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# A theorem on summable families in normed groups

*Dedicated to the professors of mathematics*
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ABSTRACT. In functional analysis the notion of a summable family (with sum $x$) is well-known. If $(x_i)_{i\in I}$ is a family of points from a normed space, $(x_i)$ is called summable to $x$ iff for each $\varepsilon > 0$ there exits a finite set $F_0 \subset I$ such that $\|x - \sum_{i\in F} x_i\| < \varepsilon$ for each finite set $F$, $F_0 \subset F \subset I$. But we can interprete this definition as the convergence of a suitable net. In a normed commutative group we characterize the fact that this net is a Cauchy net.

KEY WORDS. Commutative topological group, summable family, special nets, normed (commutative) group, Cauchy nets

## 1 Introduction

We consider a normed space $(X, \|\cdot\|)$ and let $(x_i)_{i\in I}$ be a family of points from $X, x \in X$; the notion that " $(x_i)_{i\in I}$ is a summable family with the sum $x$ " is well–known; one defines:

**1.1** For each $\varepsilon > 0$ there exists a finite set $J_0 \subset I$ such that $\|x - \sum_{i\in J} x_i\| < \varepsilon$ for each finite $J \subset I$, $J_0 \subset J$. See for instance [2], [3], [5], [7], [8].

One can prove a "Cauchy criterion", and probably at first in Hilbert spaces it has been found that for a summable family $(x_i)_{i\in I}$ (with sum $x$) for all but countably many $i \in I$ we have $x_i = 0$. But one can observe that definition 1.1 means that the following net (Moore–Smith–sequence) converges to $x$ with respect to the usual notion of convergence of nets in a topological space: let $\underline{F} = \underline{F}(I)$ be the collection of all finite nonempty subsets of $I$ and let be $x_F := \sum_{i\in F} x_i$ where $F \in \underline{F}$. If $\underline{F}$ is ordered by inclusion then $\underline{F} = (\underline{F}, \subseteq)$ is a poset which is directed: for $F', F'' \in \underline{F}$, $F' \cup F'' \in \underline{F}$ and $F' \cup F''$ is an upper bound for $F'$, $F''$. Hence $\underline{F}$ can serve as an index set for a net and the desired net is $(x_F)_{F\in\underline{F}}$. And of course "$x_F \to x$ in $(X, \|\cdot\|)$" is equivalent to definition 1.1. Clearly, the limit point $x$ is unique since $(X, \|\cdot\|)$

is a Hausdorff topological space. In [2] and [8] this net is mentioned but from this net the corresponding filter is constructed and then this filter is used. But in my opinion it is more intiutive to use directly the net $(x_F)_{F\in\underline{F}}$.

Our aim in this short note is first to emphasize that we should distinguish between the notion that the net $(x_F)_{F\in\underline{F}}$ converges and the notion that this net is a Cauchy net and second to establish the precise relationship between the convergence of the net $(x_F)_{F\in\underline{F}}$, the situation when $(x_F)$ is a Cauchy net and the fact that most members of the family $(x_i)_{i\in I}$ will vanish. The answer is given by the theorem (in section 3) and its corollary. In a second corollary we consider a short application to orthonormal families in an inner product space. Of course concerning the concrete subject of the paper both the assertions and the proofs are in some sense elementary.

## 2 Summable families of points in a commutative topological group

Let us start with a set $X$ and a family $(x_i)_{i\in I}$ of points of $X$; we consider the index set $\underline{F} = \underline{F}(I)$ as defined in section 1. If $(X, +)$ is a group we can define $x_F = \sum\limits_{i\in F} x_i$ for each $F$; if $(X, +)$ is commutative then $x_F$ is uniquely defined and $(x_F)_{F\in\underline{F}}$ is a net in $(X, +)$.

