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Approximation Properties of Multi–Scaling Functions: A Fourier Approach

Dedicated to the professors of mathematics
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ABSTRACT. In this paper, we consider approximation properties of a finite set of functions ϕ_ν $(\nu = 0, \ldots, r-1)$ which are not necessarily compactly supported, but have a suitable decay rate. Assuming that the function vector $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ is refinable, we sketch a new way, how to derive necessary and sufficient conditions for the refinement mask in Fourier domain.

KEY WORDS. Approximation Order, Refinement Mask, Strong-Fix Conditions

1 Introduction

For applications of multi–wavelets in finite element methods, the problem occurs, how to construct refinable vectors $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ $(r \in \mathbb{N})$ of functions with short support, such that algebraic polynomials of degree $< m$ $(m \in \mathbb{N})$ can be exactly reproduced by a linear combination of integer translates of ϕ_ν $(\nu = 0, \ldots, r-1)$.

In Heil, Strang and Strela [9] and in Plonka [13], the approximation properties of refinable function vectors $\boldsymbol{\phi} := (\phi_\nu)_{\nu=0}^{r-1}$ were studied in some detail. In particular, new necessary and sufficient conditions for the refinement mask of $\boldsymbol{\phi}$ could be derived. In [13], it could even be shown that the function vector $\boldsymbol{\phi}$ can only provide approximation order m if its refinement mask factorizes in a certain manner. For finding these results, [9] as well as [13] strongly used properties of doubly infinite matrices determined by the matrix coefficients occuring in the refinement equation (in time domain).

Now we want to sketch a way, how the necessary and sufficient conditions for the refinement mask of $\boldsymbol{\phi}$ can completely be derived in the Fourier domain.

As in [13], the functions ϕ_ν are allowed to have a noncompact support if they have a suitable decay rate. The main tool of our new approach is the so called superfunction, which is contained in the span of the integer translates of ϕ_ν ($\nu = 0, \ldots, r-1$) and already provides the same approximation order as $\boldsymbol{\phi}$. The results are applied to some multi–scaling functions ϕ_0, ϕ_1 first considered by Donovan, Geronimo, Hardin and Massopust [6, 7].

2 Notations

Let us introduce some notations. Consider the Hilbert space $L^2 = L^2(\mathbb{R})$ of all square integrable functions on $\mathbb{R}$. The Fourier transform of $f \in L^2(\mathbb{R})$ is defined by $\hat{f} := \int_{-\infty}^{\infty} f(x) e^{-ix\cdot}\, \mathrm{d}x$. The function vector $\boldsymbol{\phi}$ with elements in $L^2(\mathbb{R})$ is *refinable*, if $\boldsymbol{\phi}$ satisfies a refinement equation of the form

$$\boldsymbol{\phi} = \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, \boldsymbol{\phi}(2\cdot -l) \qquad (\boldsymbol{P}_l \in \mathbb{R}^{r\times r}),$$

or equivalently, if $\boldsymbol{\phi}$ satisfies the Fourier transformed refinement equation

$$\hat{\boldsymbol{\phi}} = \boldsymbol{P}(\cdot/2)\, \hat{\boldsymbol{\phi}}(\cdot/2) \tag{1}$$

with $\hat{\boldsymbol{\phi}} := (\hat{\phi}_\nu)_{\nu=0}^{r-1}$ and with the *refinement mask* (*two–scale symbol*)

$$\boldsymbol{P} = \boldsymbol{P}_{\boldsymbol{\phi}} := \frac{1}{2} \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, e^{-il\cdot}. \tag{2}$$

Note that $\boldsymbol{P}$ is an $(r\times r)$–matrix of 2π–periodic functions. The components ϕ_ν of a refinable function vector $\boldsymbol{\phi}$ are called *multi–scaling functions*.

