcal{R}$ and all elements of Y are 4-elements in $\mathcal{Q} \cup \mathcal{R}$, so that upon determination of a frequency type of a particular set of elements restricted to one of the parts $\mathcal{Q}$ or $\mathcal{R}$, the frequency type of the same set of elements in the other part immediately follows.

Consideration of $\mathcal{P} \times \mathcal{Q}$ shows that each set in $\mathcal{Q}$ contains three elements of X and two elements of Y so $V(X|_{\mathcal{R}}) = 24$ and $V(Y|_{\mathcal{R}}) = 6$. Consideration of $\mathcal{P} \times \mathcal{R}$ shows that each set in $\mathcal{R}$ contains four elements of X and one element of Y. Furthermore $V(\{A,B,C\}|_{\mathcal{R}}) = V(\{D,E,F\}|_{\mathcal{R}}) = V(\{G,H,I\}|_{\mathcal{R}}) = 4$ so each set covers at least one pair in $\mathcal{R} \times \mathcal{R}$. $V(\{J,K,L\}|_{\mathcal{R}}) = V(\{M,N,O\}|_{\mathcal{R}}) = 6$ so each of these sets covers at least three pairs in $\mathcal{R} \times \mathcal{R}$. Therefore X covers at least 9 of the 15 pairs in $\mathcal{R} \times \mathcal{R}$. Also Y covers at least one pair and $\{i,j,k\}$ covers exactly three pairs in $\mathcal{R} \times \mathcal{R}$. Thus there remain two pairs to determine in $\mathcal{R} \times \mathcal{R}$. Note that neither set $\{J,K,L\}$ nor $\{M,N,O\}$ contains two 3-elements in $\mathcal{R}$ as then X would cover twelve pairs in $\mathcal{R} \times \mathcal{R}$. There are no 3-elements from $\{A,\ldots,I\}$ in $\mathcal{R}$ as each element pairs with i,j or k in $\mathcal{P}$.

As $V(Y|_{\mathcal{R}}) = 6$, Y covers 1,2 or 3 pairs in $\mathcal{R} \times \mathcal{R}$ and no element occurs more than three times in $\mathcal{R}$, $Y|_{\mathcal{R}}$ has frequency type $1^42^1, 0^11^22^2, 0^22^3$ or $0^11^33^1$. One of these subcases is now argued for illustration.

2.2.1.1. Assume that $Y|_{\mathcal{R}}$ has frequency type $0^11^33^1$.

Then Y covers 3 pairs in $\mathcal{R} \times \mathcal{R}$ so X covers 9 pairs in $\mathcal{R} \times \mathcal{R}$. It follows that $X|_{\mathcal{R}}$ has each of the sets $\{A,B,C\}, \{D,E,F\}, \{G,H,I\}$ comprising one 2-element and two 1-elements and $\{J,K,L\}, \{M,N,O\}$ each containing three 2-elements of $\mathcal{R}$. So $X|_{\mathcal{R}}$ has frequency type 1^62^9. Let C,F,I be 2-elements in $\mathcal{R}$ and let A,B,D,E,G,H be 1-elements in $\mathcal{R}$ and thus 2-elements in $\mathcal{Q}$. It is easily seen that $X|_{\mathcal{Q}}$, $Y|_{\mathcal{Q}}$, $Z|_{\mathcal{Q}}$ have frequency types 1^92^6, $1^13^34^1$ and $1^{10}2^63^34^1$ respectively.

The following additional constraints on the variables in (3) can be noted. There is only one 4-element in $Z|_{\mathcal{Q}}$ so set type 1^34^2 is impossible in $\mathcal{Q}$ and therefore $r = 0$. All of the 3-elements of $Z|_{\mathcal{Q}}$ are elements of Y and each set in $\mathcal{Q}$ contain two elements of Y so $u = 0$. There are no 2-elements in $Y|_{\mathcal{Q}}$ so $w = 0$. Therefore (3) is uniquely satisfied by $(r, s, t, u, v, w) = (0, 3, 1, 0, 3, 0)$.

Note that the frequency types of $X|_{\mathcal{Q}}$ and $Y|_{\mathcal{Q}}$ require that the 1-element and the 4-element in the set $W \in \mathcal{Q}$ of type $1^1 2^3 4^1$ must be elements of Y. The three 2-elements in W must be in X and so they can be assumed to be A, D and G. Hence W must be of the form $hADGPT$, where P is a 1-element in $\mathcal{Q}$. It follows that P is a 3-element in $\mathcal{R}$ and each of the three sets in $\mathcal{R}$ which contain P must also include two 1-elements and two 2-elements of $X|_{\mathcal{R}}$ to satisfy $\mathcal{R} \times \mathcal{R}$. Thus all six of the 1-elements in $X|_{\mathcal{R}}$, namely A, B, D, E, G, H are in sets with P in $\mathcal{R}$ which is a contradiction given that A, D, G and P are in W. Therefore this subcase is not possible.

