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On the Gram Matrix of Translates of De la Vallée Poussin Kernels

Dedicated to the professors of mathematics
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ABSTRACT. Extending a result of A. A. Privalov [2] we prove the diagonal dominance of the Gram matrix of bases of translates of de la Vallée poussin kernels. This can be used to show the uniform boundedness of a corresponding orthogonal projection operator.

KEY WORDS. De la Vallée Poussin Kernels, Gram Matrix, Orthogonal Projection

1 Introduction

In this note we investigate arithmetic means of Dirichlet kernels, namely the de la Vallée Poussin kernels for $N, M \in \mathbb{N}$ and $1 \leq M \leq N$, defined by

$$\begin{aligned}\varphi_N^M(x) \;&:=\; \frac{1}{\sqrt{2N}} + \sqrt{\frac{2}{N}} \sum_{\ell=1}^{N-M} \cos \ell x + \sum_{\ell=N-M+1}^{N+M-1} \frac{N+M-\ell}{M\sqrt{2N}} \cos \ell x \\ &=\; \begin{cases} \dfrac{\sin Nx \sin Mx}{2M\sqrt{2N}\sin^2 \frac{x}{2}}, & \text{for} \quad x \notin 2\pi\mathbb{Z}, \\ \sqrt{2N}, & \text{for} \quad x \in 2\pi\mathbb{Z}. \end{cases}\end{aligned}$$

For $s = 0, \ldots, 2N-1$, the translates

$$\varphi_{N,s}^M(x) \;:=\; \varphi_N^M\left(x - \frac{s\pi}{N}\right)$$

satisfy the interpolatory condition $\varphi_{N,s}^M\left(\frac{k\pi}{N}\right) = \sqrt{2N}\,\delta_{k,s}$, for $k = 0, \ldots, 2N-1$. Hence, the space of translates

$$V_N^M \;:=\; \operatorname{span}\left\{\varphi_{N,s}^M : s = 0, \ldots, 2N-1\right\}$$

has the dimension $\dim V_N^M = 2N$ which is independent of M. For a further discussion of these functions and spaces and for corresponding multiresolution analyses and wavelet spaces see e.g. A. A. Privalov [2] and the authors [1].

2 The Gram matrix

Since the basis of translates $\{\varphi_{N,s}^M\}_{s=0}^{2N-1}$ is not an orthogonal one, we are interested in its Gram matrix

$$\boldsymbol{G}_N^M := \left(\left\langle \varphi_{N,r}^M , \varphi_{N,s}^M \right\rangle\right)_{r,s=0}^{2N-1},$$

with the inner product

$$\langle f , g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x)\, g(x)\, dx .$$

As discussed in [1] the circulant matrix $\boldsymbol{G}_N^M$ can be diagonalized by means of the $2N$-th Fourier matrix; i.e., $\boldsymbol{G}_N^M = \overline{\boldsymbol{F}}_{2N}\, \boldsymbol{D}_N^M\, \boldsymbol{F}_{2N}$, where

$$\boldsymbol{F}_{2N} := \frac{1}{\sqrt{2N}} \left(e^{-\frac{2\pi i r s}{2N}}\right)_{r,s=0}^{2N-1}, \qquad \overline{\boldsymbol{F}}_{2N} = \left(\boldsymbol{F}_{2N}\right)^{-1} . \tag{1}$$

The diagonal matrix $\boldsymbol{D}_N^M = \operatorname{diag}\left(d_{N,r}^M\right)_{r=0}^{2N-1}$ contains the eigenvalues

$$d_{N,r}^M = \begin{cases} \dfrac{1}{2} + \left(\dfrac{N-r}{2M^2}\right)^2, & \text{if} \quad |N-r| < M, \\ 1, & \text{otherwise.} \end{cases}$$

From this representation one rewrites immediately

$$\langle \varphi_{N,r}^M , \varphi_{N,s}^M \rangle = \delta_{s,r} - \frac{(-1)^{s-r}}{4NM^2} \sum_{k=-M+1}^{M-1} (M^2 - k^2) \cos \frac{k(s-r)\pi}{N} .$$