Let $BV(\mathbb{R})$ be the set of all functions which are of bounded variation over $\mathbb{R}$ and normalized by

$$\lim_{|x|\to\infty} f(x) = 0, \quad f(x) = \frac{1}{2} \lim_{h\to 0} (f(x+h) + f(x-h)) \quad (-\infty < x < \infty).$$

If $\hat{f} \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then the Poisson summation formula

$$\sum_{l\in\mathbb{Z}} f(l)\, e^{-iul} = \sum_{j\in\mathbb{Z}} \hat{f}(u + 2\pi j)$$

is satisfied (cf. Butzer and Nessel [3]). By $C(\mathbb{R})$, we denote the set of continuous functions on $\mathbb{R}$. For a measurable function f on $\mathbb{R}$ and $m \in \mathbb{N}$ let

$$\begin{aligned} \|f\|_p &:= \left(\int_{-\infty}^{\infty} |f(x)|^p \;\mathrm{d}x\right)^{1/p}, \\ |f|_{m,p} &:= \|\mathrm{D}^m f\|_p, \qquad \|f\|_{m,p} := \sum_{k=0}^{m} \|\mathrm{D}^k f\|_p. \end{aligned}$$

Here and in the following, D denotes the differential operator with respect to x D $:=$ d$/$dx. Let $W_p^m(\mathbb{R})$ be the usual Sobolev space with the norm $\|\cdot\|_{m,p}$. The l^p–norm of a sequence $\boldsymbol{c} := \{c_l\}_{l\in\mathbb{Z}}$ is defined by $\|\boldsymbol{c}\|_{l^p} := (\sum_{l\in\mathbb{Z}} |c_l|^p)^{1/p}$.

For $m \in \mathbb{N}$, let $E_m(\mathbb{R})$ be the space of all functions $f \in C(\mathbb{R})$ with the decay property

$$\sup_{x\in\mathbb{R}}\{|f(x)|\,(1+|x|)^{1+m+\epsilon}\} < \infty \qquad (\epsilon > 0).$$

Let $l^2_{-m} := \{\boldsymbol{c} := (c_k) : \sum_{k=-\infty}^{\infty}(1+|k|^2)^{-m}\,|c_k|^2 < \infty\}$ be a weighted sequence with the corresponding norm

$$\|\boldsymbol{c}\|_{l^2_{-m}} := \left(\sum_{l=-\infty}^{\infty}(1+|l|^2)^{-m}\,|c_l|^2\right)^{1/2}.$$

Considering the functions $\phi_\nu \in E_m(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$, we call the set $\mathcal{B}(\boldsymbol{\phi}) := \{\phi_\nu(\cdot - l) : l \in \mathbb{Z},\, \nu = 0,\ldots,r-1\}$ L^2_{-m}*–stable* if there exist constants $0 < A \leq B < \infty$ with

$$A\sum_{\nu=0}^{r-1}\|\boldsymbol{c}_\nu\|^2_{l^2_{-m}} \leq \|\sum_{\nu=0}^{r-1}\sum_{l\in\mathbb{Z}} c_{\nu,l}\,\phi_\nu(\cdot - l)\|^2_{L^2_{-m}} \leq B\sum_{\nu=0}^{r-1}\|\boldsymbol{c}_\nu\|^2_{l^2_{-m}}$$

for any sequences $\boldsymbol{c}_\nu = \{\boldsymbol{c}_{\nu,l}\}_{l\in\mathbb{Z}} \in l^2_{-m}$ $(\nu = 0,\ldots,r-1)$. Here L^2_{-m} denotes the weighted Hilbert space $L^2_{-m} = \{f : \|f\|_{L^2_{-m}} := \|(1+|\cdot|^2)^{-m/2}\,f\|_2 < \infty\}$. Note that, if the functions ϕ_ν are compactly supported, then the (algebraic) linear independence of the integer translates of ϕ_ν $(\nu = 0,\ldots,r-1)$ yields the L^2_{-m}–stability of $\mathcal{B}(\boldsymbol{\phi})$. For $m = 0$, we obtain the well–known L^2*–stability* (*Riesz stability*).