Now let $(X, +)$ be a commutative topological group and we denote by $\underline{N}(0)$ the neighbourhood filter of the zeroelement $0 \in X$. Then $\underline{V} := \{V_U |\ U \in \underline{N}(0)\}$, where $V_U = \{(x, y) \in X \times X |\ x - y \in U\}$, is a base of a natural (diagonal) uniformity for $X$. Hence we can use Cauchy nets in $X$. Let us briefly recall this notion: If $(X, \underline{V})$ is an arbitrary uniform space with diagonal structure $\underline{V}$ then a net $(x_i)_{i\in I}$ from $X$ is called Cauchy net iff for each $V \in \underline{V}$ there exists $i_V \in I$ such that $i_1, i_2 \geq i_V$ implies $(x_{i_1}, x_{i_2}) \in V$. But since for $\underline{V}$ the triangle inequality holds we also can define a Cauchy net by: For $V \in \underline{V}$ we find $i_V \in I$: for each $i \geq i_V$ holds: $(x_i, x_{i_V}) \in V$. Cauchy nets are used for instance in [4], [6]. For the proof of the theorem which we want to state in section 3 we need some lemmas. For the statements of the lemmas we assume that $(X, +)$ is a commutative topological group, $(x_i)_{i\in I}$ is a family from $X$ and $\underline{F}$ is defined as above. The main technical argument in the proofs of the lemmas is the simple fact: if $F_1,\ F_2 \subset I$ and $F_1 \cap F_2 = \emptyset$ then $x_{F_1\cup F_2} = x_{F_1} + x_{F_2}$. For this reason we prove only the first lemma (as an example).

**Lemma 1** *Equivalent are:*

(1) $(x_F)_{F\in\underline{F}}$ *is a Cauchy net.*

(2) *For each* $V \in \underline{V}$ *there exists* $F_0 \in \underline{F}$ *such that for all* $F \in \underline{F}$, $F \cap F_0 = \emptyset$ *implies* $(x_F, 0) \in V$.

**Proof:** (1) $\Longrightarrow$ (2): Let $V = V_U$, $U \in \underline{N}(0)$, be a basic element from $\underline{V}$; then by (1) we find $F_0 \in \underline{F}$ such that $F \in \underline{F}$ and $F_0 \subset F$ imply $(x_F, x_{F_0}) \in V$ and hence $x_F - x_{F_0} \in U$; now let $H \in \underline{F}$ and $H \cap F_0 = \emptyset$; then since $F_0 \subset F_0 \cup H$ we get: $x_{F_0 \cup H} - x_{F_0} = (x_{F_0} + x_H) - x_{F_0} = x_H \in U$ and thus $(x_H, 0) \in V$.

(2) $\Longrightarrow$ (1): Let $V = V_U \in \underline{V}$; by (2) we find $F_0 \in \underline{F}$ such that $(x_F, 0) \in V$ for each $F \in \underline{F}$, $F \cap F_0 = \emptyset$; now we consider an arbitrary $H \in \underline{F}$ such that $F_0 \subset H$; we can assume $F_0 \neq H$; $F_0 \cap (H \backslash F_0) = \emptyset$ implies $(x_{H\backslash F_0}, 0) \in V$ and hence $x_{H\backslash F_0} - 0 \in U$; now $x_H - x_{F_0} = x_{H\backslash F_0} + x_{F_0} - x_{F_0} = x_{H\backslash F_0} \in U$ and hence $(x_H, x_{F_0}) \in V$ showing that $(x_F)_{F\in\underline{F}}$ is a Cauchy net.

**Remarks:**

1. This result is also proved in [2], [8], where the authors used the (Cauchy) filter corresponding to $(x_F)$.
2. Since $i \notin F_0$ means $\{i\} \cap F_0 = \emptyset$ from 1 follows that if $(x_F)$ is a Cauchy net then for each neighbourhood $U$ of 0 there exists $F \in \underline{F}$ such that $x_i \in U$ for each $i \in I \backslash F$.
3. In [1] families which fulfill assertion (2) of lemma 1 are called summable, that means, $(x_i)_{i\in I}$ is summable by the definition of the author iff the net $(x_F)_{F\in\underline{F}}$ is a Cauchy net.
4. We call $(x_i)$ a Cauchy family if $(x_F)$ is a Cauchy net.