3 Case $(g_2, g_3, g_4) = (3, 6, 22)$

Since $\alpha_4 \leq 5$ for all point types in Table 1 and the volume of the quads is 88, there are two ways to configure the quads. Case 1 has sixteen points in 5 quads and two points occurring in 4 quads. Case 2 has seventeen points occurring in 5 quads and one point occurring in 3 quads. Point types with $\alpha_4 \geq 3$ must have volume 6, 7, or 8, so $d = 0$. Then there are three solutions to the point-volume equations (2) and these are now considered in turn.

Assume that $(a, b, c) = (16, 0, 2)$. This solution is impossible for Case 1 as the two 8-points must be type $2^3 3^1 4^4$ giving duplicate pairs in the doubles. Case 2 cannot be satisfied as only 6-points or 7-points have $\alpha_4 = 5$. Thus $(a, b, c) = (16, 0, 2)$ is impossible.

Assume that $(a, b, c) = (15, 2, 1)$. This solution has an 8-point, for which $\alpha_4 \leq 4$ holds. Therefore, in Case 1 the 8-point is of type $2^3 3^1 4^4$. Then the other point occurring in 4 quads is of type $2^1 3^2 4^4$. This in turn implies that there is exactly one 7-point of type $2^2 4^5$ and a contradiction is reached since duplicated pairs occur in the doubles. So Case 1 is impossible. In Case 2, noting that only 6-points or 7-points have $\alpha_4 = 5$, V consists of fifteen 6-points of type $3^1 4^5$, two 7-points of type $2^2 4^5$ and one 8-point of type $2^2 3^3 4^3$.

Assume that $(a, b, c) = (14, 4, 0)$. In Case 1, V must consist of fourteen 6-points of type $3^1 4^5$, two 7-points of type $2^2 4^5$, and two 7-points of type $2^1 3^2 4^4$. In Case 2 V consist of fourteen 6-points of type $3^1 4^5$, three 7-points of type $2^2 4^5$, and one 7-point of type $3^4 4^3$.

	Possible point types
P_1	$2^2 3^3 4^3$
P_2	$2^0 3^4 4^3$
P_3	$2^1 3^2 4^4$
P_4	$2^2 3^0 4^5$
P_5	$2^0 3^1 4^5$

Table 3: Possible point types P_i

4 Preliminaries for Computer Results

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be the set of all possible point types ($P_i = (\alpha_{(2,i)}, \alpha_{(3,i)}, \alpha_{(4,i)})$). The *point type distribution* of a (partial) PBD with respect to $\mathcal{P}$ is a vector $d = d(\mathcal{P}) = (d_1, \ldots, d_n)$ such that the entry d_i counts how many points of type P_i are in the (partial) PBD.

The case structure analysis in Sections 2 and 3 leaves four major cases to be eliminated by computation:
$(g_2, g_3, g_4) = (0, 11, 20), (a, b, c) = (13, 5, 0)$;
$(g_2, g_3, g_4) = (3, 6, 22), (a, b, c) = (14, 4, 0)$ Cases 1 and 2;
$(g_2, g_3, g_4) = (3, 6, 22), (a, b, c) = (15, 2, 1)$ Case 2.
Thus there are five possible point types to consider as listed in Table 3.

The corresponding four possible point type distributions are:
d_1 $(0, 5, 0, 0, 13)$: $5 \times 2^0 3^4 4^3, 13 \times 2^0 3^1 4^5$;
d_2 $(0, 0, 2, 2, 14)$: $2 \times 2^1 3^2 4^4, 2 \times 2^2 3^0 4^5, 14 \times 2^0 3^1 4^5$;
d_3 $(0, 1, 0, 3, 14)$: $1 \times 2^0 3^4 4^3, 3 \times 2^2 3^0 4^5, 14 \times 2^0 3^1 4^5$;
d_4 $(1, 0, 0, 2, 15)$: $1 \times 2^2 3^3 4^3, 2 \times 2^2 3^0 4^5, 15 \times 2^0 3^1 4^5$.