A. A. Privalov [2] computed the entries of $\boldsymbol{G}_N^M$ in a closed form, namely

$$\left\langle \varphi_{N,r}^M , \varphi_{N,r}^M \right\rangle = 1 - \frac{M}{3N} + \frac{1}{12NM} \tag{2}$$

and, for $r \neq s$,

$$\left\langle \varphi_{N,r}^M , \varphi_{N,s}^M \right\rangle = (-1)^{r-s} \frac{2M \cos \frac{M(r-s)\pi}{N} \sin \frac{(r-s)\pi}{2N} - \sin \frac{M(r-s)\pi}{N} \cos \frac{(r-s)\pi}{2N}}{8NM^2 \sin^3 \frac{(r-s)\pi}{2N}} . \tag{3}$$

In order to investigate the quantity of the main diagonal dominance, we will estimate the difference

$$\rho(N,M) \;:=\; \langle \varphi^M_{N,r}\,, \varphi^M_{N,r}\rangle - \sum_{\substack{s=0\\ s\neq r}}^{2N-1} \left|\langle \varphi^M_{N,r}\,, \varphi^M_{N,s}\rangle\right|\,,$$

which is independent of r due to the circulant structure of the Gram matrix.

Our main objective of this paper is to prove the following inequality.

Theorem 1 *For $N, M \in \mathbb{N}$, with $4M|N$, we have*

$$\rho(N,M) \;>\; 0.28\,.$$

For special choices of the indices N, M the value of ρ can be exactly determined.

Remark: In case of the modified Dirichlet kernel ($M = 1$), we have

$$\langle \varphi^1_{N,r}\,, \varphi^1_{N,s}\rangle \;=\; \delta_{s,r} - \frac{(-1)^{s-r}}{4N}\,,$$

which gives $\rho(N,1) = \frac{1}{2}$, for every N. For the Fejer kernel ($N = M$), from

$$\begin{aligned} \langle \varphi^N_{N,r}\,, \varphi^N_{N,r}\rangle &= \frac{2}{3} + \frac{1}{12N^2}\,, \\ \langle \varphi^N_{N,r}\,, \varphi^N_{N,s}\rangle &= \frac{1}{4N^2 \sin^2 \frac{(r-s)\pi}{2N}} \qquad \text{for } r \neq s\,, \end{aligned}$$

and from (15) one obtains $\rho(N,N) = \dfrac{1}{3} + \dfrac{1}{6N^2}$.

Note that from numerical computations one can suggest that

$$\frac{1}{3} \;<\; \rho(N,M) \;\leq\; \frac{1}{2}$$

for all $N, M \in \mathbb{N}$, and that for fixed N the function $\rho(N,M)$ is monotone decreasing with respect to M, for $1 \leq M \leq N$. In [2] A. A. Privalov proved

$$\rho(3M, M) \;>\; \frac{1}{3}\,.$$

Theorem 1 is an extension of his result to the case $4M|N$.

The main diagonal dominance of the Gram matrix shows the almost orthogonality of the basis $\{\varphi^M_{N,s}\}_{s=0}^{2N-1}$. We need Theorem 1 in order to estimate the boundedness of the orthogonal projection operator $P^M_N : C_{2\pi} \to V^M_N$. This is in particular interesting for $N = 2^j$ and $M = 2^{j-\lambda}$ for large $j \in \mathbb{N}$ and fixed $\lambda \in \mathbb{N}$.

Corollary 2 *The orthogonal projection P_N^M onto V_N^M, with $N/M = 2^\lambda \geq 4$, satisfies*

$$\|P_N^M\|_{C_{2\pi}\to C_{2\pi}} \;<\; (3\lambda + 10)^2 \,.$$

Proof of the Corollary: Writing

$$P_N^M f \;=\; \sum_{k=0}^{2N-1} \epsilon_k \, \varphi_{N,k}^M \,, \tag{4}$$

we obtain by Hölder's inequality and by a well-known inequality for trigonometric polynomials (see A. F. Timan [3], Chap. 4.9)

$$\|P_N^M f\|_\infty \;\leq\; 2N(1+\frac{5\pi}{4})\,\|\varphi_{N,0}^M\|_1 \max_{0\leq k<2N} |\epsilon_k| \;=:\; 2N(1+\frac{5\pi}{4})\,\|\varphi_{N,0}^M\|_1\, |\epsilon_\ell| \,.$$