**Lemma 2** *Let $A$ be a nonempty subset of $I$ such that $A \neq I$ and $x_i = 0$ for each $i \in I\backslash A$; let $\underline{A} = \{B \subset A \mid B \text{ finite } \}$, hence $\underline{A} \subset \underline{F}$. If $(x_F)_{F\in\underline{A}}$ is a Cauchy net then $(x_F)_{F\in\underline{F}}$ is a Cauchy net, too.*

**Lemma 3** *Let $I$ be infinite; let $A \subset I$ such that $A \neq \emptyset$, $A \neq I$ and $x_i = 0$ for each $i \in I\backslash A$; let $\underline{A}$ be defined as in lemma 2. If $(x_F)_{F\in\underline{F}}$ converges in $X$, $x_F \to x$, then $(x_F)_{F\in\underline{A}} \to x$.*

## 3 Characterizing the fact that $(x_F)_{F\in\underline{F}}$ is a Cauchy net

Here we want to consider normed groups. We define (see [8]):

**3.1** Let $(X, +)$ be a group; we call $\|\cdot\| : X \to \mathbb{R}$ a norm for $(X, +)$ iff the axioms hold:

1. $\|x\| \geq 0$ for each $x \in X$,
2. $\|x\| = 0$ iff $x = 0$,

3. $\| - x\| = \|x\|$,

4. $\|x + y\| \leq \|x\| + \|y\|$.

Then as is it easy to see, $d : d(x, y) = \|x - y\|$ defines a metric for $X$.

**Lemma 4** $(X, +, \| \cdot \|) = (X, +, d)$ *is a topological group.*

**Proof:** Each translation $x \to x + a$ is continuous and hence the topological space $X$ is homogenous; now let be $f : X \to X$, $f(x) = -x$; then if $(x_i)$ is a net from $X$ and $x_i \to 0$ we get: $\|f(x_i) - f(0)\| = \| - x_i - 0\| = \| - x_i\| = \|x_i\| = \|x_i - 0\|$ showing that $f$ is continuous; since convergence in $X \times X$ means coordinatewise convergence and by axiom 4. we see that the map $(x, y) \to x + y$ is continuous, too.

Now we want to state the main result of our paper.

**Theorem 5** *Let $(X, +, \| \cdot \|)$ be a commutative normed group; let $(x_i)_{i \in I}$ be a family of elements of $X$ and let $I$ be infinite. We consider the net $(x_F)_{F \in \underline{F}}$, where $\underline{F} = \underline{F}(I)$. Then the following statements are equivalent:*

(1) *$(x_F)$ is a Cauchy net.*

(2) (a) *There exists a countable set $I^* \subset I$ such that $x_i = 0$ for each $i \in I \backslash I^*$.*
(b) *$(x_F)_{F \in \underline{I}^*}$ is a Cauchy net.*

