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Auslander–Reiten sequences and a certain ideal in mod–FG

Dedicated to the professors of mathematics
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In [1], Auslander and Reiten defined an almost split sequence as a certain type of non–split short exact sequence of modules over artin algebras (the definition is repeated here in (19) below). Since then, this concept has been extensively used in the general representation theory of finite dimensional algebras, but also — for example by Erdmann [2] — in the much more restricted case of group algebras. In this situation, the most basic property of Auslander–Reiten sequences, namely their existence, turns out to be closely related to an old acquaintance, the so called projective maps. The purpose of the present note is to exhibit this relation, thereby giving a rather explicit construction of Auslander–Reiten sequences for group algebras. In fact, the construction can be relativized up to a point (Theorem 17). The reader not interested in this can always take $H = 1$ in what follows. For another treatment of the topic, see Greens paper [3].

Let G be a finite group, H a subgroup and F a field of characteristic $p > 0$. We consider finitely generated, right FG-modules $K, L, M, \dots$ If L and M are two such modules, denote $(L, M)_G = Hom_{FG}(L, M)$ and $(L, M)_H = Hom_{FH}(L, M)$. We will use Greek letters $\alpha, \beta, \dots$ for FG-homomorphims and Latin letters $a, b, \dots$ for FH-homomorphisms; both are written on the right. So if $\alpha : L \to M$ and $b : M \to K$, then $\alpha b \in (L, K)_H$.

The relative trace map $T_H^G : (L, M)_H \to (L, M)_G$ by $T_H^G(a) = \sum_g g^{-1}ag$, where $G = \dot{\bigcup}_g Hg$, is simply denoted by T since G and H are fixed. We will, however, sometimes use the notation $T = T(L, M)$ to emphasize the pair of FG-modules involved. Note that T is well defined, i. e. independent of the choice of coset representatives; also, $T(\alpha b\gamma) = \alpha T(b)\gamma$ whenever $\alpha b\gamma$ is defined.

Finally, the trace map from $(M, M)_1$ into F is denoted by $tr = tr_M$. Note that for $a : M \to L$ and $b : L \to M$, one has $tr_M(ab) = tr_L(ba)$.

Now we define

(1) $I(M,L) = \{\alpha : M \to L \mid tr(\alpha b) = 0 \; \forall b \in Ker\ T(L,M)\}$.

It is clear that $I(M,L)$ is an F-subspace of $(M,L)_G$ and it is a trivial exercise to show:

(2) I is a categorial ideal, i. e. for all modules K, L, M, N, one has

$$I(M,L)\ (L,K)_G \subseteq I(M,K)$$

and

$$(N,M)_G\ I(M,L) \subseteq I(N,L).$$

It is well known that $Im\ T$ is also a categorial ideal; one has

(3) $Im\ T \subseteq I$.

Proof: Fix modules M and L and let $a : M \to L$. Then $T(a)$ is a typical element in $Im\ T(M,L)$, so we have to show that $tr\,(T(a)b) = 0$ whenever $b : L \to M$ and $T(b) = 0$. Choose coset representatives $G = \dot{\bigcup}_g Hg$ such that also $G = \dot{\bigcup}_g Hg^{-1}$; this is always possible as is well known. Then $tr\,(T(a)b) = \sum_g tr(g^{-1}agb) = \sum_g tr(agbg^{-1}) = tr\,(aT(b)) = 0$ since $T(b) = 0$.

(4) If $H = 1$, then $Im\ T = I$.

Proof: Let M and L be . Denote

$$\begin{aligned} n &= \dim\,(M,L)_1 = \dim(L,M)_1, \\ k_1 &= \dim\ Ker\ T(M,L) \qquad \text{and} \\ k_2 &= \dim\ Ker\ T(L,M). \end{aligned}$$

Note that $(a,b) = tr(ab)$ defines a non-degenerate bilinear map

$$(\ )\colon\ (M,L)_1 \times (L,M)_1 \to F$$

and that by definition $I(M,L) = (M,L)_G\ \cap\ [Ker\ T(L,M)]^{\perp}$. Therefore

$$\begin{aligned} n - k_1 &= \dim(M,L)_1 - \dim\ Ker\ T(M,L) \\ &= \dim\ Im\ T(M,L) \\ &\le \dim\ I(M,L) \qquad \text{by (3)} \\ &\le \dim[\mathrm{Ker}\ T(L,M)]^{\perp} \\ &= n - k_2 \qquad \text{, since } (\ ,\ ) \text{ is non-degenerate.} \end{aligned}$$

By symmetry, equality holds and in particular, $Im\ T(M,L) = I(M,L)$.