In order to avoid unnecessary long vectors we consider whenever possible only those point types (with indices $I_d = \{i : d_i \neq 0\} = \{i_1, \ldots, i_t\}$) which really occur. Let B be any block, its *single block type* with respect to $d(\mathcal{P})$ is a vector $S = S(B) = (s_1, \ldots, s_t)$ where s_j counts how many points of type P_{i_j} occur in B. Clearly, each possible single block type S satisfies

$$k(S) := \sum_{j=1}^{t} s_j \text{ is a size from } K; \tag{4}$$

$$s_j \leq d_{i_j} \text{ for } j = 1, \ldots, t; \text{ and} \tag{5}$$

$$\alpha_{(k(S), i_j)} = 0 \text{ implies } s_j = 0 \text{ for } j = 1, \ldots, t. \tag{6}$$

Let $\mathcal{S}_d = \{S_1, \ldots, S_r\}$ be the set of all possible single block types with respect to a point type distribution $d(\mathcal{P})$ and let $I_k = \{i \in \{1, \ldots, r\} : k(S_i) = k\}$ be the set of indices i such that $S_i \in \mathcal{S}_d$ belongs to a block of size k. The *single block* *type distribution* of a (partial) PBD with point type distribution $d(\mathcal{P})$ is a vector $c_d = c_d(\mathcal{S}_d) = (c_1, \ldots, c_r)$ such that the entry c_i counts how many blocks of type S_i are in the (partial) PBD. A possible single block type distribution needs to satisfy the following conditions which are obtained by counting points of the same point type which are contained in blocks of size k in two ways (7), or by counting pairs of points of the same point type (8), or pairs of points of distinct point types (9) in two ways ($S_{h,j}$ denotes the j-th component of the possible single block type S_h).

$$\sum_{h \in I_k} c_h S_{h,j} = d_{i_j} \alpha_{(k,i_j)} \quad \text{for all } j \in \{1, \ldots, t\} \text{ and all } k \in K, \tag{7}$$

$$\sum_{h=1}^{r} c_h S_{h,j}(S_{h,j} - 1) \leq d_{i_j}(d_{i_j} - 1) \quad \text{for all } j \in \{1, \ldots, t\}, \tag{8}$$

and

$$\sum_{h=1}^{r} c_h S_{h,j} S_{h,\ell} \leq d_{i_j} d_{i_\ell} \quad \text{for all } j, \ell \in \{1, \ldots, t\}, i \neq \ell. \tag{9}$$

Note that equality holds in (8) and (9) for all $j, \ell \in \{1, \ldots, t\}$ simultaneously if and only if the partial PBD is a PBD. We give here a list of all possible single block type distributions for each of the point type distributions $d_1, \ldots, d_4$.

For the point type distribution d_1 there are 9 possible single block types $S_{d_1} = \{(3,0),\ (2,1),\ (1,2),\ (0,3),\ (4,0),\ (3,1),\ (2,2),\ (1,3),\ (0,4)\}$ which allow 3 possible single block type distributions:

$$\begin{aligned} c_{d_{11}} &= (1,7,3,0,0,0,0,15,5), \\ c_{d_{12}} &= (0,10,0,1,0,0,0,15,5), \\ c_{d_{13}} &= (0,9,2,0,0,0,1,13,6). \end{aligned}$$

For the point type distribution d_2 there are 15 possible single block types $S_{d_2} = \{(2,0,0),\ (1,1,0),\ (0,2,0),\ (2,0,1),\ (1,0,2),\ (0,0,3),\ (2,2,0),\ (2,1,1),\ (2,0,2),\ (1,2,1),\ (1,1,2),\ (1,0,3),\ (0,2,2),\ (0,1,3),\ (0,0,4)\}$ which allow 3 possible single block type distributions:

$$\begin{aligned} c_{d_{21}} &= (0,2,1,1,2,3,0,0,0,0,2,6,0,8,6), \\ c_{d_{22}} &= (0,2,1,0,4,2,0,1,0,0,0,6,0,9,6), \\ c_{d_{23}} &= (0,2,1,0,4,2,0,0,1,0,2,4,0,8,7). \end{aligned}$$

For the point type distribution d_3 there are 11 possible single block types $S_{d_3} = \{(0,2,0),\ (1,0,2),\ (0,0,3),\ (1,3,0),\ (1,2,1),\ (1,1,2),\ (1,0,3),\ (0,3,1),\ (0,2,2),$

$(0,1,3)$, $(0,0,4)\}$ which allow 1 possible single block type distributions:

$$c_{d_{31}} = (3,4,2,0,0,3,0,0,0,12,7).$$

For the point type distribution d_4 there are 10 possible single block types $S_{d_4} = \{(1,1,0),\ (0,2,0),\ (1,0,2),\ (0,0,3),\ (1,2,1),\ (1,1,2),\ (1,0,3),\ (0,2,2),\ (0,1,3),\ (0,0,4)\}$ which allow 1 possible single block type distribution:

$$c_{d_{41}} = (2,1,3,3,0,0,3,0,10,9).$$

5 Search for a PBD with Prescribed Point Type Distribution and Single Block Type Distribution

In this section, we describe the search undertaken to find a PBD$(18,\{2,3,4\})$ with one of the possible point type distributions and single block type distributions determined in the previous section. This search is split into two steps. In the first step we determine all suitable partial PBDs containing only blocks of size 2 and 3 and consider their dual. The partial PBD is called the *prestructure* and its dual is called the *dual prestructure*. In the second step it is attempted to complete the prestructures obtained with blocks of size 4.