For $\phi_\nu \in E_m(\mathbb{R})$ $(\nu = 0,\ldots,r-1)$, we say that $\boldsymbol{\phi}$ provides *controlled* L^p*–approximation order* m $(1 \leq p \leq \infty)$, if the following three conditions are satisfied:

For each $f \in W_p^m(\mathbb{R})$ there are sequences $\boldsymbol{c}_\nu^h = \{c_{\nu,l}^h\}_{l\in\mathbb{Z}}$ $(\nu = 0,\ldots,r-1; h > 0)$ such that for a constant c independent of h we have:

1^0
$$\|f - h^{-1/p}\sum_{\nu=0}^{r-1}\sum_{l\in\mathbb{Z}} c_{\nu,l}^h\,\phi_\nu(\cdot/h - l)\|_p \leq c\,h^m\,|f|_{m,p}.$$

2^0 Furthermore,

$$\|\boldsymbol{c}_\nu^h\|_{l^p} \leq c\,\|f\|_p \quad (\nu = 0,\ldots,r-1).$$

3^0 There is a constant δ independent of h such that for $l \in \mathbb{Z}$

$$\operatorname{dist}(lh, \operatorname{supp} f) > \delta \quad \Rightarrow \quad c_{\nu,l}^h = 0 \quad (\nu = 0,\ldots,r-1).$$

This notation of controlled L^p–approximation order, first introduced in Jia and Lei [11], is a generalization of the well–known definition of approximation order for compactly supported functions. In [11], the strong connection of controlled approximation order provided by $\boldsymbol{\phi}$ and the Strang–Fix conditions for $\boldsymbol{\phi}$ was shown. Note that, instead of using the definition of Jia and Lei [11], we also could take the definition of local approximation order by Halton and Light [8]. For our considerations the equivalence to the Strang–Fix conditions is important.

The theory of closed shift–invariant subspaces of $L^2(\mathbb{R})$, spanned by integer translates of a finite set of functions has been extensively studied (cf. e.g. de Boor, DeVore and Ron [1, 2]; Jia [10]). In particular, it has been shown that the approximation order provided by a vector $\boldsymbol{\phi}$ can already be realized by a finite linear combination

$$f = \sum_{\nu=0}^{r-1} \sum_{l \in \mathbb{Z}} a_{\nu l}\, \phi_\nu(\cdot - l). \qquad (a_{\nu l} \in \mathbb{R}).$$

We call f *superfunction* of $\boldsymbol{\phi}$.

3 Approximation by refinable function vectors

In this section we shall give a new approach to necessary and sufficient conditions for the refinement mask of a refinable vector $\boldsymbol{\phi}$ ensuring controlled L^p–approximation order m. In particular, we show, how a superfunction f of $\boldsymbol{\phi}$ (providing the same approximation order as $\boldsymbol{\phi}$) can be constructed by the coefficients which occur in the linear combinations of ϕ_ν reproducing the monomials.

In the following, let $r \in \mathbb{N}$ and $m \in \mathbb{N}$ be fixed. First we want to recall the result in [13] dealing with the connection between controlled L^p–approximation order, reproduction of polynomials and Strang–Fix conditions.

Theorem 1 (cf. [13]) *Let $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be a vector of functions $\phi_\nu \in E_m(\mathbb{R})$, $\hat{\phi}_\nu \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Further, let $\mathcal{B}(\boldsymbol{\phi})$ be L^2_{-m}–stable. Then the following conditions are equivalent:*

(a) *The function vector $\boldsymbol{\phi}$ provides controlled approximation order m ($m \in \mathbb{N}$).*

(b) *Algebraic polynomials of degree $< m$ can be exactly reproduced by integer translates of ϕ_ν, i.e., there are vectors $\boldsymbol{y}_l^n \in \mathbb{R}^r$ ($l \in \mathbb{Z}$; $n = 0, \dots, m-1$) such that the series $\sum_{l \in \mathbb{Z}} (\boldsymbol{y}_l^n)^T \boldsymbol{\phi}(\cdot - l)$ are absolutely and uniformly convergent on any compact interval of $\mathbb{R}$ and*

$$\sum_{l \in \mathbb{Z}} (\boldsymbol{y}_l^n)^T\, \boldsymbol{\phi}(x - l) = x^n \qquad (x \in \mathbb{R};\ n = 0, \dots, m-1).$$