[For arbitrary $H \leq G$, equality does not hold in (3) in general.]

As usual, denote $sl(M) = \{f \in (M,M)_1 \mid trf = 0\}$.

(5) $I(M,M) = (M,M)_G$ iff $Ker\ T(M,M) \subseteq sl(M)$.

In particular, this is the case if M is H–projective.

Proof: Since $I(M,M)$ is an ideal in the ring $(M,M)_G$ by (2), one has

$$\begin{aligned} I(M,M) = (M,M)_G \quad &\Leftrightarrow \quad 1_M \in I(M,M) \\ &\Leftrightarrow \quad 0 = tr(1_M\ b) = tr(b)\ \forall b \in Ker\ T(M,M) \\ &\Leftrightarrow \quad Ker\ T(M,M) \subseteq sl(M). \end{aligned}$$

If M is H–projective, then $Im\ T(M,M) = (M,M)_G$ and the assertion follows from (3).

(6) M is projective iff $Ker\ T_1^G \subseteq sl(M)$.

Proof: Clear from (4) and (5).

(7) Let U be H–projective. For every $\alpha : U \to$, there is an $a : U \to U$ such that $T(a) = \alpha$. Define $t_0(\alpha) = tr(a)$; by (5), t_0 is well defined, i. e. independent of the choice of a. Clearly $t_0 : (U,U)_G \to F$ is F-linear.

Now lef M be an FG-module and let

(8)

$$0 \longrightarrow L \xrightarrow{\alpha} U \xrightarrow{\beta} M \longrightarrow 0$$

and

$$0 \longleftarrow L \xleftarrow{a} U \xleftarrow{b} M \longleftarrow 0$$

be the H–projective cover of M, i. e. the sequences are exact, L contains no H–projective direct summand $\neq 0$, U is H–projective and $a\alpha = 1_L\ \ b\beta = 1_M$. We fix this situation for the rest of this note.

(9) For any $\lambda : M \to L$, define $t(\lambda) = t_0(\beta\lambda\alpha)$, so by (7), $t : (M,L)_G \to F$ is a well defined F-linear map.

(10) (a) Given

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & M & \longrightarrow & 0 \\ & & & & & & \Big\downarrow{\scriptstyle \rho} & & \\ & & & & & & K & & \end{array}$$

one has $\rho \in I(M,K)$ iff $t(\rho\sigma) = 0 \quad \forall \sigma : K \to L$.

(b) Given

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & M & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \sigma} & & & & & & \\ & & K & & & & & & \end{array}$$

one has $\sigma \in I(K,L)$ iff $t(\rho\sigma) = 0 \quad \forall \rho : M \to K$.

Proof: (a) "$\Rightarrow$" Let $\sigma : K \to L$. Since U is H–projective, there is an $f : U \to M$ such that $T(f) = \beta$. Define $r = \sigma\alpha f : K \to M$. Then $T(r) = \sigma\alpha T(f) = \sigma\alpha\beta = 0$. By assumption, $\rho \in I(M,K)$, hence $tr(\rho r) = 0$. Therefore $t(\rho\sigma) = t_0(\beta\rho\sigma\alpha) = tr(f\rho\sigma\alpha) = tr(\rho\sigma\alpha f) = tr(\rho r) = 0$.
"$\Leftarrow$" Let $r : K \to M$ with $T(r) = 0$. We have to show that $tr(\rho r) = 0$. Note that $rb : K \to U$, so $\eta = T(rb) : K \to U$ and $\eta\beta = T(rb\beta) = T(r) = 0$; hence $\sigma = \eta\alpha^{-1} : K \to L$ is well defined. By assumption, $0 = t(\rho\sigma) = t_0(\beta\rho\sigma\alpha) = t_0(\beta\rho\eta) = tr(\beta\rho rb) = tr(\rho rb\beta) = tr(\rho r)$.
(b) Similar.

