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Groups of prime-power order with a small number of relations

Dedicated to the professors of mathematics
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1 Introduction

The Theorem of Golod/Shafarevich (1964) (briefly GST) gives a lower bound for the number of relations needed to define a group of prime-power order. Specifically let G be a group of prime-power order and d the generator number of G, then every presentation for G has more than $d^2/4$ relations. (The generator number is the size of a minimal generating set.) The GST actually says that every pro-p-presentation for G has more than $d^2/4$ relations. The same inequality also holds for presentations of nilpotent Lie algebras with finite generator number d (Koch 1977). It is natural to ask how sharp this inequality is (see for example Kostrikin 1965). Wisliceny (1981) has shown that, for odd primes p, there are pro-p-presentations with d generators and $d^2/4 + d/2 - (7 + (-1)^d)/8$ relations which define finite pro-p-groups (and also that there are presentations for Lie algebras with the same numbers of generators and of relations which define nilpotent Lie algebras) with generator number d. The corresponding result for $p = 2$ was proved by Sauerbier (1986). Thus for $d = 3$ and $d = 4$ the bound is reached for all primes p. With the help of the p-Quotient-Program [4] Newman and Wisliceny (unpublished) showed that the bound $\lfloor d^2/4 \rfloor + 1$ is reached for finite pro-p-groups with $5 \leq d \leq 8$ for some specific primes. On the basis of these results they conjectured that the GST is sharp in the sense that:

for every prime p and every positive integer d there is a pro-p-presentation with d generators and $\lfloor d^2/4 \rfloor + 1$ relations which defines a finite pro-p-group with generator number d.

Direct use of the p-Quotient Program can only give results for a finite number of primes and generator numbers.

The main results of this paper are:

For every prime p there is a pro-p-presentation with 5 *generators and* 7 *relations which defines a finite pro-p-group with generator number* 5. (Theorem 4.2 and Remark 4.3)

For every prime p there is a pro-p-presentation with 6 *generators and* 10 *relations which defines a finite pro-p-group with generator number* 6. (Theorem 4.4 and Remark 4.5)

The results of Wisliceny (1981) are obtained by first constructing corresponding Lie algebra presentations and then modifying them to construct suitable pro-p-presentations. We follow a similar path here though with a significant variation for $d = 5$. The main problem is *finding* suitable presentations. Showing they have the required properties is then relatively straight-forward. The search for presentations for Lie algebras was, in part, done with a program written in the computer algebra system GAP (M. Schönert et al. 1995).

We have found a nilpotent Lie algebra with generator number 6 defined by a presentation with 6 generators and 10 relations (3.3).

For $d = 5$ we have been unable to find a nilpotent Lie algebra with generator number 5 which has a presentation with 5 generators and 7 relations. In fact we expect none exist - at least none with homogeneous relations. We prove a partial result in this direction in Section 3. The Lie algebras associated with groups given by some presentations with 5 generators and 7 relations can be shown to have additional relations which are enough to prove nilpotence.

The relevant definitions are recalled briefly in Section 2 (more detail is given, for instance, in Wisliceny 1981). We deal with results for Lie algebras in Section 3. In Section 4 we exhibit the finite pro-p-groups.

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2 Preliminaries

Let $\mathcal{L}(X)$ or simply $\mathcal{L}$ be the free K-Lie algebra or $K[\pi]$-Lie algebra with the set $X = \{x_1, \dots, x_d\}$ of free generators over an arbitrary field K given with the usual gradation $\mathcal{L} = \sum_{m=1}^{\infty} \mathcal{L}^m$ $(\deg(x_i) = \deg(\pi) = 1)$. In the case of the free K-Lie algebra the vector space $\mathcal{L}^m$ is generated by all the left-normed Lie products of the form $x_{i_1} \dots x_{i_m}$; in the case of the free $K[\pi]$-Lie algebra $\mathcal{L}^m$ is generated by all products of the form $\pi^k x_{i_1} \dots x_{i_l}$ with $l > 0$, $k \geq 0$ and $k + l = m$. Let $I(R)$ be the ideal in $\mathcal{L}$ which is generated by a set $R \subseteq \mathcal{L}^2$. Then the factor Lie algebra $L = \mathcal{L}/I(R)$ is a graded K-algebra or $K[\pi]$-algebra (respectively) $L = \sum_{m=1}^{\infty} L^m$ with the homogeneous components L^m which are generated by products of degree m. We denote the dimension of L^m by $\dim L^m$. We define the dimension-vector of L by $\dim L := (\dim L^1, \dim L^2, \dots)$. We call a Lie algebra L *dimension-periodic*, if $\dim L$ is eventually periodic. We consider nilpotent Lie algebras as special cases of dimension-periodic Lie algebras.