5.1 Search for a Prestructure

Let $d = d(\mathcal{P}) \in \{d_1,\dots,d_4\}$ be a prescribed possible point type distribution, let $c_d = c_d(\mathcal{S}_d)$ be one possible single block type distribution and let $\mathcal{P}^p = \{P_1^p,\dots,P_n^p\}$ where P_j^p is the restriction of point type P_j with respect to the partial block size set $\{2,3\}$, that is, $P_j^p = 2^{\alpha(2,j)}3^{\alpha(3,j)}$. Furthermore, let $(V,\mathcal{B}^p)$ be a partial PBD$(v,\{2,3\})$ with point type distribution d^p with respect to $\mathcal{P}^p$ and single block type distribution c_d^p computed with respect to $\mathcal{S}_d^p = \{S_i \in \mathcal{S}_d : i \in I_2 \cup I_3\}$. Now, $(V,\mathcal{B}^p)$ is called a *suitable prestructure* with respect to d and c_d if $\forall x \in V : P(x) \in \mathcal{P}^p$, $\forall B \in \mathcal{B}^p : S(B) \in \mathcal{S}_d^p$, $d^p = d$ and $c_d^p = ((c_d)_i : i \in I_2 \cup I_3)$.

The task is to find all suitable, non-isomorphic prestructures with respect to prescribed $d(\mathcal{P})$ and $c_d(\mathcal{S}_d)$. When trying to search for these non-isomorphic prestructures on 18 points we found out that this search was particularly slow whenever there is one special point type that occurs very often since any isomorphism maps points of a certain point type to points of the same point type. That was the reason why we decided to search instead for the dual prestructure. The advantage is that the dual prestructure has only $g_2+g_3 \leq 11$ points, thus isomorphism testing is much faster. The disadvantage, however, is that this approach requires some additional computations.

The dual prestructure is again a partial PBD as noted in the introduction. Thus all notation introduced for partial PBDs in Section 4 and the (in)equalities (7), (8), (9) are also valid for the dual prestructure. In order to distinguish between primal and dual prestructure we will use (in accordance with the terminology given in Section 1) the words *element type* (rather than point type), *element type distribution* (rather than point type distribution), *single set type* (rather than single block type) and so on whenever we refer to the dual prestructure.

The parameters of the dual of a suitable prestructure with prescribed $d(\mathcal{P}^p)$ and $c_d(\mathcal{S}_d^p)$ are as follows. The set sizes are from

$$K^* = \{k_j^* := \alpha_{(2,i_j)} + \alpha_{(3,i_j)} : j = 1, \ldots, t\}.$$

Note that the set sizes k_j^* are pairwise distinct for each point type distribution $d \in \{d_1, \ldots, d_4\}$. Thus we get a different set size from each point type. The $g_2 + g_3$ elements are of type

$$P_i^* = (k_1^*)^{S_{i,1}} \ldots (k_t^*)^{S_{i,t}} \text{ for } i = 1, \ldots, |\mathcal{S}_d^p|.$$

These element types are also pairwise distinct. Therefore, it is easily checked that we have exactly $(c_d)_i$ elements of type P_i^*, which means that the element type distribution $d^* = c_d$. Unfortunately, although the dual of the dual prestructure gives the primal prestructure, we can not conclude that the single set type distribution $c_d^* = d$ since distinct element types may correspond to the same block size. So it is necessary to compute all possible single set types $\mathcal{S}_d^*$ and all possible single set type distributions $c_d^*(\mathcal{S}_d^*)$ using (7), (8), (9) and the fact that the dual of a single set type corresponds to some point type. The single set types and possible single set type distributions obtained this way are presented in Appendix A.