(11) For any two modules, denote

$$\overline{(K,N)} = (K,N)_G \;/\; I(K,N)\ .$$

Write $S_K = \overline{(K,K)}$ and note that by (2), S_K is a ring and $\overline{(K,N)}$ is an S_K–S_N–bimodule.

(12) Given

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & M & \longrightarrow & 0 \\ & & \vdots\,\varphi^* & & \vdots\,\varphi_1 & & \Big\downarrow\varphi & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & L & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & M & \longrightarrow & 0\ , \end{array}$$

the map $f = \beta\varphi b : U \to U$ satisfies $f\beta = \beta\varphi$. Since U is H–projective, there exists $\varphi_1 : U \to U$ such that $\varphi_1\beta = \beta\varphi$. Then $\varphi^* = \alpha\varphi_1\alpha^{-1} : L \to L$ is well defined, and the above diagram commutes.
Note that φ is H–projective iff φ^* is H–projective:
If $\varphi = T(f)$ is H-projective, then $\varphi_2 = \beta T(fb) : U \to U$ and $\alpha\varphi_2 = 0$. Moreover, $\varphi_2\beta = \beta T(fb\beta) = \beta T(f) = \beta\varphi = \varphi_1\beta$, so $\eta = (\varphi_1 - \varphi_2)\alpha^{-1} : U \to L$ is well defined and H-projective since U is H-projective. It follows that $\alpha\eta\alpha = \alpha(\varphi_1 - \varphi_2) = \alpha\varphi_1 = \varphi^*\alpha$, so $\alpha\eta = \varphi^*$, since α is a monomorphism. Therefore φ^* is H-projective. The converse is shown

similarly.
Therefore, the map $\varphi \mapsto \varphi^*$ induces a ring isomorphism

$$\hat{\omega}: \ \hat{S}_M = (M, M)_G \ / Im \ T(M, M) \to \hat{S}_L.$$

By (2) and (3),

$$\hat{I}_M = I(M, M) \ / \ Im \ T(M, M)$$

is an ideal of $\hat{S}_M$. Not surprisingly,

(13)

$$\hat{\omega}(\hat{I}_M) = \hat{I}_L.$$

We show first

(14) Given the commutative diagram in (12) and $\lambda : M \to L$, one has

$$t(\varphi\lambda) = t(\lambda\varphi^*).$$

Proof: Take $f : U \to U$ such that $T(f) = \varphi_1$. Then $t(\varphi\lambda) = t_0(\beta\varphi\lambda\alpha) = t_0(\varphi_1\beta\lambda\alpha) = tr(f\beta\lambda\alpha) = tr(\beta\lambda\alpha f) = t_0(\beta\lambda\alpha\varphi_1) = t_0(\beta\lambda\varphi^*\alpha) = t(\lambda\varphi^*)$.

Proof: of (13). By (10a) with $K = M$, we have $\varphi \in I(M, M)$ iff $tr(\varphi\lambda) = 0$ for all $\lambda : M \to L$. By (14), this is equivalent to $t(\lambda\varphi^*) = 0$ for all such λ, and by (10b) with $K = L$, this is the case iff $\varphi^* \in I(L, L)$. The assertion follows.

From (13), it is clear that
(15) $S_M \simeq S_L$; the map $\varphi \mapsto \varphi^*$ induces a ring isomorphism $\omega : S_M \to S_L$.

(16) As usual, we denote $V^* = Hom_F(V, F)$ the dual of the F–space V. Note that if V is an R–S–bimodule for rings R and S, then V^* is an S–R–bimodule by

$$v(sw^*) = (vs)w^*$$

and

$$v(w^*r) = (rv)w^*$$

for $v \in V$, $w^* \in V^*$, $r \in R$ and $s \in S$.

