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An application of Dilworth's Theorem to a problem on free Lie-algebras

Dedicated to the professors of mathematics
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1 Introduction

A generalization of a theorem of Golod, Šafarevič [3] after an idea by Koch [5] gives the following lower bound for the number r of relations given by sums of Lie-monoms of degree m which are necessary to obtain a nilpotent Lie-algebra as a factor-algebra of a free Lie-algebra with d generators [8]:

$$\frac{(m-1)^{m-1}}{m^m} d^m < r.$$

In order to study the sharpness of this bound one has to construct nilpotent Lie-algebras with a small number of relations. Wisliceny [9] described a class of systems of relations, called increasing systems (Erhöhungssysteme), which yield nilpotent Lie-algebras. This was first done for the case $m = 2$ [9] and later this was generalized by him [10] and Sauerbier [7], [8] to any $m \geq 2$.
In the next section we will explain that increasing systems can be described in a purely combinatorial way. Thus we are led to the combinatorial problem of the minimization of the size of increasing systems for given parameters d and m. (Of course, in general there may exist other systems than increasing systems yielding nilpotent Lie-algebras, and these systems may have still smaller sizes. One of them has been constructed by Sauerbier in the case $m = 3$ [8]. Another example is known in the case $m = 2$ [11]. But we regard only increasing systems).
After some preparations we will see that we have to find the minimum number of chains in a chain decomposition of a certain interval order which equals by Dilworth's Theorem [3]

the maximum size of an antichain in that order. We are able to determine this size in a short and elementary way. Thus we obtain a new short proof for the minimum size of increasing systems and a simpler formula for this number than given in [8].

2 The order-theoretic formulation

Let $\mathcal{L}(X)$ be the free LIE-algebra with the set $X = \{x_1, \dots, x_d\}, d \in \{3, 4, \dots\}$, of free generators over an arbitrary field K. We write $\mathcal{L}(X)$ as a graded algebra $\mathcal{L}(X) = \sum_{m=1}^{\infty} \mathcal{L}^m(X)$ where $\mathcal{L}^m(X)$ denotes the K-vector space which is generated by the LIE-monoms of degree m. For a subset $R \subseteq \mathcal{L}^m(X)$ let $I(R)$ be the ideal in $\mathcal{L}(X)$ generated by R and let $L_R := \mathcal{L}(X)/I(R)$ be the corresponding factor algebra. Let B be any basis of the K-vector space $\mathcal{L}(X)$ which contains LIE-monoms only. Clearly, $B^m := B \cap \mathcal{L}^m(X)$ is then a basis of $\mathcal{L}^m(X)$. Now we come to the main definition (cf. [8]). A subset R of $\mathcal{L}^m(X)$ is called an *increasing system of degree m relative to the basis B^m* if the following two conditions hold:

1. Any element of R is a finite sum of elements of B^m and the summands can be ordered in such a way that every index of a generator appearing as a factor in one summand is smaller than every index in the succeeding summand.

2. For every $b \in B^m$ there exists exactly one sum in R containing b as a summand.

As a specified basis we take the HALL-basis $B^m = H^m$ of $\mathcal{L}^m(X)$ [2]. We will not give its original definition. Instead we use the fact that there exists a bijection between H^m and the set of non-periodic cyclic words with m letters over the alphabet X [5]. To be precise we will define such words. For the sake of brevity we work only with the indices, i.e. we put $X := \{1, 2, \dots, d\}$. Let $W_{m,d}$ be the set of all words with m letters from X. For $\boldsymbol{a} = a_1 a_2 \cdots a_m \in W_{m,d}$ let $\{\boldsymbol{a}\} = \{a_1, a_2, \dots, a_m\}$. In $W_{m,d}$ we introduce an equivalence relation by $a_1 \cdots a_m \sim b_1 \cdots b_m$ iff there is some $p \in \{0, \dots, m-1\}$ such that $a_i = b_{i+p}$ holds for all i where the addition of the indices is modulo m. Clearly, for $\boldsymbol{a} \sim \boldsymbol{b}$ we have $\{\boldsymbol{a}\} = \{\boldsymbol{b}\}$. The class containing $\boldsymbol{a}$ is denoted by $[\boldsymbol{a}]$. Let $\min[\boldsymbol{a}] := \min\{l : l \in \{\boldsymbol{a}\}\}$ and $\max[\boldsymbol{a}]$ be defined analogously. Let $C_{m,d}$ be the set of all equivalence classes of size m. The elements of this set can be interpreted as the non-periodic cyclic words with m letters from X.

Given $b \in H^m$ one obtains a representative $\boldsymbol{a}$ of the bijectively associated element $[\boldsymbol{a}] \in C_{m,d}$ by omitting the brackets. Thus the bijection between $C_{m,d}$ and H^m has the property that for $[\boldsymbol{a}] \in C_{m,d}$ the corresponding basis element b in H^m has exactly $\{\boldsymbol{a}\}$ as the set of indices of the factors whose LIE-product is b. Standard application of MÖBIUS-inversion (cf. [1])

gives WITT's formula (cf. [2]):

$$| C_{m,d} | = \frac{1}{m} \sum_{t|m} \mu(t) d^{\frac{m}{t}} =: f_m(d)$$

where μ is the number theoretic MÖBIUS-function. Our set $C_{m,d}$ becomes a poset (in particular an interval order, but we do not need this fact) if we define an ordering by $[\boldsymbol{a}] < [\boldsymbol{b}]$ iff $\max[\boldsymbol{a}] < \min[\boldsymbol{b}]$.
By condition 1. of an increasing system R, every element of R corresponds to a chain in $C_{m,d}$ (the summands correspond to the elements of the chain). Moreover, condition 2. says that the whole set R corresponds to a decomposition of $C_{m,d}$ into $| R |$ chains.