Let $\mathcal{F}(X)$ or simply $\mathcal{F}$ be the free pro-p-group freely generated by the set X. Further let $\{X; \mathcal{R}_{\mathcal{F}}\}$ be a pro-p-presentation with $\mathcal{R}_{\mathcal{F}} \subseteq \mathcal{F}_2 = \mathcal{F}^p(\mathcal{F}, \mathcal{F})$. Let G be the pro-p-group given by this presentation. We will, in the usual way, use the graded object $gr\, G = \sum_{m=1}^{\infty} G_m/G_{m+1}$ with respect to the p-central series

$$G_1 := G, \qquad G_{m+1} := G_m^p(G_m, G)$$

of G as an $F_p[\pi]$-Lie algebra where F_p is the field with p elements.

There is an isomorphism between $gr\, \mathcal{F} = \sum_{m=1}^{\infty} \mathcal{F}_m/\mathcal{F}_{m+1}$ and the free $F_p[\pi]$-Lie algebra $\mathcal{L} = \sum_{m=1}^{\infty} \mathcal{L}^m$. Let $R = h(\mathcal{R}_{\mathcal{F}})$ be the image under the canonical homomorphism $h : \mathcal{F}_2 \to \mathcal{F}_2/\mathcal{F}_3 \cong \mathcal{L}^2$. This map h assigns group products to sums, p-th-powering to multiplication by π and commutators to Lie products. For example $h(x_1 \cdot x_2^p \cdot (x_3, x_4)) = x_1 + \pi x_2 + x_3 x_4$. Let J be the ideal in $\mathcal{L}$ with $gr\, G \cong \mathcal{L}/J$. Then we have $I(R) \subseteq J$.

We will use that the structure of the m-th homogeneous component of $gr\, G$ is

$$G_m/G_{m+1} = V^m \oplus \pi V^{m-1} \oplus \pi^2 V^{m-2} \oplus \ldots \oplus \pi^{m-2} V^2 \oplus \pi^{m-1} V,$$

where V^n $(1 \leq n \leq m)$ denotes the vector space which is generated by Lie monomials $x_{i_1} \cdots x_{i_n}$.

3 Lie Algebras

Theorem 3.1 *Let K be a field with characteristic zero. Then the K-Lie algebra $L5$ defined by*

$$\{x_1, x_2, x_3, x_4, x_5;$$
$$x_1x_2, x_1x_3, x_3x_5, x_4x_5, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_2x_5 - x_3x_4\}$$

is dimension-periodic and $\dim L5 = (5, 3, 5, 3, \dots)$.

Remark 3.2 If $char(K) = p$, then the dimension-vector depends on p. It seems to be dimension-periodic with period $2p$: $\dim L5 = (5, 3, ..., 5, 3, 5, 4, ...)$. In the proof below we simply show that $\dim L5 = (5, 3, 5, 3, 5, 3, 5, 3, ...)$ for $p \geq 5$ which is used later.

Proof: of Theorem 3.1 and Remark 3.2.
We have to show that for m odd $\dim L5^m = 5$ and for m even $\dim L5^m = 3$.
For $i = 1, \dots, 5$ we set

$$i^{(1)} := x_i.$$

For $k \geq 1$ we set recursively:

$$\begin{array}{rcl} a^{(k)} & = & 2^{(k)}3^{(1)}, \\ b^{(k)} & = & 2^{(k)}4^{(1)}, \\ c^{(k)} & = & 4^{(k)}3^{(1)}, \end{array} \qquad \begin{array}{rcl} 1^{(k+1)} & = & a^{(k)}2^{(1)}, \\ 2^{(k+1)} & = & a^{(k)}3^{(1)}, \\ 3^{(k+1)} & = & a^{(k)}4^{(1)}, \\ 4^{(k+1)} & = & c^{(k)}3^{(1)}, \\ 5^{(k+1)} & = & c^{(k)}4^{(1)}. \end{array}$$

The system of relations of $L5$ is invariant with respect to the following transformation:
$1^{(k)} \longleftrightarrow 5^{(k)}$, $2^{(k)} \longleftrightarrow 4^{(k)}$, $3^{(k)} \longleftrightarrow 3^{(k)}$,
$a^{(k)} \longleftrightarrow c^{(k)}$, $b^{(k)} \longleftrightarrow -b^{(k)}$.
Our aim is to show that $\{1^{(k)}, 2^{(k)}, 3^{(k)}, 4^{(k)}, 5^{(k)}\}$ is a basis of $L5^m$ for $m = 2k-1$ and $\{a^{(k)}, b^{(k)}, c^{(k)}\}$ is a basis of $L5^m$ for $m = 2k$.
It is easy to see this for $m = 1$ and $m = 2$. By induction, for $m > 2$ we get the following equations.
$m = 2k$, $k \geq 2$, $i + j = k$:

$$a^{(i)}a^{(j)} = b^{(i)}b^{(j)} = c^{(i)}c^{(j)} = 0.$$

$m = 2k$, $k \geq 2$, $i + j = k + 1$:

$$1^{(i)}1^{(j)} = 2^{(i)}2^{(j)} = 3^{(i)}3^{(j)} = 4^{(i)}4^{(j)} = 5^{(i)}5^{(j)} = 0,$$

