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Basic properties of Fejer monotone sequences

Dedicated to the professors of mathematics
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Summary. *We consider certain classes of Fejer monotone sequences and their basic properties. These properties are the starting point of a convergence theory for corresponding iterative methods which are widely used to solve convex problems.*

1 Introduction

The theory of Fejer monotone sequences and mappings has a long history. It started already in the sixteeth of our century and was closely connected with the Russian mathematician Eremin ([4],[5]). The monography [6] contains a summarizing description for finite-dimensional spaces with various applications to the iterative solution of convex problems. Parallel to this development the basic ideas of Fejer theory also occured in concrete applications (e.g. [9],[1],[7],[12]). This process continues up to present time. Often the authors do not know the original literature and the developed theory (e.g.[18],[10],[8],[3]). The great interest in Fejer methods is caused by the wide range of applications. At present they play an important part in computerized tomography and image recovery (see e.g. [19]). Consequently, there are strong requirements to expand and to complete the Fejer theory. My paper [13] supplies a unifying concept for the application of Fejer methods on the level of Hilbert space assuming somewhat different conditions than in [6]. Finally I develop in [17] a theory of strong convergent Fejer methods using some basic properties of Fejer monotone sequences given here. Such sequences (x_k) can be generated by Fejer monotone mappings g ([16],[17]). The aim is to win iterative Fejer methods

$$x_{k+1} \in g(x_k)$$

which converge to an element x^* of a convex and closed problem set M.

2 Fejer monotone sequences

Let H be a Hilbert space. We consider a sequence (x_k) in H. Its set of weak and strong accumulation values is denoted by $A^w(x_k)$ and $A^s(x_k)$, respectively. As usual the number

$$\rho(y,M) = \inf\{\|y-x\| : x \in M\}$$

gives the distance between $y \in H$ and $M \subseteq H$. If M is convex and closed, then there is a unique metric (orthogonal) projector P_M onto M with

$$P_M y = \operatorname{argmin}\{\|y-x\| : x \in M\}.$$

Moreover, we have the relation $\rho(y,M) = \|y - P_M y\|$. Finally $B(x,r)$ denotes the **closed ball** with midpoint x and radius r.

Definition 2.1 *Let M be a nonempty subset of H. The sequence (x_k) is said to be* M-**Fejer-monotone** *iff*

$$\|x_{k+1} - x\| \leq \|x_k - x\| \quad \forall k \in \mathbb{N},\ \forall x \in M.$$

It is said to be **regularly** M-**Fejer monotone** *iff additionally*

$$x_{k+1} \neq x_k \quad \forall k \in \mathbb{N} : x_k \notin M.$$

It is said to be **strictly** M-**Fejer monotone** *iff additionally*

$$\|x_{k+1} - x\| < \|x_k - x\| \quad \forall k \in \mathbb{N} : x_k \notin M\,,\ \forall x \in M.$$

It is called **(regularly, strictly) Fejer monotone** *iff it is (regularly, strictly) M-Fejer monotone for any subset M. The set*

$$C(x_k) = \{x \in H : \|x_{k+1} - x\| \leq \|x_k - x\| \quad \forall k \in \mathbb{N}\}$$

is called the **Fejer carrier** *of (x_k).*

Remark 2.2 Originally strictly M-Fejer monotone sequences (x_k) with $x_k \notin M$ were called Fejer monotone (see [6, subsection 1.1]). But we want to start with the above weaker version of the concept (see also [2, p. 22],[13, p. 886]). So the distance between the iterates of a M-Fejer monotone sequence and each fixed element in M does not expand. In the strict case this distance is reduced step by step for all iterates outside of M.

The above definitions have some easy consequences. The assertions listed in the first Lemma need not be proven. They are obvious.