Theorem *In the case of d generators, the minimum size of an increasing system of degree m relative to the* HALL*-basis H^m equals*

$$f_m(d) - f_m\left(\left\lfloor \frac{d-1}{2} \right\rfloor\right) - f_m\left(\left\lceil \frac{d-1}{2} \right\rceil\right).$$

Proof: From above we know that the minimum size of an increasing system equals the minimum number of chains in a chain decomposition of $C_{m,d}$ and by DILWORTH' Theorem [3] this equals the maximum size of an antichain in $C_{m,d}$. Thus we study in the following only the size of antichains.
For $u \in \{1, \ldots, d\}$ let

$$A_u := \{[\boldsymbol{a}] \in C_{m,d} : \min[\boldsymbol{a}] \leq u \leq \max[\boldsymbol{a}]\}.$$

Here both inequalities cannot be satisfied simultaneously with equality since we are dealing with "non-periodic words" and thus need at least two different letters. It is easy to see that A_u is an antichain in $C_{m,d}$ for each u. Moreover, if A is any antichain in $C_{m,d}$, let

$$u := \max\{\min[\boldsymbol{a}] : [\boldsymbol{a}] \in A\}.$$

Then, for all $[\boldsymbol{a}] \in A$,

$$\begin{aligned} \min[\boldsymbol{a}] &\leq u && \text{(by definition of } u\text{)}, \\ \max[\boldsymbol{a}] &\geq u && \text{(since } A \text{ is an antichain)}, \end{aligned}$$

thus $A \subseteq A_u$. Consequently, it is enough to look for the maximum size of the antichains $A_u, u = 1, 2, \ldots, d$. For that we introduce the following sets (omitting the parameters m and d):

$$\begin{aligned} C^{\leq}[i,j] &:= \{[\boldsymbol{a}] \in C_{m,d} : i \leq \min[\boldsymbol{a}] \quad \text{and} \quad j \geq \max[\boldsymbol{a}]\}, \\ C^{=}[i,j] &:= \{[\boldsymbol{a}] \in C_{m,d} : i = \min[\boldsymbol{a}] \quad \text{and} \quad j = \max[\boldsymbol{a}]\}. \end{aligned}$$

Since $C^{\leq}[i,j]$ is constructed using the $j-i+1$ letters $\{i, i+1, \ldots, j\}$ we have

$$| \, C^{\leq}[i,j] \, | = f_m(j-i+1). \tag{1}$$

Moreover, in view of

$$C^{=}[i,j] = C^{\leq}[i,j] \setminus \left(C^{\leq}[i+1,j] \cup C^{\leq}[i,j-1] \right)$$

and

$$C^{\leq}[i+1,j] \cap C^{\leq}[i,j-1] = C^{\leq}[i+1,j-1]$$

it holds

$$| \, C^{=}[i,j] \, | = f_m(j-i+1) - 2f_m(j-i) + f_m(j-i-1), \tag{2}$$

i. e. the size of $C^{=}[i,j]$ depends only on the difference $j-i$.
Obviously,

$$\begin{aligned} A_u \setminus A_{u-1} &= \{[\boldsymbol{a}] \in C_{m,d} : \min[\boldsymbol{a}] = u \leq \max[\boldsymbol{a}]\} \\ &= C^{=}[u,u+1] \uplus C^{=}[u,u+2] \uplus \ldots \uplus C^{=}[u,d] \end{aligned}$$

and

$$\begin{aligned} A_{u-1} \setminus A_u &= \{[\boldsymbol{a}] \in C_{m,d} : \min[\boldsymbol{a}] \leq u-1 = \max[\boldsymbol{a}]\} \\ &= C^{=}[u-2,u-1] \uplus C^{=}[u-3,u-1] \uplus \ldots \uplus C^{=}[1,u-1] \end{aligned}$$

where $\uplus$ denotes a disjoint union.
Because of (2) we have

$$\begin{aligned} | \, C^{=}[u,u+1] \, | &= | \, C^{=}[u-2,u-1] \, |, \\ | \, C^{=}[u,u+2] \, | &= | \, C^{=}[u-3,u-1] \, |, \ldots \end{aligned}$$

Moreover, in the partition of $A_u \setminus A_{u-1}$ there are $d-u$ sets, and in the partition of $A_{u-1} \setminus A_u$ there are $u-2$ sets. Thus $| \, A_u \, | \geq | \, A_{u-1} \, |$ iff $d-u \geq u-2$ iff $u \leq \dfrac{d+2}{2}$, and $| \, A_u \, |$ is maximum if $u = \left\lfloor \dfrac{d+1}{2} \right\rfloor$.
Finally,

$$A_u = C_{m,d} \setminus \left(C^{\leq}[1,u-1] \uplus C^{\leq}[u+1,d] \right),$$

consequently, using (1),

$$| \, A_u \, | = f_m(d) - f_m(u-1) - f_m(d-u),$$

and with the optimal $u = \left\lfloor \dfrac{d+1}{2} \right\rfloor$ we obtain the formula from the theorem.

Remark: The theorem remains true if the HALL-basis H^m is replaced by any other basis B^m which contains LIE-monoms only. This follows from [7, Theorem 6].

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