(c) *The function vector $\boldsymbol{\phi}$ satisfies the Strang–Fix conditions of order m, i.e., there is a finitely supported sequence of vectors $\{\boldsymbol{a}_l\}_{l \in \mathbb{Z}}$, such that*

$$f := \sum_{l \in \mathbb{Z}} \boldsymbol{a}_l^T\, \boldsymbol{\phi}(\cdot - l)$$

satisfies

$$\hat{f}(0) \neq 0; \quad \mathrm{D}^n \hat{f}(2\pi l) = 0 \quad (l \in \mathbb{Z} \setminus \{0\}; n = 0, \ldots, m-1).$$

The equivalence of (a) and (c) is already shown in Jia and Lei [11], Theorem 1.1. Further, (b) follows from (c) by [11], Corollary 2.3. For showing that (b) yields (c), in [13] the function

$$f := \sum_{k=0}^{m-1} \boldsymbol{a}_k^{\mathrm{T}} \, \boldsymbol{\phi}(\cdot + k), \tag{3}$$

is introduced. Here, the coefficient vectors $\boldsymbol{a}_k$ are determined by

$$(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{m-1}) := (\boldsymbol{y}_0^0, \ldots, \boldsymbol{y}_0^{m-1}) \, \boldsymbol{V}^{-1}$$

with the Vandermonde matrix $\boldsymbol{V} := (k^n)_{k,n=0}^{m-1}$. Hence we have

$$\boldsymbol{y}_0^n = \sum_{k=0}^{m-1} k^n \, \boldsymbol{a}_k \qquad (n = 0, \ldots, m-1). \tag{4}$$

By Fourier transform of (3) we obtain

$$\hat{f}(u) = \boldsymbol{A}(u)^{\mathrm{T}} \, \hat{\boldsymbol{\phi}}(u)$$

with

$$\boldsymbol{A}(u) := \sum_{k=0}^{m-1} \boldsymbol{a}_k \, e^{iuk}. \tag{5}$$

That means, $\boldsymbol{A}(u)$ is an $(r \times r)$–matrix of trigonometric polynomials. Observe that by (4)

$$(\mathrm{D}^n \boldsymbol{A})(0) = \sum_{k=0}^{m-1} (ik)^n \, \boldsymbol{a}_k = i^n \, \boldsymbol{y}_0^n \quad (n = 0, \ldots, m-1).$$

Using the Poisson summation formula it can be shown that f satisfies the conditions

$$(\mathrm{D}^\mu \hat{f})(2\pi l) = \delta_{0,l} \, \delta_{0,\mu} \quad (l \in \mathbb{Z}; \, \mu = 0, \ldots, m-1)$$

and hence the Strang–Fix conditions of order m (cf. [13]). Observe that f in (3) is a superfunction of $\boldsymbol{\phi}$.

In the new proof for the following theorem, this superfunction will be the main tool.

Theorem 2 *Let $\boldsymbol{\phi} = (\phi_\nu)_{\nu=0}^{r-1}$ be a refinable vector of functions $\phi_\nu \in E_m(\mathbb{R})$, $\hat{\phi}_\nu \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Further, let $\mathcal{B}(\boldsymbol{\phi})$ be L^2_{-m}–stable. Then the function vector $\boldsymbol{\phi}$ provides L^p–controlled approximation order m if and only if the refinement mask $\boldsymbol{P}$ of $\boldsymbol{\phi}$ in (2) satisfies the following*

conditions:
There are vectors $\boldsymbol{y}_0^k \in \mathbb{R}^r$; $\boldsymbol{y}_0^0 \neq \mathbf{0}$ $(k = 0, \ldots, m-1)$ *such that for* $n = 0, \ldots, m-1$ *we have*

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} \, (2i)^{k-n} \, (\mathrm{D}^{n-k} \boldsymbol{P})(0) \;=\; 2^{-n} \, (\boldsymbol{y}_0^n)^{\mathrm{T}}, \tag{6}$$

$$\sum_{k=0}^{n} \binom{n}{k} (\boldsymbol{y}_0^k)^{\mathrm{T}} \, (2i)^{k-n} \, (\mathrm{D}^{n-k} \boldsymbol{P})(\pi) \;=\; \mathbf{0}^{\mathrm{T}}, \tag{7}$$

where $\mathbf{0}$ *denotes the zero vector.*

Proof: Note that the conditions (6)–(7) can also be written in the form

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=0} \;=\; (\mathrm{D}^n \boldsymbol{A})^T(0) \quad (n = 0, \ldots, m-1), \tag{8}$$