$$1^{(i)}2^{(j)} = 1^{(i)}3^{(j)} = 5^{(i)}4^{(j)} = 5^{(i)}3^{(j)} = 0.$$

$m = 2k$, $k \geq 2$:

$$\begin{array}{rccccccccc}
a^{(k)} & = & 2^{(k)}3^{(1)} & = & -\frac{1}{3}*2^{(k-1)}3^{(2)} & = & \cdots & = & -\frac{1}{3}*2^{(1)}3^{(k)} \\
 & = & \frac{1}{2}*1^{(k)}4^{(1)} & = & \cdots & = & \frac{1}{2}*1^{(2)}4^{(k-1)} & = & 1^{(1)}4^{(k)} \\
 & = & -a^{(k-1)}b^{(1)} & = & \cdots & = & -a^{(1)}b^{(k-1)}, & & \\
b^{(k)} & = & 2^{(k)}4^{(1)} & = & \cdots & = & 2^{(1)}4^{(k)} & = & \frac{1}{2}*1^{(k)}5^{(1)} \\
 & = & \frac{1}{4}*1^{(k-1)}5^{(2)} & = & \cdots & = & \frac{1}{4}*1^{(2)}5^{(k-1)} & = & \frac{1}{2}*1^{(1)}5^{(k)} \\
 & = & -a^{(k-1)}c^{(1)} & = & \cdots & = & -a^{(1)}c^{(k-1)}, & & \\
c^{(k)} & = & 4^{(k)}3^{(1)} & = & -\frac{1}{3}*4^{(k-1)}3^{(2)} & = & \cdots & = & -\frac{1}{3}*4^{(1)}3^{(k)} \\
 & = & \frac{1}{2}*5^{(k)}2^{(1)} & = & \cdots & = & \frac{1}{2}*5^{(2)}2^{(k-1)} & = & 5^{(1)}2^{(k)} \\
 & = & c^{(k-1)}b^{(1)} & = & \cdots & = & c^{(1)}b^{(k-1)}. & &
\end{array}$$

$m = 2k + 1$, $k \geq 2$, $i + j = k + 1$:

$$a^{(i)}1^{(j)} = b^{(i)}3^{(j)} = c^{(i)}5^{(j)} = 0.$$

$m = 2k+1$, $k \geq 2$:

$$\begin{aligned}
1^{(k+1)} &= a^{(k)}2^{(1)} = \cdots = a^{(1)}2^{(k)} = b^{(k)}1^{(1)} \\
&= \frac{1}{2} * b^{(k-1)}1^{(2)} = \cdots = \frac{1}{2} * b^{(1)}1^{(k)}, \\
2^{(k+1)} &= a^{(k)}3^{(1)} = -\frac{1}{3} * a^{(k-1)}3^{(2)} = \cdots = -\frac{1}{3} * a^{(1)}3^{(k)} \\
&= b^{(k)}2^{(1)} = \cdots = b^{(1)}2^{(k)} = -c^{(k)}1^{(1)} = -\frac{1}{2} * c^{(k-1)}1^{(2)} \\
&= \cdots = -\frac{1}{2} * c^{(1)}1^{(k)}, \\
3^{(k+1)} &= a^{(k)}4^{(1)} = \cdots = a^{(1)}4^{(k)} = c^{(k)}2^{(1)} = \cdots = c^{(1)}2^{(k)}, \\
4^{(k+1)} &= c^{(k)}3^{(1)} = -\frac{1}{3} * c^{(k-1)}3^{(2)} = \cdots = -\frac{1}{3} * c^{(1)}3^{(k)} \\
&= -b^{(k)}4^{(1)} = \cdots = -b^{(1)}4^{(k)} = -a^{(k)}5^{(1)} \\
&= -\frac{1}{2} * a^{(k-1)}5^{(2)} = \cdots = -\frac{1}{2} * a^{(1)}5^{(k)}, \\
5^{(k+1)} &= c^{(k)}4^{(1)} = \cdots = c^{(1)}4^{(k)} = -b^{(k)}5^{(1)} \\
&= -\frac{1}{2} * b^{(k-1)}5^{(2)} = \cdots = -\frac{1}{2} * b^{(1)}5^{(k)}.
\end{aligned}$$

It is routine to check that there are no further relations among the given generators.