Taking the inner product with $\varphi_{N,\ell}^M$ in (4), we estimate

$$\begin{aligned} |\langle P_N^M f\,,\, \varphi_{N,\ell}^M\rangle| \;&=\; \left|\sum_{k=0}^{2N-1} \epsilon_k \langle \varphi_{N,k}^M\,,\, \varphi_{N,\ell}^M\rangle\right| \\ &\geq\; |\epsilon_\ell| \left(\langle \varphi_{N,\ell}^M\,,\, \varphi_{N,\ell}^M\rangle - \sum_{\substack{k=0\\ k\neq\ell}}^{2N-1} |\langle \varphi_{N,k}^M\,,\, \varphi_{N,\ell}^M\rangle|\right) . \end{aligned}$$

Applying Theorem 1 yields

$$|\epsilon_\ell| \;<\; 3.6\,|\langle P_N^M f\,,\, \varphi_{N,\ell}^M\rangle| \;=\; 3.6\,|\langle f\,,\, \varphi_{N,\ell}^M\rangle| \;\leq\; 3.6\,\|f\|_\infty\, \|\varphi_{N,\ell}^M\|_1 \,.$$

Hence,

$$\|P_N^M f\|_\infty \;<\; 7.2\, N(1+\frac{5\pi}{4})\,\|\varphi_{N,0}^M\|_1^2\, \|f\|_\infty \,.$$

Using known results on the Lebesgue constants with respect to the de la Vallée Poussin kernel (see e.g. [1]) the Corollary is proved. ■

3 Proof of the Theorem

The proof is based on three Lemmata concerning the function

$$f_m(x) \;:=\; \frac{2m\cos 2mx \sin x - \sin 2mx \cos x}{\sin^3 x} \tag{5}$$

for any $m \in \mathbb{N}$, $m \geq 2$. Since $\cos k(\pi - x) = (-1)^k \cos kx$ and $\sin k(\pi - x) = (-1)^{k-1} \sin kx$ for all $k \in \mathbb{N}$, we have

$$f_m(\pi - x) \;=\; f_m(x) \,, \tag{6}$$

i.e., f_m is symmetric in $[0, \pi]$ with respect to $\frac{\pi}{2}$.
Using the rule of l'Hôpital, we compute

$$f_m(0) \quad = \quad \frac{2m(1-4m^2)}{3}\,. \tag{7}$$

The first Lemma describes the behaviour of f_m in the interval $\left[0, \frac{\pi}{2m}\right]$.

Lemma 3 *For all $m \in \mathbb{N}$, $m \geq 2$, the function f_m is negative and monotone increasing in $\left[0, \frac{\pi}{2m}\right]$.*

Proof: In order to prove the monotonicity of f_m we show that for all $0 < x < \frac{\pi}{2m}$

$$f'_m(x) \quad > \quad 0\,.$$

We have

$$f'_m(x) \quad = \quad \frac{g_m(x)}{\sin^4 x}\,,$$

with

$$g_m(x) \quad = \quad 3\sin 2mx\cos^2 x - 6m\cos 2mx\cos x\sin x - (4m^2-1)\sin 2mx\sin^2 x\,,$$

which has the Taylor expansion

$$g_m(x) \quad = \quad \frac{8m(4m^4-5m^2+1)}{15}\,x^5 + \frac{g_m^{(7)}(\theta)}{5040}\,x^7$$

for a certain $\theta \in \left(0, \frac{\pi}{2m}\right)$. We have to verify that $g_m(x) > 0$ in $\left(0, \frac{\pi}{2m}\right)$. If $g_m^{(7)}(\theta) \geq 0$, then it is evident. If $g_m^{(7)}(\theta) < 0$, then with

$$\begin{aligned} g_m^{(7)}(\theta) \quad = \quad & (2m^2-2)\big\{(11(2m)^6 + 100(2m)^4 - 48(2m)^2 - 64)\sin 2m\theta\sin 2\theta \\ & +(-48(2m)^5 - 80(2m)^3 + 128(2m))\cos 2m\theta\cos 2\theta + 2(2m)^7\cos 2m\theta\sin^2\theta\big\}, \end{aligned}$$

we estimate

$$-g_m^{(7)}(\theta) \quad < \quad \begin{cases} (2m^2-2)(2m)^5(48+5)\,, & \text{for } 0 < \theta \leq \frac{\pi}{4m}\,, \\ (2m^2-2)(2m)^5 2\pi^2\,, & \text{for } \frac{\pi}{4m} < \theta < \frac{\pi}{2m}\,. \end{cases}$$