Lemma 2.3 *The following properties hold:*

(P1) (x_k) *Fejer monotone* $\Longleftrightarrow C(x_k) \neq \emptyset$,

(P2) (x_k) *M-Fejer monotone* $\Longrightarrow M \subseteq C(x_k)$,

(P3) (x_k) *Fejer monotone* $\Longrightarrow (x_k)$ $C(x_k)$*-Fejer monotone,*

(P4) (x_k) *M-Fejer monotone* , $\emptyset \neq N \subseteq M \Longrightarrow (x_k)$ *N-Fejer monotone,*

(P5) (x_k) *regularly M-Fejer monotone* $\Longrightarrow (x_k)$ *M-Fejer monotone,*

(P6) (x_k) *(strictly) Fejer monotone* $\Longrightarrow$ *subsequence* $(x_{k'})$ *(strictly) Fejer monotone.*

If a sequence (x_k) is M-Fejer monotone, but not strictly, it can be strictly Fejer monotone relative to a subset N. If (x_k) is strictly M-Fejer monotone, it need not be strictly $C(x_k)$-Fejer monotone (compare (P3)).

Lemma 2.4 *Let be* $k_0 \in \mathbb{N}$ *fixed. Then there holds*

(P7) (x_k) *M-Fejer monotone,* $x_{k_0} \in M \Longrightarrow x_{k_0+1} = x_{k_0}$,

(P8) (x_k) *regularly M-Fejer monotone,* $x_{k_0+1} = x_{k_0} \Longrightarrow x_{k_0} \in M$,

(P9) (x_k) *strictly M-Fejer monotone* $\Longrightarrow$ (x_k) *regularly M-Fejer monotone.*

Proof: Let (x_k) be M-Fejer monotone. Assuming $x_{k_0} \in M$ the property (P7) follows immediately if $k = k_0$ and $x = x_{k_0}$ are chosen in the defining inequality. Let (x_k) be regularly M-Fejer monotone. Then $x_{k_0+1} = x_{k_0}$ generates a contradiction if x_{k_0} is supposed to be not in M. So (P8) arises. If (x_k) is strictly M-Fejer monotone and x_{k_0} is not in M for any index k_0 then $\|x_{k_0+1} - x\| < \|x_{k_0} - x\|$ and consequently $x_{k_0+1} \neq x_{k_0}$. Thus (x_k) is regularly M-Fejer monotone as asserted in (P9). ■

By Lemma 2.4 the set M of a M-Fejer monotone sequence (x_k) contains at most stationary points of (x_k) . In the regular case all stationary points of (x_k) lie in M. Now we introduce the abbreviation

$$d_k(x) := \|x_k - x\|^2 - \|x_{k+1} - x\|^2 \tag{1}$$

and list some properties of $d_k(x)$.

Lemma 2.5 *The functional $d_k(x)$ has the representations*

$$\begin{aligned} d_k(x) &= (x_k - x_{k+1}, x_k + x_{k+1} - 2x) &(2)\\ &= 2(x_k - x_{k+1}, x_k - x) - \|x_k - x_{k+1}\|^2. &(3) \end{aligned}$$

The sequence (x_k) is M-Fejer monotone iff

$$d_k(x) \geq 0 \quad \forall k \in \mathbb{N},\ \forall x \in M.$$

Besides, it is

$$C(x_k) = \{x \in H : d_k(x) \geq 0 \quad \forall k \in \mathbb{N}\}.$$

Proof: The transformation of (1) supplies

$$\begin{aligned} d_k(x) &= \|x_k\|^2 - \|x_{k+1}\|^2 - 2(x_k, x) + 2(x_{k+1}, x)\\ &= (x_k - x_{k+1}, x_k + x_{k+1}) - 2(x_k - x_{k+1}, x)\\ &= (x_k - x_{k+1}, x_k + x_{k+1} - 2x)\\ &= (x_k - x_{k+1}, 2(x_k - x) - (x_k - x_{k+1}))\\ &= 2(x_k - x_{k+1}, x_k - x) - \|x_k - x_{k+1}\|^2. \end{aligned}$$

The next assertions are immediate consequences of Fejer monotony (see Definition 2.1). ■

We turn to a remarkable property of the set $C(x_k)$.