(17) Theorem Let (8) be the H–projective cover of M and let K be an FG–module. Then

$$[\ ,\ ]: \ \overline{(M, K)} \times \overline{(K, L)} \to F \qquad \text{by} \qquad [\overline{\gamma}, \overline{\delta}] = t(\gamma\delta)$$

for $\gamma \in (M,K)_G$ and $\delta \in (K,L)_G$ is a well defined non–degenerate bilinear map. The map

$$\zeta : \overline{(K,L)} \to \overline{(M,K)}^{\,*} \qquad \text{by} \qquad (\overline{\gamma})(\overline{\delta}\zeta) = [\overline{\gamma}, \overline{\delta}]$$

is an S_K–S_M–bimodule isomorphism. Also

$$\xi : \overline{(M,K)} \to \overline{(K,L)}^{\,*} \qquad \text{by} \qquad (\overline{\delta})(\overline{\gamma}\xi) = [\overline{\gamma}, \overline{\delta}]$$

is an S_M–S_K–bimodule isomorphism.

Proof: The first statement is immediate from (10a) and (10b). It is then obvious that ζ is an F–isomorphism . The S_K–S_M structure needs some comment: $\overline{(M,K)}$ is an S_M–S_K–bimodule by (11), hence $\overline{(M,K)}^{\,*}$ is an S_K–S_M–bimodule by (16). Again by (11), $\overline{(K,L)}$ is an S_K–S_L–bimodule; now $S_M \simeq S_L$ by (15), and we can use the isomorphism $\omega : S_M \to S_L$ to define an S_K–S_M–structure on $\overline{(K,L)}$.

Now let $\gamma \in (M,K)_G,\ \delta \in (K,L)_G,\ \eta \in (K,K)_G$ and $\varphi \in (M,M)_G$. Then

$$(\overline{\gamma})[(\overline{\eta}\overline{\delta})\zeta] = [\overline{\gamma}, \overline{\eta}\overline{\delta}] = t(\gamma\eta\delta)$$

and

$$\begin{aligned}
(\overline{\gamma})[\overline{\eta}(\overline{\delta}\zeta)] &= (\overline{\gamma}\ \overline{\eta})(\overline{\delta}\zeta) \qquad [\text{see } (16)] \\
&= [\overline{\gamma\eta}, \overline{\delta}] \\
&= t(\gamma\eta\delta)
\end{aligned}$$

so $(\overline{\eta}\overline{\delta})\zeta = \overline{\eta}(\overline{\delta}\zeta)$ and ζ is S_K–linear.

To see the S_M–linearity, note that by definition

$$\overline{\delta}\overline{\varphi} = \overline{\delta}(\overline{\varphi}\omega) = \overline{\delta}\overline{\varphi^*}\ .$$

Therefore

$$\begin{aligned}
(\overline{\gamma})[(\overline{\delta}\overline{\varphi})\zeta] &= (\overline{\gamma})[(\overline{\delta}\overline{\varphi^*})\zeta] \\
&= [\overline{\gamma}, \overline{\delta}\overline{\varphi^*})] \\
&= t(\gamma\delta\varphi^*)
\end{aligned}$$

while

$$\begin{aligned}
(\overline{\gamma})[(\overline{\delta}\zeta)\overline{\varphi}] &= (\overline{\varphi\gamma})(\overline{\delta}\zeta) \qquad [\text{see } (16)] \\
&= [\overline{\varphi\gamma}, \overline{\delta}] \\
&= t(\varphi\gamma\delta)\ .
\end{aligned}$$

Equality follows now from (14) with $\lambda = \gamma\delta : M \to L$.
The proof for ξ is similar.

(18) Corollary Let M be indecomposable and non–projective. Then there exists a non–projective map $\tau : M \to \Omega M$ such that for every module K and every map $\eta : K \to M$ which is not a split–epimorphism, the map $\eta\tau : K \to \Omega M$ is projective.