Hence,

$$\begin{aligned} g_m(x) \quad > \quad & \frac{8m(4m^4-5m^2+1)}{15}x^5 - \frac{53(2m)^5(2m^2-2)}{5040}x^7 \\ = \quad & \frac{m(m^2-1)x^5}{1260}\left(672(4m^2-1) - 53(2m)^4x^2\right) \\ \geq \quad & \frac{m(m^2-1)x^5}{1260}\left(672(4m^2-1) - 53\pi^2(2m)^2\right) \quad > \quad 0, \end{aligned}$$

which proves that f_m is monotone increasing in the interval $\left[0, \frac{\pi}{2m}\right]$.

Since $f_m(0) < 0$ (see (7)) and

$$f_m\left(\frac{\pi}{2m}\right) \;=\; -\frac{2m}{\sin^2 \frac{\pi}{2m}} \;<\; 0\,,$$

the monotonicity of f_m assures further that f_m is strictly negative in the interval $\left[0, \frac{\pi}{2m}\right]$. ■

In the second Lemma, we analyze the sign behaviour of f_m in the interval $\left[\frac{\pi}{2m}, \pi - \frac{\pi}{2m}\right]$.

Lemma 4 *Let $m \in \mathbb{N}$, with $m \geq 2$. Then for $\ell = 1, \ldots, m-1$, we have*

$$\begin{aligned}
\left|f_m\left(\tfrac{\ell\pi}{2m}+x\right)\right| &= (-1)^\ell\ f_m\left(\tfrac{\ell\pi}{2m}+x\right) && \textit{for } 0 \leq x \leq \tfrac{\pi}{8m}\,, && (8)\\
\left|f_m\left(\tfrac{\ell\pi}{2m}+x\right)\right| &\leq (-1)^\ell\ f_m\left(\tfrac{\ell\pi}{2m}+x\right) + \tfrac{4m}{\pi\sin^2\left(\frac{\ell\pi}{2m}+x\right)} && \textit{for } \tfrac{\pi}{8m} < x \leq \tfrac{\pi}{4m}\,, && (9)\\
\left|f_m\left(\tfrac{\ell\pi}{2m}+x\right)\right| &= (-1)^{\ell+1}\ f_m\left(\tfrac{\ell\pi}{2m}+x\right) && \textit{for } \tfrac{\pi}{4m} \leq x \leq \tfrac{\pi}{2m}\,, && (10)
\end{aligned}$$

and for $\ell = m, \ldots, 2m-2$, we have

$$\begin{aligned}
\left|f_m\left(\tfrac{\ell\pi}{2m}+x\right)\right| &= (-1)^\ell\ f_m\left(\tfrac{\ell\pi}{2m}+x\right) && \textit{for } 0 \leq x \leq \tfrac{\pi}{4m}\,, && (11)\\
\left|f_m\left(\tfrac{\ell\pi}{2m}+x\right)\right| &\leq (-1)^{\ell+1}\ f_m\left(\tfrac{\ell\pi}{2m}+x\right) + \tfrac{4m}{\pi\sin^2\left(\frac{\ell\pi}{2m}+x\right)} && \textit{for } \tfrac{\pi}{4m} < x < \tfrac{3\pi}{8m}\,, && (12)\\
\left|f_m\left(\tfrac{\ell\pi}{2m}+x\right)\right| &= (-1)^{\ell+1}\ f_m\left(\tfrac{\ell\pi}{2m}+x\right) && \textit{for } \tfrac{3\pi}{8m} \leq x \leq \tfrac{\pi}{2m}\,. && (13)
\end{aligned}$$

Proof: By (5) we can write

$$f_m\left(\frac{\ell\pi}{2m}+x\right) \;=\; \frac{(-1)^\ell\,(a-b)}{\sin^3\left(\frac{\ell\pi}{2m}+x\right)}\,,$$

with

$$a = 2m\ \cos 2mx\ \sin\left(\tfrac{\ell\pi}{2m}+x\right), \quad b = \sin 2mx\ \cos\left(\tfrac{\ell\pi}{2m}+x\right).$$

For all $\ell = 1, \ldots, m-1$ and $0 \leq x \leq \frac{\pi}{2m}$, we have

$$\frac{\pi}{2m} \;\leq\; \frac{\ell\pi}{2m}+x \;\leq\; \frac{\pi}{2}\,.$$

We use that $\sin x \geq \frac{2x}{\pi}$, and consider three intervals for x.