Now, an exhaustive search technique (backtracking) was applied to search for a dual prestructure with prescribed parameter. We do this by systematically building up a feasible partial PBD. A partial PBD$(g_2 + g_3, K^*)$ on the element set $V^* = \{0, 1, \ldots, g_2+g_3-1\}$ with sets $\mathcal{B}^*$ is called *feasible* with respect to given $d^*(\mathcal{P}^*)$ and $c_d^*(\mathcal{S}_d^*)$ if

(i) if $(k_1^*)^{\alpha_1} \ldots (k_t^*)^{\alpha_t}$ is the current element type of $x \in V^*$ and $(k_1^*)^{S_{j,1}} \ldots (k_t^*)^{S_{j,t}}$ is the desired element type of x where j is uniquely determined by $\sum_{i=1}^{j-1}(d^*)_i \leq x < \sum_{i=1}^{j}(d^*)_i$, then $\alpha_i \leq S_{j,i}$ for $i = 1, \ldots, t$;

(ii) $S(B) \in \mathcal{S}_d^*$ for all $B \in \mathcal{B}^*$;

(iii) $|\{B \in \mathcal{B}^* : S(B) = S_i \in \mathcal{S}_d^*\}| \leq (c_d^*)_i$ for all $i = 1, \ldots, |\mathcal{S}_d^*|$;

(iv) $\mathcal{B}^* \leq_{\text{lex}} \pi(\mathcal{B}^*)$ for every element type preserving permutation $\pi : V^* \to V^*$, i.e. π satisfies: if $x \in V^*$ and $\sum_{i=1}^{j-1}(d^*)_i \leq x < \sum_{i=1}^{j}(d^*)_i$, then $\sum_{i=1}^{j-1}(d^*)_i \leq \pi(x) < \sum_{i=1}^{j}(d^*)_i$.

Let $\mathcal{X} = \{B \subseteq V^* : S(B) \in \mathcal{S}_d^*\}$. We try to construct feasible partial PBD $(V^*, \mathcal{B}_m^*)$ for $m = 0, 1, \ldots, 18$ with $\mathcal{B}_0^* = \emptyset$ and $\mathcal{B}_m^* \in \mathcal{X}^m = \mathcal{X} \times \mathcal{X} \times \ldots \times \mathcal{X}$. Clearly, if $(V^*, \mathcal{B}_{18}^*)$ is a feasible dual prestructure, then its dual is a suitable primal prestructure. Note, that in condition (i) it is checked that the current element type of each element x can be extended to the desired element type. For that purpose we fix an order of the elements types and want the first $(d^*)_1$ elements to have the desired element type P_1^*, the next $(d^*)_2$ elements to have the desired element type P_2^*, and so on until we want the last $(d^*)_{|P^*|}$ elements to have the desired element type $P_{|P^*|}^*$. Conditions (ii) and (iii) ensure that the current set type distribution can be extended to the desired set type distribution. Finally, condition (iv) ensures that we construct from each class of isomorphic dual prestructures exactly one representative prestructure. For more information on search techniques used in design theory see for example [1] or [4]. The backtracking algorithm is given below.

Backtracking algorithm to find a feasible dual prestructure with respect to prescribed $d^* = d^*(\mathcal{P}^*)$ and $c_d^* = c_d^*(\mathcal{S}_d^*)$

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1. **procedure** Search(d^*,c_d^*,$\mathcal{B}^*$)
2. **begin**
3. **if** $|\mathcal{B}^*| = 18$
4. **then** compute primal prestructure $(V^*, \mathcal{B}^*)^*$ and save solution
5. **else**
6. **for each** $B \in \mathcal{X}$ **do**
7. **if** $(V^*, \mathcal{B}^* \cup \{B\})$ is a feasible partial PBD with respect to d^*,c_d^*
8. **then** Search(d^*,c_d^*,$\mathcal{B}^* \cup \{B\}$)
9. **end**

Running the algorithm with Search($d^*, c_d^*, \emptyset$) where $(d^*, c_d^*) \in \{(d_{11}^*, c_{d_{111}}^*), (d_{11}^*, c_{d_{112}}^*), (d_{11}^*, c_{d_{113}}^*), (d_{11}^*, c_{d_{114}}^*), (d_{11}^*, c_{d_{115}}^*), (d_{12}^*, c_{d_{121}}^*), (d_{13}^*, c_{d_{131}}^*), (d_{13}^*, c_{d_{132}}^*), (d_{21}^*, c_{d_{211}}^*), (d_{22}^*, c_{d_{221}}^*), (d_{23}^*, c_{d_{231}}^*), (d_{31}^*, c_{d_{311}}^*), (d_{41}^*, c_{d_{411}}^*)\}$ as given in Appendix A, we found that there exist exactly 8 suitable primal prestructures whose blocks are listed in Appendix B.

5.2 Search for a Completion of a Given Prestructure

Given a suitable primal prestructure with respect to prescribed $d(\mathcal{P})$ and $c_d(\mathcal{S}_d)$ we want to find a completion of the prestructure with blocks of size 4 such that the

PBD obtained has point type distribution $d(\mathcal{P})$ and single block type distribution $c_d(\mathcal{S}_d)$. The algorithm used is basically the same as the one in the previous section. We just need to alter the feasibility predicate and the search space $\mathcal{X}$ in accordance with our new demands.