Here we will only prove the equation

$$b^{(i)}b^{(j)} = 0 \quad \text{for} \quad m = 2k, \quad k \geq 2 \quad \text{and} \quad i + j = k.$$

In this part of the proof we need that $char(K) = 0$. Moreover the proof of the other equations is easier.
For $k = 2$, and thus $m = 4$, we have $b^{(1)}b^{(1)} = 0$ because of anticommutativity. For $k = 3$, and thus $m = 6$:

$$\begin{aligned}
2^{(3)}4^{(1)} &= b^{(2)}2^{(1)}4^{(1)} = b^{(2)}4^{(1)}2^{(1)} + 4^{(1)}2^{(1)}b^{(2)} \\
&= -4^{(3)}2^{(1)} - b^{(1)}b^{(2)} = b^{(1)}4^{(2)}2^{(1)} + b^{(2)}b^{(1)} \\
&= b^{(1)}2^{(1)}4^{(2)} + 2^{(1)}4^{(2)}b^{(1)} + b^{(2)}b^{(1)} \\
&= 2^{(2)}4^{(2)} + 2 * b^{(2)}b^{(1)} = -4^{(2)}2^{(2)} + 2 * b^{(2)}b^{(1)} \\
&= b^{(1)}4^{(1)}2^{(2)} + 2 * b^{(2)}b^{(1)} \\
&= b^{(1)}2^{(2)}4^{(1)} + 2^{(2)}4^{(1)}b^{(1)} + 2 * b^{(2)}b^{(1)} \\
&= 2^{(3)}4^{(1)} + 3 * b^{(2)}b^{(1)}
\end{aligned}$$

and therefore $b^{(2)}b^{(1)} = 0$. This also holds for $char(K) > 3$ which justifies the claim in Remark 3.2.

Assuming that the given equations are true for all $m < 2k$ we show $b^{(i)}b^{(j)} = 0$ for $i + j = k$, $k \geq 3$. We get

$$\begin{aligned} 2^{(k)}4^{(1)} &= b^{(i)}2^{(j)}4^{(1)} \\ &= b^{(i)}4^{(1)}2^{(j)} + 4^{(1)}2^{(j)}b^{(i)} \\ &= -4^{(i+1)}2^{(j)} - b^{(j)}b^{(i)}, \end{aligned}$$

moreover, if $\mu + \nu = k + 1$, $i < \mu$:

$$\begin{aligned} 2^{(\mu)}4^{(\nu)} &= b^{(i)}2^{(\mu-i)}4^{(\nu)} \\ &= b^{(i)}4^{(\nu)}2^{(\mu-i)} + 4^{(\nu)}2^{(\mu-i)}b^{(i)} \\ &= -4^{(i+\nu)}2^{(\mu-i)} - b^{(\mu+\nu-i-1)}b^{(i)} \\ &= 2^{(\mu-i)}4^{(i+\nu)} + b^{(i)}b^{(j)}, \end{aligned}$$

on the other hand, if $i \geq \mu$ and consequently $j < \nu$:

$$\begin{aligned} 2^{(\mu)}4^{(\nu)} &= -4^{(\nu)}2^{(\mu)} \\ &= b^{(j)}4^{(\nu-j)}2^{(\mu)} \\ &= b^{(j)}2^{(\mu)}4^{(\nu-j)} + 2^{(\mu)}4^{(\nu-j)}b^{(j)} \\ &= 2^{(j+\mu)}4^{(\nu-j)} + b^{(i)}b^{(j)}. \end{aligned}$$

Thus there are k_1, k_2 with $k_1 < k_2$ and s, t with $s + t = k + 1$ so, that

$$2^{(s)}4^{(t)} + k_1 * b^{(i)}b^{(j)} = 2^{(s)}4^{(t)} + k_2 * b^{(i)}b^{(j)}$$

and since $char(K) = 0$ follows $b^{(i)}b^{(j)} = 0$.

With the help of a program written in the computer algebra system GAP we have found that the following holds

3.3 *Let K be a field with characteristic different from 2. Then the K-Lie algebra $L6$ defined by the presentation*

$$\begin{aligned} &\{x_1, x_2, x_3, x_4, x_5, x_6; \\ &x_1x_2, x_1x_3, x_3x_4, x_4x_6, x_5x_6, x_1x_4 + x_2x_3, \\ &x_1x_5 + x_2x_4, x_1x_6 + x_2x_5, x_2x_6 + x_3x_5, x_3x_6 + x_4x_5\} \end{aligned}$$

is nilpotent of class 5 and

$$\dim L6 = (6, 5, 10, 12, 10, 0) \quad for \quad char(K) = 3,$$

$$\dim L6 = (6,5,10,12,8,0) \quad for \quad char(K) > 3 \quad \text{or} \quad = 0.$$

We now examine the question of the existence of nilpotent K-Lie algebras L with $d(L) = 5$ and $r(L) = 7$.

In the papers of BRAHANA (1951,1958) quotient groups G/N with $N \subseteq (G,G)$ of the group G of the order p^{15} which can be given by the presentation $\{X; R\}$ with

$$\begin{aligned} X &= \{x_1, x_2, \ldots, x_{15}\} \quad \text{and} \\ R &= \{x_i^p \ (i = 1, \ldots, 15), \\ &\quad (x_2, x_1) = x_6, \ldots, (x_5, x_4) = x_{15}, \\ &\quad (x_j, x_i) \ (15 \geq j > i \geq 1,\ j > 5)\}. \end{aligned}$$

are studied. Using BRAHANA's classification of types of invariant subgroups N we can prove the following

Theorem 3.4 *There are no nilpotent Lie algebras with generator number* 5 *which can be defined by* 7 *homogeneous relations of degree* 2.