(i) For $0 \leq x \leq \frac{\pi}{8m}$, we estimate

$$\begin{aligned}
a-b &= 2m\ \cos 2mx\ \sin\left(\frac{\ell\pi}{2m}+x\right) - \sin 2mx\ \cos\left(\frac{\ell\pi}{2m}+x\right)\\
&\geq 2m\ \sin 2mx\ \sin\frac{\pi}{2m} - \sin 2mx\ \cos\left(\frac{\ell\pi}{2m}+x\right)\\
&\geq \sin 2mx\ \left(2-\cos\left(\frac{\ell\pi}{2m}+x\right)\right) \;>\; 0\,,
\end{aligned}$$

which gives (8).
(ii) For $\frac{\pi}{8m} < x \le \frac{\pi}{4m}$, we have that $a, b > 0$ and so we get

$$\begin{aligned}\left| f_m\left(\frac{\ell\pi}{2m} + x\right)\right| &= \frac{|a-b|}{\sin^3\left(\frac{\ell\pi}{2m}+x\right)} \le \frac{a-b+2b}{\sin^3\left(\frac{\ell\pi}{2m}+x\right)} \\ &\le (-1)^\ell\, f_m\left(\frac{\ell\pi}{2m}+x\right) + \frac{2\cos\left(\frac{\ell\pi}{2m}+x\right)}{\sin^3\left(\frac{\ell\pi}{2m}+x\right)}.\end{aligned}$$

Further we estimate

$$\cot\left(\frac{\ell\pi}{2m}+x\right) < \frac{1}{\frac{\ell\pi}{2m}+x} \le \frac{2m}{\pi},$$

from which (9) follows.
(iii) For $\frac{\pi}{4m} \le x \le \frac{\pi}{2m}$, we have $a \le 0$ and $b \ge 0$, i.e., $a - b \le 0$, which yields (10).

In order to obtain (11)–(13), by means of the symmetry (6) we rewrite

$$\begin{aligned} f_m\left(\frac{\ell\pi}{2m}+x\right) &= f_m\left(\pi - \left(\frac{\ell\pi}{2m}+x\right)\right) \\ &= f_m\left(\frac{(2m-(\ell+1))\pi}{2m} + \left(\frac{\pi}{2m} - x\right)\right).\end{aligned}$$

Thereby, (11), (12) and (13) follow from (10), (9) and (8), respectively. ■

The third Lemma contains special sums of values of f_m.

Lemma 5 *For all $m \in \mathbb{N}$, $m \ge 2$, and for $0 \le x \le \frac{\pi}{2m}$, we have*

$$\sum_{\ell=1}^{2m-2} (-1)^\ell\, f_m\left(\frac{\ell\pi}{2m}+x\right) = |f_m(x)| - \left| f_m\left(\frac{\pi}{2m}-x\right)\right|,$$

and

$$\sum_{\ell=1}^{2m-1} \left| f_m\left(\frac{\ell\pi}{2m}\right)\right| = |f_m(0)|.$$

Proof: We use the well-known equality

$$\sum_{\ell=0}^{2m-1} \frac{1}{\sin^2(\frac{x}{2}+\frac{\ell\pi}{2m})} = \frac{4m^2}{\sin^2 mx}, \tag{14}$$

from which, by differentiation,

$$\sum_{\ell=0}^{2m-1} \frac{\cos(\frac{x}{2}+\frac{\ell\pi}{2m})}{\sin^3(\frac{x}{2}+\frac{\ell\pi}{2m})} = \frac{8m^3\cos mx}{\sin^3 mx}$$