A partial PBD$(18, \{2, 3, 4\})$ on the point set $V = \{0, 1, \ldots, 17\}$ with block set $\mathcal{B}$ is called *feasible* with respect to given suitable prestructure $(V, \mathcal{B}^p)$, and distribution vectors $d(\mathcal{P})$ and $c_d(\mathcal{S}_d)$ if

(i) if $2^{\alpha_2}3^{\alpha_3}4^{\alpha_4}$ is the point type of $x \in V$, then $\alpha_4 \leq \alpha_{(4,j)}$ where j is uniquely determined by $\sum_{i=1}^{j-1} d_i \leq x < \sum_{i=1}^{j} d_i$;

(ii) $S(B) \in \mathcal{S}_d$ for all $B \in \mathcal{B}$ and $\mathcal{B}^p \subseteq \mathcal{B}$;

(iii) $|\{B \in \mathcal{B} : S(B) = S_i \in \mathcal{S}_d\}| \leq (c_d)_i$ for all $i = 1, \ldots, |\mathcal{S}_d|$;

(iv) $\mathcal{B} \leq_{\text{lex}} \pi(\mathcal{B})$ for every point type preserving permutation π which is an isomorphism of the prestructure.

We define the search space to be $\mathcal{X} = \{B \subseteq V : |B| = 4, S(B) \in \mathcal{S}_d\}$.

Backtracking algorithm to complete a given prestructure $(V, \mathcal{B}^p)$ with respect to prescribed $d = d(\mathcal{P})$ and $c_d = c_d(\mathcal{S}_d)$

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procedure Search($\mathcal{B}^p$,d,c_d,$\mathcal{B}$)
begin
  if $|\mathcal{B}| = 31$
  then save solution and stop
  else
    for each $B \in \mathcal{X}$ **do**
      if $(V, \mathcal{B} \cup \{B\})$ is a feasible partial PBD with respect to $\mathcal{B}^p$,d,c_d
      then Search($\mathcal{B}^p$,d,c_d,$\mathcal{B} \cup \{B\}$)
end

We started the algorithm above with Search$(\mathcal{B}^p, d, c_d, \mathcal{B}^p)$ for each triple $(d, c_d, \mathcal{B}^p) \in \{(d_1, c_{d_{11}}, \mathcal{B}^p_{d_{111}})$ $(d_1, c_{d_{12}}, \mathcal{B}^p_{d_{121}})$, $(d_1, c_{d_{13}}, \mathcal{B}^p_{d_{131}})$, $(d_2, c_{d_{21}}, \mathcal{B}^p_{d_{211}})$, $(d_2, c_{d_{22}}, \mathcal{B}^p_{d_{221}})$, $(d_2, c_{d_{23}}, \mathcal{B}^p_{d_{231}})$, $(d_3, c_{d_{31}}, \mathcal{B}^p_{d_{311}})$, $(d_4, c_{d_{41}}, \mathcal{B}^p_{d_{411}})\}$. We found that none of the triples was completable to a PBD with 31 blocks.

6 Conclusion

We have computed in the previous sections all suitable prestructures for a PBD$(18, \{2, 3, 4\})$ with exactly 31 blocks and have shown that none of these prestructures is completable. Therefore, we have established:

Theorem 6.1. *There does not exist a PBD on 18 points with 31 blocks of size at most* 4.

A Possible Single Set Type Distributions of the Dual Prestructure

Here is listed the parameters of all possible primal and dual prestructures. The parameter sets are labeled by d_{ij} which indicates that the primal prestructure belongs to the restriction to blocks of size 2 and 3 of the point type distributions d_i with single block type distribution $c_{d_{ij}}$.

$\mathbf{d}_{11}$: Parameter of primal prestructure: $d_1^p: \ 5 \times 2^0 3^4, 13 \times 2^0 3^1$
$S_{d_1}^p = \{(3,0),(2,1),(1,2),(0,3)\}$,
$c_{d_{11}}^p = (1,7,3,0)$

Parameter of dual prestructure: d_{11}^*: $1 \times 1^0 4^3, 7 \times 1^1 4^2, 3 \times 1^2 4^1$
$\mathcal{S}_{d_{11}}^* = \{(0,1,0),(0,0,1),(1,3,0),(1,2,1),(1,1,2),(1,0,3),(0,4,0),$
$(0,3,1),(0,2,2),(0,1,3)\}$,
$c_{d_{111}}^* = (7,6,0,3,0,0,2,0,0,0)$,
$c_{d_{112}}^* = (7,6,1,1,1,0,2,0,0,0)$,
$c_{d_{113}}^* = (7,6,2,0,0,1,2,0,0,0)$,
$c_{d_{114}}^* = (7,6,1,2,0,0,1,1,0,0)$,
$c_{d_{115}}^* = (7,6,2,0,1,0,1,1,0,0)$