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=\pi} \;=\; \mathbf{0}^T \quad (n = 0, \ldots, m-1), \tag{9}$$

where $\boldsymbol{A}$, defined in (5), is the symbol of a superfunction f of $\boldsymbol{\phi}$ in (3). From Theorem 1 we know that $\boldsymbol{\phi}$ provides controlled approximation order m if and only if f satisfies the Strang–Fix conditions of order m. Hence we only have to prove:
The relations (8)–(9) are satisfied if and only if f satisfies the Strang–Fix conditions of order m, i.e.,

$$(\mathrm{D}^n \hat{f})(2\pi l) = c_n \, \delta_{0,l} \quad (n = 0, \ldots, m-1) \tag{10}$$

with constants $c_n \in \mathbb{R}$ and $c_0 \neq 0$.

1. We show that the relations (8)–(9) are satisfied if we have (10).

Note that by (1)

$$\hat{f}(2u) = \boldsymbol{A}^T(2u)\, \hat{\boldsymbol{\phi}}(2u) = \boldsymbol{A}^T(2u)\, \boldsymbol{P}(u) \hat{\boldsymbol{\phi}}(u).$$

Taking the derivatives, it follows on the one hand

$$\begin{aligned}(\mathrm{D}^n \hat{f})(u) &= \mathrm{D}^n[\boldsymbol{A}^T(u)\, \hat{\boldsymbol{\phi}}(u)] \\ &= \sum_{k=0}^{n} \binom{n}{k} (\mathrm{D}^k \boldsymbol{A}^T)(u)\, (\mathrm{D}^{n-k} \hat{\boldsymbol{\phi}})(u)\end{aligned}$$

and on the other hand

$$\begin{aligned}2^n \, (\mathrm{D}^n \hat{f})(2u) &= \mathrm{D}^n[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)\, \hat{\boldsymbol{\phi}}(u)] \\ &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^k[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]\, (\mathrm{D}^{n-k} \hat{\boldsymbol{\phi}})(u).\end{aligned}$$

2. Let us first show that the conditions (9) are satisfied. For all $l \in \mathbb{Z}$ we find by (10) that

$$0 = \hat{f}(4\pi l + 2\pi) = \boldsymbol{A}^T(4\pi l + 2\pi)\, \boldsymbol{P}(2\pi l + \pi)\, \hat{\boldsymbol{\phi}}(2\pi l + \pi) = \boldsymbol{A}^T(0)\, \boldsymbol{P}(\pi)\, \hat{\boldsymbol{\phi}}(2\pi l + \pi).$$

Hence, linear independence of the sequences $\{\hat{\boldsymbol{\phi}}_\nu(\pi + 2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ gives

$$\boldsymbol{A}^T(0)\, \boldsymbol{P}(\pi) = \boldsymbol{0}^T.$$

Note that the linear independence of the sequences $\{\hat{\boldsymbol{\phi}}_\nu(u + 2\pi l)\}_{l\in\mathbb{Z}}$ for all $u \in \mathbb{R}$ and for $\nu = 0, \ldots, r-1$ is satisfied if and only if the integer translates of ϕ_ν form a L^2–stable basis of their closed span (cf. Jia and Micchelli [12]). This was the first step of the induction proof.