follows. Then,

$$\begin{aligned}
&\sum_{\ell=0}^{2m-1} (-1)^\ell \; f_m\left(\frac{\ell\pi}{2m}+x\right) \\
&= \sum_{\ell=0}^{2m-1} (-1)^\ell \; \frac{2m\cos(\ell\pi+2mx)\sin\left(\frac{\ell\pi}{2m}+x\right)-\sin(\ell\pi+2mx)\cos\left(\frac{\ell\pi}{2m}+x\right)}{\sin^3\left(\frac{\ell\pi}{2m}+x\right)} \\
&= 2m\cos 2mx \sum_{\ell=0}^{2m-1} \frac{1}{\sin^2\left(\frac{\ell\pi}{2m}+x\right)} - \sin 2mx \sum_{\ell=0}^{2m-1} \frac{\cos\left(\frac{\ell\pi}{2m}+x\right)}{\sin^3\left(\frac{\ell\pi}{2m}+x\right)} \\
&= \frac{2m\cos 2mx \; 4m^2}{\sin^2 2mx} - \frac{\sin 2mx \; 8m^3\cos 2mx}{\sin^3 2mx} \quad = \quad 0\,.
\end{aligned}$$

From Lemma 3 and from the symmetry (6) we further deduce

$$\begin{aligned}
\sum_{\ell=1}^{2m-2} (-1)^\ell \; f_m\left(\frac{\ell\pi}{2m}+x\right) &= f_m\left(\frac{(2m-1)\pi}{2m}+x\right) - f_m(x) \\
&= |f_m(x)| - \left|f_m\left(\frac{\pi}{2m}-x\right)\right|.
\end{aligned}$$

Using Lemma 3 and (8), (11) and (13) from Lemma 4, we conclude

$$\sum_{\ell=1}^{2m-1} \left|f_m\left(\frac{\ell\pi}{2m}\right)\right| = \sum_{\ell=1}^{2m-1} (-1)^\ell \; f_m\left(\frac{\ell\pi}{2m}\right) = -f_m(0) = |f_m(0)|\,. \qquad \blacksquare$$

Proof of Theorem 1 From (3) and definition (5) of f_m, for all $s = 1, \ldots, 2N-1$, we have

$$|\langle \varphi_{N,0}^M, \varphi_{N,s}^M\rangle| = \left|\frac{2M\cos\frac{Ms\pi}{N}\sin\frac{s\pi}{2N} - \sin\frac{Ms\pi}{N}\cos\frac{s\pi}{2N}}{8NM^2\sin^3\frac{s\pi}{2N}}\right| = \frac{1}{8NM^2}\left|f_M\left(\frac{s\pi}{2N}\right)\right|.$$

In view of the Remark on page 107 we can assume $M \geq 2$, and we set $K := N/M$. Replacing $s = K\ell + k$, where $k, \ell \in \mathbb{N}_0$, $0 \leq k \leq K-1$ and $0 \leq \ell \leq 2M-1$, we can split

$$\begin{aligned}
\rho(N,M) &= \left\langle \varphi_{N,0}^M, \varphi_{N,0}^M\right\rangle - \sum_{s=1}^{2N-1} \left|\left\langle \varphi_{N,0}^M, \varphi_{N,s}^M\right\rangle\right| \\
&= \left\langle \varphi_{N,0}^M, \varphi_{N,0}^M\right\rangle - \sum_{\ell=1}^{2M-1} |\langle \varphi_{N,0}^M, \varphi_{N,K\ell}^M\rangle| - \sum_{k=1}^{K-1}\sum_{\ell=0}^{2M-1} |\langle \varphi_{N,0}^M, \varphi_{N,K\ell+k}^M\rangle|\,.
\end{aligned}$$

By Lemma 5 we have

$$\sum_{\ell=1}^{2M-1} |\langle \varphi_{N,0}^M, \varphi_{N,K\ell}^M\rangle| = \frac{1}{8NM^2}\sum_{\ell=1}^{2M-1}\left|f_M\left(\frac{\ell\pi}{2M}\right)\right| = \frac{1}{8NM^2}\,|f_M(0)|\,.$$