$\mathbf{d}_{12}$: Parameter of primal prestructure: $d_1^p: \ 5 \times 2^0 3^4, 13 \times 2^0 3^1$
$S_{d_1}^p = \{(3,0),(2,1),(1,2),(0,3)\}$,
$c_{d_{12}}^p = (0,10,0,1)$

Parameter of dual prestructure: d_{12}^*: $10 \times 1^1 4^2, 1 \times 1^3 4^0$
$\mathcal{S}_{d_{12}}^* = \{(1,0),(0,1),(4,0)\}$,
$c_{d_{121}}^* = (10,3,5)$

$\mathbf{d}_{13}$: Parameter of primal prestructure: $d_1^p: \ 5 \times 2^0 3^4, 13 \times 2^0 3^1$
$S_{d_1}^p = \{(3,0),(2,1),(1,2),(0,3)\}$,
$c_{d_{13}}^p = (0,9,2,0)$

Parameter of dual prestructure: d_{13}^*: $9 \times 1^1 4^2, 2 \times 1^2 4^1$
$\mathcal{S}_{d_{13}}^* = \{(1,0),(0,1),(4,0),(3,1),(2,2)\}$
$c_{d_{131}}^* = (9,4,3,2,0)$, $c_{d_{132}}^* = (9,4,4,0,1)$

$\mathbf{d}_{21}$: Parameter of primal prestructure: $d_2^p: \ 2 \times 2^1 3^2, 2 \times 2^2 3^0, 14 \times 2^0 3^1$
$S_{d_2}^p = \{(2,0,0),(1,1,0),(0,2,0),(2,0,1),(1,0,2),(0,0,3),\}$
$c_{d_{21}}^p = (0,2,1,1,2,3)$

Parameter of dual prestructure: d^*_{21}: $2 \times 1^0 2^1 3^1, 1 \times 1^0 2^2 3^0, 1 \times 1^1 2^0 3^2, 2 \times 1^2 2^0 3^1, 3 \times 1^3 2^0 3^0$
$\mathcal{S}^*_{d_{21}} = \{(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),(2,0,0,0,0),(1,1,0,0,0),(1,0,1,1,0),(1,0,0,2,0)\}$,
$c^*_{d_{211}} = (1,4,9,0,2,2,0)$

$\mathbf{d}_{22}$: Parameter of primal prestructure: d^p_2 : $2 \times 2^1 3^2, 2 \times 2^2 3^0, 14 \times 2^0 3^1$
$S^p_{d_2} = \{(2,0,0),(1,1,0),(0,2,0),(2,0,1),(1,0,2),(0,0,3)\}$
$c^p_{d_{22}} = (0,2,1,0,4,2)$

Parameter of dual prestructure: d^*_{22}: $2 \times 1^0 2^1 3^1, 1 \times 1^0 2^2 3^0, 4 \times 1^2 2^0 3^1, 2 \times 1^3 2^0 3^0$
$\mathcal{S}^*_{d_{22}} = \{(0,0,1,0),(0,0,0,1),(2,0,0,0),(1,1,0,0),(1,0,2,0)\}$,
$c^*_{d_{221}} = (8,6,0,2,2)$

$\mathbf{d}_{23}$: Parameter of primal prestructure: d^p_2 : $2 \times 2^1 3^2, 2 \times 2^2 3^0, 14 \times 2^0 3^1$
$S^p_{d_2} = \{(2,0,0),(1,1,0),(0,2,0),(2,0,1),(1,0,2),(0,0,3)\}$
$c^p_{d_{23}} = (0,2,1,0,4,2)$

Parameter of dual prestructure: d^*_{23}: $2 \times 1^0 2^1 3^1, 1 \times 1^0 2^2 3^0, 4 \times 1^2 2^0 3^1, 2 \times 1^3 2^0 3^0$
$\mathcal{S}^*_{d_{23}} = \{(0,0,1,0),(0,0,0,1),(2,0,0,0),(1,1,0,0),(1,0,2,0)\}$,
$c^*_{d_{231}} = (8,6,0,2,2)$

$\mathbf{d}_{31}$: Parameter of primal prestructure: d^p_3 : $1 \times 2^0 3^4, 3 \times 2^2 3^0, 14 \times 2^0 3^1$
$S^p_{d_3} = \{(0,2,0),(1,0,2),(0,0,3))\}$,
$c^p_{d_{31}} = (3,4,2)$