Theorem 2.6 *The Fejer carrier $C(x_k)$ of (x_k) is convex and closed.*

Proof: By Lemma 2.5 the Fejer carrier of (x_k) can be represented in the form

$$C(x_k) = \{x \in H : (x_k - x_{k+1}, x_k + x_{k+1} - 2x) \geq 0 \quad \forall k \in \mathbb{N}\}.$$

Now we fix arbitrary elements $x, x' \in C(x_k)$ and arbitrary numbers α, α' satisfying $\alpha \geq 0$, $\alpha' \geq 0$, $\alpha + \alpha' = 1$. Using the abbreviations $u_k = x_k - x_{k+1}$ and $v_k = x_k + x_{k+1}$ we get at first $\alpha(u_k, v_k - 2x) \geq 0$, $\alpha'(u_k, v_k - 2x') \geq 0$ and then by addition of both inequalities

$$(u_k, (\alpha + \alpha')v_k - 2\alpha x - 2\alpha' x') = (u_k, v_k - 2(\alpha x + \alpha' x')) \geq 0.$$

This means $\alpha x + \alpha' x' \in C(x_k)$. Hence $C(x_k)$ is convex. Finally we consider a sequence (y_n) in $C(x_k)$ converging to $y \in H$. Consequently we conclude with the above abbreviations

$$(u_k, v_k - 2y_n) \geq 0 \quad \forall k, n \in \mathbb{N}$$

and, since the scalar product is continuous, also

$$(u_k, v_k - 2y) \geq 0 \quad \forall k \in \mathbb{N}.$$

This means $y \in C(x_k)$. Hence $C(x_k)$ is closed. ■

Theorem 2.7 *Let M be nonempty and (x_k) be M-Fejer monotone. Then the following statements hold:*

a) (x_k) *is bounded and therefore weakly precompact* ($A^w(x_k) \neq \emptyset$),

b) $\|x' - x\| = \inf_k \|x_k - x\| = \lim_k \|x_k - x\| \quad \forall x' \in A^s(x_k),\ \forall x \in M$,

c) $A^s(x_k)$ *ist a singleton in* $M \iff A^s(x_k) \cap M \neq \emptyset$,

d) $\lim_k x_{k'} = x^* \in M$ *for any subsequence* $(x_{k'}) \implies \lim_k x_k = x^* \in M$,

e) $\lim_k d_k(x) = 0 \quad \forall x \in M$.

Proof: Let (x_k) be M-Fejer monotone.
a) We choose a fixed $x \in M$. Then we get by repeated use of definition

$$\|x_k - x\| \leq \|x_0 - x\|$$

and hence

$$\|x_k\| \leq \|x_k - x\| + \|x\| \leq \|x_0 - x\| + \|x\|$$

for all $k \in \mathbb{N}$. Thus (x_k) is bounded and weakly precompact.
b) We consider the sequence $r_k = \|x_k - x\|$ for a certain fixed $x \in M$. Obviously (r_k) is monotone decreasing and bounded from below (by 0). Consequently, (r_k) converges, and we have

$$r := \lim_k r_k = \inf_k r_k \,.$$

Let be $x' \in A^s(x_k)$ and choose any subsequence $(x_{k'})$ of (x_k) with limit x'. Since the norm is continuous, we obtain

$$r = \lim_{k'} \|x_{k'} - x\| = \|x' - x\| \,.$$

c) Let be $x^* \in A^s(x_k) \cap M$. Then b) implies for $x' = x = x^*$

$$0 = \|x^* - x^*\| = \lim_k \|x_k - x^*\| \,.$$

This means $\lim_k x_k = x^* \in M$. The reversion is trivial.
d) Since the sequence (r_k) with members $r_k = \|x_k - x\|$ is convergent for each $x \in M$, we get immediately

$$\lim_k \|x_{k'} - x\| = \lim_k \|x_k - x\|$$

for any subsequence $(x_{k'})$ of (x_k). The assertion follows if we substitute the limit x^* of $(x_{k'})$ for x.
e) Evidently, $d_k(x) = r_k^2 - r_{k+1}^2$ tends to 0 because (r_k) is convergent. ∎

Theorem 2.8 *Let M be nonempty, convex and closed. Further, let (x_k) be M-Fejer monotone. Then the following conditions are equivalent:*

a) $A^s(x_k)$ *is a singleton in* M.

b) $\lim_k \rho(x_k, M) = 0$.