Proof: Take $H = 1$ in the above discussion; so in particular, in (8), $L = \Omega M$. By (4), $S_M \neq 0$ since M is not projective. But M is indecomposable, so S_M is local. Choose $0 \neq f \in (S_M)^*$ such that $J(S_M)f = 0$. By the theorem — with $K = M$ — , there is a $\tau : M \to \Omega M$ such that $\overline{\tau}\zeta = f$. Clearly, τ is not projective, since otherwise $\overline{\tau} = 0$, hence $f = 0$. If $\eta : K \to M$ is not a split–epimorphism and $\rho : M \to K$, then $\rho\eta : M \to M$ and in fact $\rho\eta \in J(M, M)_G$. Hence

$$0 = (\overline{\rho\eta})f = (\overline{\rho\eta})(\overline{\tau}\zeta) = [\overline{\rho\eta}, \overline{\tau}] = t(\rho\eta\tau)\ .$$

This holds for all $\rho : M \to K$, so by (10b) with $\sigma = \eta\tau : K \to \Omega M = L$, we conclude that $\eta\tau \in I(K, \Omega M)$. By (4), this means that $\eta\tau$ is projective.

(19) An *Auslander–Reiten sequence* is a non–split short exact sequence

$$0 \longrightarrow N \xrightarrow{\varphi} E \xrightarrow{\psi} M \longrightarrow 0$$

such that every map $\eta : K \to M$ which is not a split–epimorphism admits a factorization over ψ and such that M and N are indecomposable.

(20) Theorem (Auslander–Reiten) Let M be indecomposable and non-projective. Then there exists a unique (up to isomorphism of short exact sequences) Auslander–Reiten sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0\ .$$

Moreover, $N \simeq \Omega^2 M$.

Proof: The uniqueness is an easy exercise, so it is enough to produce an Auslander –Reiten sequence.

Consider

$$0 \longrightarrow \Omega M \longrightarrow P \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow \Omega^2 M \xrightarrow{\mu} Q \xrightarrow{\pi} \Omega M \longrightarrow 0$$

the projective covers, so $\Omega^2 M$ is indecomposable. Let τ be as in (18) and consider

$$\begin{tikzcd} Q \arrow[r, "\pi"] & \Omega M \\ E \arrow[u, "\tau_1"'] \arrow[r, "\psi"] & M \arrow[u, "\tau"'] \end{tikzcd}$$

the pull–back of (π, τ) , i. e.

$$\begin{aligned} E &= \{(q,m) \in Q \oplus M \mid q\pi = m\tau\} \\ (q,m)\tau_1 &= q \\ (q,m)\psi &= m \ . \end{aligned}$$

Note that ψ is an epimorphism since π is. Define $\varphi : \Omega^2 M \to E$ by $x\varphi = (x\mu, 0)$. Then the diagram

$$\begin{tikzcd} & 0 \arrow[r] & \Omega^2 M \arrow[r, "\mu"] & Q \arrow[r, "\pi"] & \Omega M \arrow[r] & 0 \\ (*) & 0 \arrow[r] & \Omega^2 M \arrow[u, equal] \arrow[r, "\varphi"] & E \arrow[u, "\tau_1"'] \arrow[r, "\psi"] & M \arrow[u, "\tau"'] \arrow[r] & 0 \end{tikzcd}$$

is exact and commutative. Moreover, $(*)$ is non–split, since otherwise, there is a $\sigma : M \to E$ such that $\sigma\psi = 1_M$, but then $(\sigma\tau_1)\pi = \sigma\psi\tau = \tau$, i. e. τ factors over Q, hence is projective, a contradiction. Finally, if $\eta : K \to M$ is not a split–epimorphism, then $\eta\tau : K \to \Omega M$ is projective by (18), so $\eta\tau$ factors over π :

$$\begin{tikzcd} & Q \arrow[r, "\pi"] & \Omega\ M \\ & E \arrow[u, "\tau_1"'] \arrow[r, "\psi"'] & M \arrow[u, "\tau"'] \\ K \arrow[uur, "\gamma"] \arrow[ur, dotted, "\delta"] \arrow[urr, "\eta"'] & & \end{tikzcd}$$

Since E is the pull–back, there is $\delta : K \to E$ such that $\delta\psi = \eta$ (and $\delta\tau_1 = \gamma$) . This shows that $(*)$ is an Auslander–Reiten sequence.

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