Let now $\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=\pi} = \boldsymbol{0}^T$ be satisfied for $\mu = 0, \ldots, n-1$ $(n < m)$, and observe that by assumption (10) $(\mathrm{D}^n \hat{f})(4\pi l + 2\pi) = 0$ for all $l \in \mathbb{Z}$. Then, by the linear independence of $\{\hat{\boldsymbol{\phi}}_\nu(\pi + 2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ and by

$$\begin{aligned} 0 &= 2^n\, (\mathrm{D}^n \hat{f})(4\pi l + 2\pi) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^k[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=\pi}\, (\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(\pi + 2\pi l) \\ &= \mathrm{D}^n[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=\pi}\, \hat{\boldsymbol{\phi}}(\pi + 2\pi l) \end{aligned}$$

it follows that

$$\mathrm{D}^n[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=\pi} = \boldsymbol{0}^T.$$

Thus, the relations (9) are satisfied.

3. Now we show that (10) yields (8). Let $u = 2\pi l$ $(l \in \mathbb{Z})$. Then we have on the one hand by the Strang–Fix conditions

$$\hat{f}(4\pi l) = \boldsymbol{A}^T(0)\, \boldsymbol{P}(0)\, \hat{\boldsymbol{\phi}}(2\pi l) = c_0\, \delta_{0,l}$$

and on the other hand

$$\hat{f}(2\pi l) = \boldsymbol{A}^T(0)\, \hat{\boldsymbol{\phi}}(2\pi l) = c_0\, \delta_{0,l}.$$

By linear independence of $\{\hat{\boldsymbol{\phi}}_\nu(2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ we obtain

$$\boldsymbol{A}^T(0)\, \boldsymbol{P}(0) = \boldsymbol{A}^T(0).$$

Again, we proceed by induction. Let now $\mathrm{D}^\mu[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=0} = (\mathrm{D}^\mu \boldsymbol{A}^T)(0)$ be satisfied for $\mu = 0, \ldots, n-1$ $(n < m)$ and observe that by assumption $(\mathrm{D}^n \hat{f})(2\pi l) = c_n\, \delta_{0,l}$ $(l \in \mathbb{Z},\ c_0 \neq 0)$. Then we find for all $l \in \mathbb{Z}$

$$\begin{aligned} 2^n (\mathrm{D}^n \hat{f})(4\pi l) &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^k[\boldsymbol{A}(2u)^T\, \boldsymbol{P}(u)]|_{u=0}\, (\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (\mathrm{D}^k \boldsymbol{A}^T)(0)\, (\mathrm{D}^{n-k}\hat{\boldsymbol{\phi}})(2\pi l) \\ &\quad + \mathrm{D}^n[\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=0}\, \hat{\boldsymbol{\phi}}(2\pi l) = c_n\, \delta_{0,l}\,. \end{aligned}$$

On the other hand, for $l \in \mathbb{Z}$

$$(\mathrm{D}^n \hat{f})(2\pi l) = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{D}^k \boldsymbol{A}^T)(0)\, (\mathrm{D}^{n-k} \hat{\boldsymbol{\phi}})(2\pi l) = c_n\, \delta_{0,l}.$$

Hence, a comparison yields

$$\mathrm{D}^n [\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=0}\, \hat{\boldsymbol{\phi}}(2\pi l) = (\mathrm{D}^n \boldsymbol{A}^T)(0)\, \hat{\boldsymbol{\phi}}(2\pi l)\,.$$

By linear independence of $\{\hat{\boldsymbol{\phi}}_\nu(2\pi l)\}_{l\in\mathbb{Z}}$ for $\nu = 0, \ldots, r-1$ we obtain

$$\mathrm{D}^n [\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=0} = (\mathrm{D}^n \boldsymbol{A}^T)(0).$$

Now the proof by induction is complete.

4. We are going to prove the reverse direction. Assume that the relations (8)–(9) are satisfied. We show that then the conditions $(\mathrm{D}^n \hat{f})(2\pi l) = c_n\, \delta_{0,l}$ $(n = 0, \ldots, m-1)$ hold, where $c_0 \neq 0$.