Moreover, under a) *or* b) *the estimates*

$$\|P_M x_k - x^*\| \leq \rho(x_k, M) \quad , \quad \|x_k - x^*\| \leq 2\rho(x_k, M)$$

hold, where P_M is the metric projector onto M and $x^* \in M$ is the limit of (x_k).

Proof: We consider a M-Fejer monotone sequence (x_k). At first we show the implication b) $\Rightarrow$ a). The reversion is obvious. By repeated use of definition we get

$$\|x_m - x\| \leq \|x_k - x\| \quad \forall m \geq k \ , \ \forall x \in M \ .$$

If we put $x = P_M x_k$, it follows

$$\|x_m - P_M x_k\| \leq \|x_k - P_M x_k\| = \rho(x_k, M) \quad \forall m \geq k \ .$$

Hence

$$\|x_k - x_m\| \leq \|x_k - P_M x_k\| + \|P_M x_k - x_m\| \leq 2\rho(x_k, M) \quad \forall m \geq k \ .$$

Let us now suppose $\lim_k \rho(x_k, M) = 0$. Then (x_k) turns out to be a Cauchy sequence as the last relation supplies $\|x_k - x_m\| \to 0$ for $k \to \infty$. Thus (x_k) converges to a certain element x^* in H because H is complete. Observing $\lim_k \rho(x_k, M) = 0$ and the closedness of M , $x^* \in M$ follows. If we finally let m tend to infinity in the second and third above estimate, then also the asserted error estimates for (x_k) are shown. ■

Remark 2.9 The convergence assertion of Theorem 2.8 without error estimates is already given in [7, Lemma 6]. But the proof is more troublesome. Moreover, the well-known theorem had to be used that the intersection of a descending sequence of convex, closed and bounded balls is nonempty. Corresponding error estimates as in Theorem 2.8 under more general and more special conditions can be found in [14] and [15].

Theorem 2.10 *Let (x_k) be M-Fejer monotone. Then (x_k) converges weakly to an element x^* in M iff $A^w(x_k) \subseteq M$.*

Proof: The first part of the equivalence is trivial. We show the second. We consider a M-Fejer monotone sequence (x_k) with $A^w(x_k) \subseteq M$. By Theorem 2.7a) the set $A^w(x_k)$ is nonempty. We assume that there are at least two different elements x' and x'' in $A^w(x_k)$ which lie consequently also in M . Hence the limits

$$r' = \lim_k \|x_k - x'\| \quad , \quad r'' = \lim_k \|x_k - x''\|$$

exist in view of Theorem 2.7b). Without loss of generality we suppose $r'' \leq r'$. We choose a subsequence $(x_{k'})$ of (x_k) with $\mathrm{wlim}_k\, x_{k'} = x'$. Using the so-called Opial's condition (see [11]) which is satisfied in H we attain

$$\begin{aligned} r' &= \lim_k \|x_k - x'\| = \lim_k \|x_{k'} - x'\| \\ &< \lim_k \|x_{k'} - x''\| = \lim_k \|x_k - x''\| = r'' \,. \end{aligned}$$

Generally Opial's condition holds for the limit superior. But in view of the Fejer monotony the limit superior coincides with the usual limit. So the last estimate leads to a contradiction. Hence all weak accumulation values of (x_k) are equal (to the weak limit x^* in M).