Parameter of dual prestructure: d^*_{31}: $3 \times 1^0 2^2 4^0, 4 \times 1^2 2^0 4^1, 2 \times 1^3 2^0 4^0$
$\mathcal{S}^*_{d_{31}} = \{(0,1,0),(0,0,1),(2,0,0),(0,4,0)\}$,
$c^*_{d_{311}} = (8,6,3,1)$

$\mathbf{d}_{41}$: Parameter of primal prestructure: d^p_4 : $1 \times 2^2 3^3, 2 \times 2^2 3^0, 15 \times 2^0 3^1$
$S^p_{d_4} = \{(1,1,0),(0,2,0),(1,0,2),(0,0,3)\}$,
$c^p_{d_{41}} = (2,1,3,3)$

Parameter of dual prestructure: d^*_{41}: $2 \times 1^0 2^1 5^1, 1 \times 1^0 2^2 5^0, 3 \times 1^2 2^0 5^1, 3 \times 1^3 2^0 5^0$
$\mathcal{S}^*_{d_{41}} = \{(0,0,1,0),(0,0,0,1),(2,0,0,0),(1,1,0,0),(2,0,3,0)\}$,
$c^*_{d_{411}} = (6,9,0,2,1)$

B Dual and Primal Prestructures

Here is listed all of the sets of the dual prestructures without sets of size 1 (which are easy to complete) and the corresponding blocks of the primal prestructures.

$\mathbf{d}_{111}$: $\mathcal{B}^* = \{\{0,1,2,8\},\{0,3,4,9\},\{0,5,6,10\},\{1,3,5,7\},\{2,4,6,7\}\}$;
$\mathcal{B}^p_{d_{111}} = \{\{0,1,2\},\{0,3,5\},\{0,4,6\},\{1,3,7\},\{1,4,8\},\{2,3,9\},\{2,4,10\}$,

$\{3,4,11\},\{0,12,13\},\{1,14,15\},\{2,16,17\}\}$

$\mathbf{d}_{121}$: $\mathcal{B}^* = \{\{0,1,2,3\},\{0,4,5,6\},\{1,4,7,8\},\{2,5,7,9\},\{3,6,8,9\}\}$;
$\mathcal{B}^p_{d_{121}} = \{\{0,1,5\},\{0,2,6\},\{0,3,7\},\{0,4,8\},\{1,2,9\},\{1,3,10\},\{1,4,11\},$
$\{2,3,12\},\{2,4,13\},\{3,4,14\},\{15,16,17\}\}$

$\mathbf{d}_{131}$: $\mathcal{B}^* = \{\{0,1,2,3\},\{0,4,5,6\},\{1,4,7,8\},\{2,5,7,9\},\{3,6,8,10\}\}$;
$\mathcal{B}^p_{d_{131}} = \{\{0,1,5\},\{0,2,6\},\{0,3,7\},\{0,4,8\},\{1,2,9\},\{1,3,10\},\{1,4,11\},$
$\{2,3,12\},\{2,4,13\},\{3,14,15\},\{4,16,17\}\}$

$\mathbf{d}_{211}$: $\mathcal{B}^* = \{\{0,2\},\{0,3,4\},\{1,2\},\{1,3,5\}\}$,
$\mathcal{B}^p_{d_{211}} = \{\{0,3\},\{1,2\},\{2,3\},\{0,1,4\},\{0,5,6\},\{1,7,8\},\{9,10,11\},\{12,13,14\},\{15,16,17\}\}$

$\mathbf{d}_{221}$: $\mathcal{B}^* = \{\{0,2\},\{0,3,4\},\{1,2\},\{1,5,6\}\}$,
$\mathcal{B}^p_{d_{221}} = \{\{0,3\},\{1,2\},\{2,3\},\{0,4,5\},\{0,6,7\},\{1,8,9\},\{1,10,11\},\{12,13,14\},\{15,16,17\}\}$

$\mathbf{d}_{231}$: Same prestructure as d_{221}

$\mathbf{d}_{311}$: $\mathcal{B}^* = \{\{0,1\},\{0,2\},\{1,2\},\{3,4,5,6\}\}$,
$\mathcal{B}^p_{d_{311}} = \{\{1,3\},\{2,3\},\{1,2\},\{0,4,5\},\{0,6,7\},\{0,8,9\},\{0,10,11\},\{12,13,14\},\{15,16,17\}\}$

$\mathbf{d}_{411}$: $\mathcal{B}^* = \{\{0,1,3,4,5\},\{0,2\},\{1,2\}\}$,
$\mathcal{B}^p_{d_{411}} = \{\{0,1\},\{0,2\},\{1,2\},\{0,3,4\},\{0,5,6\},\{0,7,8\},\{9,10,11\},\{12,13,14\},\{15,16,17\}\}$

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