**Proof:** By lemma 4 we see that we can use the lemmas of section 2 for the proof.
(1) $\Longrightarrow$ (2): By lemma 1 we get: for each $n \in \mathbb{N}$, $n \geq 1$ there exists $F_n \in \underline{F}$ such that $d(x_F, 0) = \|x_F\| < \dfrac{1}{n}$ for each $F \in \underline{F}$, $F \cap F_n = \emptyset$; hence: $i \in T \backslash \bigcup\limits_{n=1}^{\infty} F_n$ implies $\{i\} \cap F_n = \emptyset$ for each $n$ and thus $\|x_i\| = \|x_{\{i\}}\| < \dfrac{1}{n}$ for each $n$ showing that $x_i = 0$ holds. $I^* = \bigcup\limits_{n=1}^{\infty} F_n$ is countable and so (a) holds. To prove also (b) let $\varepsilon > 0$ be given. We assume that $I^* \neq I$, otherwise there is nothing to prove. Since $(x_F)$ is a Cauchy net we find $F_\varepsilon \in \underline{F}$: $\|x_F - x_{F_\varepsilon}\| < \varepsilon$ for each $F \in \underline{F}$, $F_\varepsilon \subset F$; now $F_\varepsilon \cap I^* = \emptyset$ or $F_\varepsilon \cap I^* \neq \emptyset$; if $F_\varepsilon \cap I^* = \emptyset$ let $F_1 \in \underline{I}^*$ be fixed and $H \in \underline{I}^*$, $F_1 \subset H$; then $F_\varepsilon \subset F_1 \cup F_\varepsilon \subset H \cup F_\varepsilon$ and hence: $\|x_H - x_{F_1}\| = \|x_{H \cup F_\varepsilon} - x_{F_1 \cup F_\varepsilon}\| = \|x_{H \cup F_\varepsilon} - x_{F_\varepsilon} + x_{F_\varepsilon} - x_{F_1 \cup F_\varepsilon}\| < 2\varepsilon$; if $F_\varepsilon \cap I^* \neq \emptyset$ let be $H \in \underline{I}^*$ and $F_\varepsilon \cap I^* \subset H$; then $F_\varepsilon \subset H \cup F_\varepsilon \backslash I^*$ implies: $\|x_H - x_{F_\varepsilon \cap I^*}\| = \|x_{H \cup (F_\varepsilon \backslash I^*)} - x_{(F_\varepsilon \cap I^*) \cup (F_\varepsilon \backslash I^*)}\| = \|x_{H \cup (F_\varepsilon \backslash I^*)} - F_\varepsilon\| < \varepsilon$.
(2) $\Longrightarrow$ (1): Setting $A = I^*$ in lemma 2 we get the proof.

**Corollary 6** *Under the assumptions of the theorem it holds: If the net $(x_F)_{F \in \underline{F}}$ converges in $X$, $x_F \to x$, then we get:*

(a) *There exists a countable set* $I^* \subset I$ *such that* $x_i = 0$ *for each* $i \in I \backslash I^*$.

(b) $(x_F)_{F \in \underline{I^*}} \to x$

(c) *For each enumeration of* $I^*$, $I^* = \{i_0, i_1, \ldots\}$ *holds that the sequence* $\left(\sum_{j=0}^{k} x_{i_j}\right)_{k \in \mathbb{N}} = (x_{\{i_0, i_1, \ldots, i_k\}})_{k \in \mathbb{N}}$ *converges to* $x$.

**Proof:** $x_F \to x$ implies that $(x_F)$ is a Cauchy net and hence we get (a) by theorem 1; then we get assertion (b) by lemma 3 if we substitute $A = I^*$ in the lemma. If $I^* = \{i_0, i_1, \ldots\}$ and $F_k = \{i_0, \ldots, i_k\}$ for each $k \in \mathbb{N}$ then $(\{F_k\}, \subseteq)$ clearly is a cofinal subset of $\underline{I^*}$ and hence we get (c).

**Remark:** Assertion (a) of corollary 6 is proved in [2], [8] and is stated (without proof) in [3].

**Corollary 7** *Let* $(X, (\cdot, \cdot))$ *be an inner product space and* $(x_i)_{i \in I}$ *an arbitrary orthonormal family in* $X$*; let* $x$ *be an arbitrary point in* $X$*; then there exists a countable set* $I^* \subset I$ *such that* $(x, x_i) = 0$ *for each* $i \in I \backslash I^*$.

**Proof:** In [7] it is shown that $((x, x_i)x_i)_{i \in I}$ is a Cauchy family; since $(X, +, \|\cdot\|)$ is a commutative normed group by theorem 1 we find a countable $I^* \subset I$ such that $(x, x_i)x_i = 0$ for each $i \in I \backslash I^*$; but $x_i$ being a member of an orthonormal family yields $x_i \neq 0$ and hence $(x, x_i) = 0$.

**Remark:** This result one finds in [5] (with another proof).

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