Proof: BRAHANA's classification of the planes in a 9-dimensional projective space into 22 types yields that we have only to investigate the nilpotence of the Lie algebras with relations:

$$\begin{aligned} a)\, L_a &: x_1x_2 - x_3x_4 + x_3x_5, x_1x_3 - x_2x_5, x_2x_4 - x_3x_5, x_1x_4, x_1x_5, x_2x_3, x_4x_5, \\ b)\, L_b &: x_1x_2 - x_3x_4, x_1x_3 - x_2x_5, x_1x_4 - x_2x_3, x_1x_5, x_2x_4, x_3x_5, x_4x_5, \\ c)\, L_c &: x_1x_2 - x_3x_4, x_1x_3 - x_2x_5, x_2x_3 - x_4x_5, x_1x_4, x_1x_5, x_2x_4, x_3x_5. \end{aligned}$$

Again let $i := x_i$ in the following calculations.
a) For $n \geq 1$ we consider the Lie products

$$\begin{aligned} ak_1l_1 &:= 12\cdots 23\cdots 3\,, \\ bk_2l_2 &:= 42\cdots 23\cdots 3\,, \\ ck_3l_3 &:= 52\cdots 23\cdots 3\,, \end{aligned}$$

where k_i denotes the number of the factors „2" and l_i denotes the number of the factors „3" with $k_i + l_i = n$, $k_i = 0, \ldots, n$. The number of these Lie products is $3n+3$. From the relation $23 = 0$ and the Jacobi identity we get the following system of equations:

$$ak_1l_1 + b(k_1-1)(l_1+1) - c(k_1-1)(l_1+1) = 0, \text{ where } k_1 \geq 1, k_1 + l_1 = n\,,$$

$$ak_2l_2 + c(k_2+1)(l_2-1) = 0, \text{ where } l_2 \geq 1, k_2 + l_2 = n\,,$$

$$ck_3l_3 - b(k_3+1)(l_3-1) = 0, \text{ where } l_3 \geq 1, k_3 + l_3 = n.$$

This is a system of $3n$ homogeneous linear equations in $3n+3$ unknowns. Therefore the dimension of the space of solutions is at least 3 and consequently the Lie algebra L_a is not nilpotent.

b) Mapping $5 \to 0$ we get the Lie algebra $12 = 34$, $14 = 23$, 13, 24 with $d = 4$ and $r = 4$ as a homomorphic image of L_b. It is a consequence of GST that this Lie algebra, and therefore the Lie algebra L_b, is not nilpotent.

c) The transformation $1 \to 3$, $2 \to 2$, $3 \to 4$, $4 \to 1$, $5 \to 5$ shows that $L_c \cong L5$.

4 Finite pro-p-groups

Let G be the group defined by the following presentation:

$$\begin{aligned}
&\{x_1,\ x_2,\ x_3,\ x_4,\ x_5;\\
&(x_1,x_2),\ (x_1,x_3),\ (x_5,x_4),\ (x_5,x_3),\\
&(x_1,x_4) = (x_2,x_3),\ (x_1,x_5) = (x_2,x_4),\\
&(x_2,x_5) = (x_3,x_4)\}.
\end{aligned}$$

The Nilpotent Quotient Program (Nickel 1993) shows that G has a largest nilpotent quotient and it has class 6; moreover modulo its 2-torsion it has class 5. Hence for p odd the pro-p-group G_p defined by the above presentation, viewed as a pro-p-presentation, is nilpotent of class 5. It follows that the Lie ring associated with the p-central series of G_p is nilpotent. Since by Theorem 3.1 this is not the case for $L5$ we need to find additional relations in the associated Lie ring.

4.1 *For $p \geq 3$ the space V^n in $gr\, G_p$ is zero for $n \geq 7$.*

Proof: The F_p-Lie algebra $gr\, G_p$ is a homomorphic image of the dimension-periodic Lie algebra $L5$ for $p \geq 3$.

If $gr\, G_p = \mathcal{L}/J$ and $L5 = \mathcal{L}/I$ then $I \subseteq J$. Let $\mathcal{L}_n := \sum_{m=n}^{\infty} \mathcal{L}^m$. We want to find elements of $J \setminus I$. For this we have to calculate in $gr\, G_p$ with regard to higher commutators. Instead of the Jacobi identity we use the rules:

$$abc = acb + cba + (ab)(bc) + (bc)(ca) + (ca)(ab),$$

$$a_1 \cdots a_k(ba) = a_1 \cdots a_k ba - a_1 \cdots a_k ab + a_1 \cdots a_k b(ab) + a_1 \cdots a_k a(ab)$$