From Lemmata 4 and 5 and by symmetry (6) we obtain

$$
\begin{aligned}
8NM^2 \sum_{k=1}^{K-1}\sum_{\ell=0}^{2M-1} |\langle \varphi_{N,0}^M, \varphi_{N,K\ell+k}^M\rangle| \;&=\; \sum_{k=1}^{K-1}\sum_{\ell=0}^{2M-1} \left| f_M\left(\frac{\ell\pi}{2M}+\frac{k\pi}{2N}\right)\right| \\
= \;& \sum_{k=1}^{K-1}\left(\left|f_M\left(\frac{k\pi}{2N}\right)\right| + \left|f_M\left(\frac{(2M-1)\pi}{2M}+\frac{k\pi}{2N}\right)\right|\right) + \sum_{k=1}^{K-1}\sum_{\ell=1}^{2M-2}\left|f_M\left(\frac{\ell\pi}{2M}+\frac{k\pi}{2N}\right)\right| \\
\le \;& \sum_{k=1}^{K-1}\left(\left|f_M\left(\frac{k\pi}{2N}\right)\right| + \left|f_M\left(\frac{\pi}{2M}-\frac{k\pi}{2N}\right)\right|\right) \\
& + \sum_{k=1}^{\frac{K}{2}}\sum_{\ell=1}^{2M-2}(-1)^\ell\, f_M\left(\frac{\ell\pi}{2M}+\frac{k\pi}{2N}\right) - \sum_{k=\frac{K}{2}+1}^{K-1}\sum_{\ell=1}^{2M-2}(-1)^\ell\, f_M\left(\frac{\ell\pi}{2M}+\frac{k\pi}{2N}\right) \;+\; S \\
= \;& \sum_{k=1}^{K-1}\left(\left|f_M\left(\frac{k\pi}{2N}\right)\right| + \left|f_M\left(\frac{\pi}{2M}-\frac{k\pi}{2N}\right)\right|\right) + \sum_{k=1}^{\frac{K}{2}}\left(\left|f_M\left(\frac{k\pi}{2N}\right)\right| - \left|f_M\left(\frac{\pi}{2M}-\frac{k\pi}{2N}\right)\right|\right) \\
& - \sum_{k=\frac{K}{2}+1}^{K-1}\left(\left|f_M\left(\frac{k\pi}{2N}\right)\right| - \left|f_M\left(\frac{\pi}{2M}-\frac{k\pi}{2N}\right)\right|\right) \;+\; S \\
= \;& 4\sum_{k=1}^{\frac{K}{2}-1}\left|f_M\left(\frac{k\pi}{2N}\right)\right| + 2\left|f_M\left(\frac{\pi}{4M}\right)\right| \;+\; S \;\;\le\;\; 2(K-1)\,|f_M(0)| \;+\; S\,,
\end{aligned}
$$

where

$$
S \;=\; \frac{4M}{\pi}\left(\sum_{\ell=1}^{M-1}\sum_{k=\frac{K}{4}+1}^{\frac{K}{2}} \frac{1}{\sin^2\left(\frac{\ell\pi}{2M}+\frac{k\pi}{2N}\right)} + \sum_{\ell=M}^{2M-2}\sum_{k=\frac{K}{2}+1}^{\frac{3K}{4}-1} \frac{1}{\sin^2\left(\frac{\ell\pi}{2M}+\frac{k\pi}{2N}\right)}\right).
$$

From (14) we deduce

$$
\sum_{\ell=1}^{2M-1}\frac{1}{\sin^2\frac{\ell\pi}{2M}} \;=\; \frac{4M^2-1}{3} \;=\; 1 + 2\sum_{\ell=1}^{M-1}\frac{1}{\sin^2\frac{\ell\pi}{2M}}\,, \tag{15}
$$

which gives

$$
\sum_{\ell=1}^{M-1}\frac{1}{\sin^2\frac{\ell\pi}{2M}} \;=\; \frac{2}{3}(M^2-1)\,.
$$

Hence,

$$
S \;\le\; \left(\frac{K}{2}-1\right)\frac{8M(M^2-1)}{3\pi}\,.
$$

In view of the monotonicity of f_M in $\left[0, \frac{\pi}{2M}\right]$ (see Lemma 3), together with (2), (7) and

$K \geq 4$, we finally estimate

$$\begin{aligned}
\rho(N,M) &\geq 1 - \frac{M}{3N} + \frac{1}{12NM} - \frac{1}{8NM^2}\left((2K-1)|f_M(0)| + S\right) \\
&\geq 1 - \frac{M}{3N} + \frac{1}{12NM} - \frac{(2K-1)(4M^2-1)}{12NM} - \frac{(K-2)(M^2-1)}{6\pi NM} \\
&= \frac{1}{3} - \frac{1}{6\pi} + \frac{1}{6M^2} + \frac{M}{3\pi N} + \frac{K-2}{6\pi NM} \quad > \quad 0.28\,. \qquad \blacksquare
\end{aligned}$$

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