For the μ-th derivative of $\hat{f}$ we find

$$\begin{aligned}
2^\mu\, (\mathrm{D}^\mu \hat{f})(4\pi l) &= \sum_{k=0}^{\mu} \binom{\mu}{k} \mathrm{D}^\mu [\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=0}\, (\mathrm{D}^{\mu-k} \hat{\boldsymbol{\phi}})(2\pi l) \\
&= \sum_{k=0}^{\mu} \binom{\mu}{k} (\mathrm{D}^\mu \boldsymbol{A})(0)\, (\mathrm{D}^{\mu-k} \hat{\boldsymbol{\phi}})(2\pi l) \\
&= (\mathrm{D}^\mu \hat{f})(2\pi l)
\end{aligned}$$

and

$$\begin{aligned}
2^\mu\, (\mathrm{D}^\mu \hat{f})(4\pi l + 2\pi) &= \sum_{k=0}^{\mu} \binom{\mu}{k} \mathrm{D}^\mu [\boldsymbol{A}^T(2u)\, \boldsymbol{P}(u)]|_{u=\pi}\, (\mathrm{D}^{\mu-k} \hat{\boldsymbol{\phi}})(2\pi l + \pi) \\
&= 0.
\end{aligned}$$

Thus, we indeed obtain $(\mathrm{D}^n \hat{f})(2\pi l) = c_n\, \delta_{0,l}$. It only remains to show that $c_0 \neq 0$. By Poisson summation formula and using the L^2–stability of $\boldsymbol{\phi}$ we have

$$\hat{\phi}(0) = \boldsymbol{A}^T(0)\, \hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^T\, \hat{\boldsymbol{\phi}}(0) = (\boldsymbol{y}_0^0)^T \sum_{l\in\mathbb{Z}} \boldsymbol{\phi}(\cdot - l) \neq 0.$$

Hence f satisfies the Strang–Fix conditions of order m. ∎

Remark: For proving the second direction in Theorem 2 we do not need any stability condition if we assume that $(\boldsymbol{y}_0^0)^T\, \hat{\boldsymbol{\phi}}(0) \neq 0$. Since $\boldsymbol{y}_0^0$ and $\hat{\boldsymbol{\phi}}(0)$ are a left and a right eigenvector of $\boldsymbol{P}(0)$, respectively, this assumption is satisfied if the eigenvalue 1 of $\boldsymbol{P}(0)$ is simple.

4 The GHM–multi–scaling functions

We consider the example of a vector of two multi–scaling functions $\boldsymbol{\phi} := (\phi_0, , \phi_1)^T$ treated in Donovan, Geronimo, Hardin and Massopust ([6, 7]). In the special case $s = s_0 = s_1$ (with $s \in [-1, 1]$) of their construction, the refinement equation of $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\phi}(x) = \boldsymbol{P}_0\, \boldsymbol{\phi}(2x) + \boldsymbol{P}_1\, \boldsymbol{\phi}(2x-1) + \boldsymbol{P}_2\, \boldsymbol{\phi}(2x-2) + \boldsymbol{P}_3\, \boldsymbol{\phi}(2x-3), \tag{11}$$

where

$$\begin{aligned}
\boldsymbol{P}_0 &:= \begin{pmatrix} -\dfrac{s^2-4s-3}{2\,(s+2)} & 1 \\ -\dfrac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & \dfrac{3s^2+s-1}{2\,(s+2)} \end{pmatrix}, \\
\boldsymbol{P}_1 &:= \begin{pmatrix} -\dfrac{s^2-4s-3}{2\,(s+2)} & 0 \\ -\dfrac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & 1 \end{pmatrix}, \\
\boldsymbol{P}_2 &:= \begin{pmatrix} 0 & 0 \\ -\dfrac{3(s-1)(s+1)(s^2-s+3)}{4\,(s+2)^2} & \dfrac{3s^2+s-1}{2\,(s+2)} \end{pmatrix}, \\
\boldsymbol{P}_3 &:= \begin{pmatrix} 0 & 0 \\ -\dfrac{3(s-1)(s+1)(s^2-3s-1)}{4\,(s+2)^2} & 0 \end{pmatrix}.
\end{aligned}$$

For the refinement mask $\boldsymbol{P}$ we have

$$\boldsymbol{P}(u) := \frac{1}{2}\,(\boldsymbol{P}_0 + \boldsymbol{P}_1\, e^{-iu} + \boldsymbol{P}_2\, e^{-2iu} + \boldsymbol{P}_3\, e^{-3iu}).$$