$(k > 1)$ and calculate modulo $\mathcal{L}_{m+2} \cup I$ if the product on the left is an element of $\mathcal{L}_m$. We use again the symbols from the proof of (3.1). We get:

$$\begin{aligned}
0 &= 2^{(1)}3^{(1)}1^{(1)}5^{(1)}1^{(1)} \\
&= 1^{(1)}4^{(1)}1^{(1)}5^{(1)}1^{(1)} \\
&= 1^{(1)}4^{(1)}(1^{(1)}5^{(1)})1^{(1)} + 1^{(1)}4^{(1)}5^{(1)}1^{(1)}1^{(1)} \\
&\quad + 1^{(1)}4^{(1)}5^{(1)}(1^{(1)}5^{(1)})1^{(1)} + 1^{(1)}4^{(1)}1^{(1)}(1^{(1)}5^{(1)})1^{(1)} \\
&= 1^{(1)}4^{(1)}(1^{(1)}5^{(1)})1^{(1)} + 1^{(1)}4^{(1)}5^{(1)}1^{(1)}1^{(1)} + a^{(3)} + 0 \\
&= 5^{(1)}1^{(1)}(2^{(1)}3^{(1)})1^{(1)} + 2^{(1)}3^{(1)}5^{(1)}1^{(1)}1^{(1)} + a^{(3)},
\end{aligned}$$

further

$$\begin{aligned}
(5^{(1)}1^{(1)})(2^{(1)}3^{(1)})1^{(1)} &= 5^{(1)}1^{(1)}1^{(1)}(2^{(1)}3^{(1)}) + 1^{(1)}(2^{(1)}3^{(1)})(5^{(1)}1^{(1)}) \\
&= 5^{(1)}1^{(1)}1^{(1)}(2^{(1)}3^{(1)}) + 0 \\
&= 5^{(1)}1^{(1)}1^{(1)}2^{(1)}3^{(1)} - 5^{(1)}1^{(1)}1^{(1)}3^{(1)}2^{(1)} \\
&\quad + 5^{(1)}1^{(1)}1^{(1)}2^{(1)}(3^{(1)}2^{(1)}) + 5^{(1)}1^{(1)}1^{(1)}3^{(1)}(3^{(1)}2^{(1)}) \\
&= 5^{(1)}1^{(1)}1^{(1)}2^{(1)}3^{(1)} - 5^{(1)}1^{(1)}1^{(1)}3^{(1)}2^{(1)} + 0 + 0 \\
&= 4^{(1)}3^{(1)}1^{(1)}1^{(1)}3^{(1)} + 0 \\
&= 4^{(1)}1^{(1)}3^{(1)}1^{(1)}3^{(1)} + (1^{(1)}4^{(1)})(4^{(1)}3^{(1)})1^{(1)}3^{(1)} \\
&= 3^{(1)}2^{(1)}3^{(1)}1^{(1)}3^{(1)} + 0 = 0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0 &= 2^{(1)}3^{(1)}5^{(1)}1^{(1)}1^{(1)} + a^{(3)} \\
&= 2^{(1)}5^{(1)}3^{(1)}1^{(1)}1^{(1)} + (5^{(1)}2^{(1)})(2^{(1)}3^{(1)})1^{(1)}1^{(1)} + a^{(3)} \\
&= 3^{(1)}4^{(1)}3^{(1)}1^{(1)}1^{(1)} + a^{(3)} \\
&= 3^{(1)}4^{(1)}1^{(1)}1^{(1)}3^{(1)} + a^{(3)} \\
&= 1^{(1)}4^{(1)}3^{(1)}1^{(1)}3^{(1)} + (3^{(1)}4^{(1)})(4^{(1)}1^{(1)})1^{(1)}3^{(1)} + a^{(3)} \\
&= 2^{(1)}3^{(1)}3^{(1)}1^{(1)}3^{(1)} + 0 + a^{(3)} = a^{(3)}.
\end{aligned}$$

The symmetry of the presentation of G_p gives also $c^{(3)} = 0$. It follows $1^{(4)} = 2^{(4)} = \ldots = 5^{(4)} = 0$ and thus $V^7 = 0$.
(We can also show $b^{(3)} = 0$ so even $V^6 = 0$.)

Theorem 4.2 *The pro-p-presentation*

$$\begin{aligned}
\{x_1, x_2, x_3, x_4, x_5; \quad & (x_1, x_2),\ (x_1, x_3) = x_3^{-p}, \\
& (x_5, x_4) = x_4^p,\ (x_5, x_3) = x_1^{-p}, \\
& (x_1, x_4) = (x_2, x_3) \cdot x_2^p \cdot x_3^p, \\
& (x_1, x_5) = (x_2, x_4), \\
& (x_2, x_5) = (x_3, x_4) \cdot x_5^{-p}\}
\end{aligned}$$

defines a finite group for all primes $p > 2$.

Remark 4.3 We denote the group H_p. We can see with the help of the p-Quotient Program:

$$\dim gr\, H_2 = (5, 8, 13, 16, 11, 4, 3, 3, 3, 3, \dots),$$

$$\dim gr\, H_3 = (5, 8, 13, 16, 10, 4, 0),$$

$$\dim gr\, H_p = (5, 8, 13, 16, 10, 3, 0) \quad \text{for} \quad p = 5,\ 7,\ 11\,.$$

This is presumably true for all $p \geq 5$; that, however, would require more work to show. It seems likely that H_2 is not a finite 2-group. The p-Quotient Program shows that the following slightly modified presentation does define a 2-group with class 6 and order 2^{57}:

$$\begin{array}{ll} \{x_1, x_2, x_3, x_4, x_5; & (x_1, x_2),\ (x_1, x_3) = x_3^{-2}, \\ & (x_5, x_4) = x_4^{-2},\ (x_5, x_3) = x_1^{-2}, \\ & (x_1, x_4) = (x_2, x_3) \cdot x_2^{-2} \cdot x_3^2, \\ & (x_1, x_5) = (x_2, x_4), \\ & (x_2, x_5) = (x_3, x_4) \cdot x_5^{-2}\} \end{array}$$

Proof: of Theorem 4.2: Clearly H_p is finite iff $gr\, H_p$ is finite. Using (4.1), Remark 3.2 and the proof of Theorem 3.1 it is enough to show that there is a natural number l with $\pi^l x_i = 0$ for $i = 1, 2, \dots, 5$ in $gr\, H_p$. Let L be the $F_p[\pi]$-Lie algebra defined by the following relations:

$$x_1 x_2 = 0, \tag{1}$$
$$x_3 x_1 = \pi x_3, \tag{2}$$
$$x_5 x_4 = \pi x_4, \tag{3}$$
$$x_3 x_5 = \pi x_1, \tag{4}$$
$$x_1 x_4 = x_2 x_3 + \pi x_2 + \pi x_3, \tag{5}$$
$$x_1 x_5 = x_2 x_4, \tag{6}$$
$$x_2 x_5 = x_3 x_4 - \pi x_5. \tag{7}$$

We show that in L and therefore also in $gr\, H_p$:

$$\pi^5 x_1 = 0,\ \pi^6 x_3 = 0,\ \pi^7 x_2 = 0,\ \pi^8 x_5 = 0,\ \pi^9 x_4 = 0.$$

Again let $i := x_i$ in the following calculations. By (5), (1) and (2), it follows that

$$1412 = \pi 232 + \pi^2 32. \tag{8}$$

Further we have $4112 = 4211 = 5111$ and

$$\begin{aligned} 51113 &= 511(13) + 51131 = -\pi 5113 + 51(13)1 + 51311 \\ &= -\pi 51(13) - \pi 5131 - \pi 5131 + 5(13)11 + 53111 \\ &= \pi^2 513 - 2 * 5(13)1 - 2 * \pi 5311 - \pi 1111 \\ &= \pi^2 5(13) + \pi^2 531 + 2 * \pi^2 531 - \pi 5311 \\ &= -\pi^3 53 = \pi^3 35 = \pi^4 1. \end{aligned}$$

Therefore

$$14123 = -\pi^4 1. \tag{9}$$

By (8) and (9) we have

$$\pi 23231 - \pi^2 2331 = 0. \tag{10}$$

Further

$$\begin{aligned} 2331 &= 23(31) + 2313 = \pi 233 + 2(31)3 = 2 * \pi 233 \\ &\text{and} \\ 23231 &= 232(31) + 23213 = \pi 2323 + 23123 = \pi 2323 + \pi 2323 = 2 * \pi 2323. \end{aligned}$$

By (10), (8) and (9) it follows

$$0 = 2 * \pi^2 2323 - 2 * \pi^3 233 = 2 * \pi(\pi 2323 - \pi^2 233) = 2 * \pi 14123 = -2 * \pi^5 1.$$

Therefore it is $\pi^5 1 = 0$. Further it follows $\pi^6 3 = 0$ by (2), $\pi^7 2 = 0$ by (5), $\pi^8 5 = 0$ by (7) and $\pi^9 4 = 0$ by (3).
Consequently, we can choose $l = 9$.

Theorem 4.4 *The pro-p-presentation*

$$\begin{aligned} \{x_1, x_2, x_3, x_4, x_5, x_6; \quad & (x_1, x_2), \\ & (x_1, x_3) = x_3^{-p}, (x_3, x_4) = x_1^{-p}, \\ & (x_1, x_4) = (x_3, x_2) \cdot x_2^p \cdot x_3^p, (x_1, x_5) = (x_4, x_2), \\ & (x_2, x_5) = (x_6, x_1) \cdot x_6^p, (x_3, x_5) = (x_6, x_2), \\ & (x_4, x_5) = (x_6, x_3), (x_4, x_6) = x_4^p, \\ & (x_5, x_6) = x_5^p\} \end{aligned}$$

defines a finite group for all $p > 2$.