Applying the result of Theorem 2 we can show that $\boldsymbol{\phi}$ provides the controlled L^p–approximation order $m = 2$:

Observing that

$$\begin{aligned}
\boldsymbol{P}(0) &= \begin{pmatrix} \dfrac{-s^2+4s+3}{2(s+2)} & \dfrac{1}{2} \\ \dfrac{-3(s-1)^3(s+1)}{2(s+2)^2} & \dfrac{3s^2+2s+1}{2(s+2)} \end{pmatrix}, \\
(\mathrm{D}\boldsymbol{P})(0) &= i \begin{pmatrix} \dfrac{s^2-4s-3}{4(s+2)} & 0 \\ \dfrac{9(s-1)^3(s+1)}{4(s+2)^2} & \dfrac{-(3s^2+2s+1)}{2(s+2)} \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{P}(\pi) &= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{3(s^2-1)}{2(s+2)} \end{pmatrix}, \\
(\mathrm{D}\boldsymbol{P})(\pi) &= i \begin{pmatrix} \frac{-s^2+4s+3}{4(s+2)} & 0 \\ \frac{3(s^2-1)(-s^2+4s+3)}{4(s+2)^2} & \frac{-3(s^2-1)}{2(s+2)} \end{pmatrix},
\end{aligned}$$

we find with

$$\boldsymbol{y}_0^0 = \left(\frac{-3(s^2-1)}{s+2}, 1\right), \qquad \boldsymbol{y}_0^1 = \left(\frac{-3(s^2-1)}{2(s+2)}, 1\right)$$

the relations

$$(\boldsymbol{y}_0^0)^T \, \boldsymbol{P}(0) = (\boldsymbol{y}_0^0)^T, \quad (\boldsymbol{y}_0^0)^T \, \boldsymbol{P}(\pi) = \mathbf{0}^T$$

and

$$\begin{aligned}
(2i)^{-1} \, (\boldsymbol{y}_0^0)^T \, (\mathrm{D}\boldsymbol{P})(0) + (\boldsymbol{y}_0^1)^T \, \boldsymbol{P}(0) &= 2^{-1} \, (\boldsymbol{y}_0^1)^T, \\
(2i)^{-1} \, (\boldsymbol{y}_0^0)^T \, (\mathrm{D}\boldsymbol{P})(\pi) + (\boldsymbol{y}_0^1)^T \, \boldsymbol{P}(\pi) &= \mathbf{0}^T.
\end{aligned}$$

Hence, (8)–(9) are satisfied for $m = 2$. Knowing $\boldsymbol{y}_0^0$ and $\boldsymbol{y}_0^1$, we can construct a superfunction f of $\boldsymbol{\phi}$ (as defined in (3)) by

$$f(x) = (\boldsymbol{y}_0^0 - \boldsymbol{y}_0^1)^T \, \boldsymbol{\phi}(x) + (\boldsymbol{y}_0^1)^T \, \boldsymbol{\phi}(x+1)$$

obtaining

$$f(x) = \frac{3(1-s^2)}{2(s+2)}(\phi_0(x) + \phi_0(x+1)) + \phi_1(x+1).$$

Application of the refinement equation (11) on the right hand side yields

$$\begin{aligned}
f(x) &= \frac{9(1-s^2)}{4(s+2)}(\phi_0(2x) + \phi_0(2x+1)) + \frac{3(1-s^2)}{4(s+2)}(\phi_0(2x-1) + \phi_0(2x+2)) \\
&\quad + \frac{1}{2}(\phi_1(2x+2) + \phi_1(2x)) + \phi_1(2x+1) \\
&= \frac{1}{2}f(2x-1) + f(2x) + \frac{1}{2}f(2x+1).
\end{aligned}$$

That means, f itself satisfies the refinement equation of the hat–function $h(x) := \max\{(1 - |x|), 0\}$. Hence, taking a proper normalization constant, the superfunction f coincides with the hat function h. Indeed, in [6] the approximation order 2 provided by $\boldsymbol{\phi}$ was derived by showing that the hat–function h lies in the span of the integer translates of ϕ_0, ϕ_1.

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