Proof: We denote the group M_p. The Lie algebra $gr\, M_p$ is a homomorphic image of the following $F_p[\pi]$-Lie algebra L:

$$\begin{aligned}
x_2x_1 &= 0, && (11)\\
x_3x_1 &= \pi x_3, && (12)\\
x_4x_3 &= \pi x_1, && (13)\\
x_4x_1 &= x_2x_3 - \pi x_2 - \pi x_3, && (14)\\
x_1x_5 &= x_4x_2, && (15)\\
x_2x_5 &= x_6x_1 + \pi x_6, && (16)\\
x_3x_5 &= x_6x_2, && (17)\\
x_4x_5 &= x_6x_3, && (18)\\
x_4x_6 &= \pi x_4, && (19)\\
x_5x_6 &= \pi x_5. && (20)
\end{aligned}$$

We show that in L and therefore also in $gr\, M_p$:

$$\pi^5 x_1 = 0, \quad \pi^6 x_3 = 0, \quad \pi^7 x_2 = 0, \quad \pi^8 x_6 = 0, \quad \pi^9 x_4 = \pi^9 x_5 = 0.$$

Again let $i := x_i$ in the following calculation. By the Jacobi identity, (12) and (13) it is $413 = 4(13) + 431 = -\pi 43 = -\pi^2 1$. It follows $4131 = 0$. On the other hand we have $413 = 233 - \pi 23$ by (14) and so

$$\begin{aligned}
0 &= 2331 - \pi 231 && (21)\\
&= 23(31) + 2313 - \pi 2(31) - \pi 213\\
&= \pi 233 + 2(31)3 + 2133 - \pi^2 23 = \pi 233 + \pi 233 - \pi^2 23\\
&= \pi 233 + \pi(233 - \pi 23) = \pi 233 - \pi^3 1.
\end{aligned}$$

That is

$$\pi^3 1 = \pi 233, \qquad 0 = \pi 2331 \tag{22}$$

and so $0 = \pi 2331 = \pi^2 231 = \pi^2 2(31) = \pi^3 23$ by (21). By (22) we then have $\pi^5 1 = \pi^3 233 = 0$. It follows that $\pi^6 3 = 0$ by (12), $\pi^7 2 = 0$ by (14), $\pi^8 6 = 0$ by (16), $\pi^9 4 = 0$ by (19) and $\pi^9 5 = 0$ by (20).
Now Theorem 4.4 follows from (3.3).

Remark 4.5 Here again the p-Quotient Program gives the group M_p has class 5 and order p^{94} for small odd primes. It can also be used to show that the following pro-2-presentation defines a finite 2-group with class 6 and order 2^{125}.

$$\begin{aligned}\{x_1, x_2, x_3, x_4, x_5, x_6; \quad & (x_1, x_2) \cdot x_2^2,\ (x_1, x_3) \cdot x_3^2,\ (x_4, x_1) = (x_3, x_2) \cdot x_6^2, \\ & (x_5, x_1) = (x_4, x_2),\ (x_5, x_2) = (x_4, x_3), \\ & (x_5, x_3) = (x_6, x_2),\ (x_5, x_4) = (x_6, x_3), \\ & (x_1, x_6) = x_1^2,\ (x_4, x_6) = x_4^2,\ (x_5, x_6) = x_5^2\}\end{aligned}$$

References

[1] **Brahana, H.R.** : *Finite metabelian groups and the lines of a projective four-space.* Amer. J. Math. **73**, 539-555 (1951)

[2] **Brahana, H.R.** : *Metabelian p-groups with five generators and orders* p^{12} *and* p^{11}*.* Illinois J. Math. **2**, 641-717 (1958)

[3] **Golod, E.S.** and **Shafarevich, I.R.** : *On the class field tower.* Izv. Akad. Nauk SSSR Ser. Mat. **28**, 261-272 (1964)

[4] **Havas, G., Newman, M.F.** and **O'Brien, E.A.** : *ANU p-Quotient Program (Version* 1.3*), written in* `C` *(1995), available from* `maths.anu.edu.au` *by* `anonymous ftp` *in the directory* `pub/PQ`*, as a share library with* GAP 3.4 *and as part of* `Magma`*.*

[5] **Koch, H.** : *Erzeugenden- und Relationenrang für endlich dimensionale nilpotente Liesche Algebren.* Algebra i Logika **16**, 364-374 (1977)

[6] **Kostrikin, A.I.** : *On presenting groups by generators and defining relations.* Izv. Akad. Nauk SSSR Ser. Mat. **29**, 1119-1122 (1965)

[7] **Nickel, W.** : *A Nilpotent Quotient Program (Version* 1.1*), written in* `C`*.* (1993)

[8] **Sauerbier, S.** : *Zur Darstellung von Pro-2-Gruppen durch Erzeugende und Relationen.* Wiss. Z. Päd. Hochsch. Güstrow, Math.-Nat.Fak. **1**, 27-38 (1986)

[9] **Schönert, M. et al.** : *GAP – Groups, Algorithms, and Programming.* (Release 3.4, 1995). Lehrstuhl D für Mathematik, Rheinisch-Westfälische Technische Hochschule Aachen, Germany

[10] **Wisliceny, J.** : *Zur Darstellung von Pro-p-Gruppen und Lieschen Algebren durch Erzeugende und Relationen.* Math. Nachr. **102**, 57-78 (1981)

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