3 Geometrical aspects

If we want to study the geometrical aspects of Fejer monotone sequences, there are two possible points of view. The first point of view assumes that the set M and an iterate x_k are given. Then we are interested in the **restriction set** which describes the admissible location of the next iterate x_{k+1}. The second point of view starts from the given sequence (x_k) and looks for sets which enclose and generate $C(x_k)$. Both aspects are considered successively.

Lemma 3.1 *Let (x_k) be Fejer monotone. Then there holds with $M = C(x_k)$*

$$x_{k+1} \in \bigcap_{x \in M} B(x, \|x_k - x\|) \subseteq B(x, \rho(x_k, M)) \quad \forall k \in \mathbb{N} \,.$$

Proof: By Lemma 2.3 (x_k) is M-Fejer monotone with $M = C(x_k)$. This implies immediately the ball condition

$$x_{k+1} \in B(x, \|x_k - x\|)$$

for all k and all $x \in M$. So x_{k+1} must lie also in the intersection of these balls taken over all $x \in M$. One special ball is obtained for $x = P_M x_k$, namely $B(x, \rho(x_k, M))$. ■

Remark 3.2 Obviously, the first restriction set given in Lemma 3.1 is convex and closed, since all balls have this property. Without doubt the shape of this set depends on the special properties of (x_k) and the shape of $C(x_k)$, respectively.

Now we turn to the second point of view. We assume the sequence (x_k) to be given and look for the generation of $C(x_k)$. We consider the sets

$$\begin{aligned} Q_k &:= \{x \in H : \|x_{k+1} - x\| \leq \|x_k - x\|\} , \\ H_k &:= \{x \in H : \|x_{k+1} - x\| = \|x_k - x\|\} , \end{aligned}$$

which are only of interest in the case $x_k \neq x_{k+1}$, since they otherwise coincide with the whole space H. If the definition of $C(x_k)$ is observed, we get the relation

$$C(x_k) = \bigcap_k Q_k .$$

In the following we use the abbreviation

$$y_k := \frac{1}{2}(x_k + x_{k+1}) . \tag{4}$$

.

Theorem 3.3 *Let (x_k) be Fejer monotone.*
Then there holds for any k with $x_k \neq x_{k+1}$: The set Q_k is a nonempty halfspace with the boundary hyperplane H_k. H_k has the normal direction $x_k - x_{k+1}$ and contains the element y_k which is the orthogonal projection of x_k onto H_k. The iterate x_{k+1} is the mirror point of x_k with respect to H_k.

Proof: Let (x_k) be Fejer monotone and consider an index k with $x_k \neq x_{k+1}$. Then Q_k is nonempty as this holds already for $C(x_k)$. The expression (1) can be written in the form

$$d_k(x) = (x_k - x_{k+1}, x_k + x_{k+1} - 2x)$$

(see (2) in Lemma 2.5). Just for the elements of Q_k both sides of the equation are nonnegative. Hence we can write by use of (4)

$$Q_k = \{x \in H : (x_k - x_{k+1}, x) \leq (x_k - x_{k+1}, y_k)\} , \tag{5}$$

$$H_k = \{x \in H : (x_k - x_{k+1}, x) = (x_k - x_{k+1}, y_k)\} . \tag{6}$$

The condition for k ensures that Q_k is a halfspace whose boundary hyperplanes H_k contains the element y_k and has the normal direction $x_k - x_{k+1}$ which is orthogonal to H_k. In view of $x_k - y_k = \frac{1}{2}(x_k - x_{k+1})$ this difference is also orthogonal to H_k. Consequently, y_k is the orthogonal projection of x_k onto H_k. Besides y_k turns out to be the midpoint of the segment generated by the convex hull of $\{x_k, x_{k+1}\}$. Hence x_{k+1} is the mirror point of x_k with respect to H_k. ■

Lemma 3.4 *Let (x_k) be M-Fejer monotone. Then there holds*

a) $(x_{k+1} - x_k, x - x_k) \geq 0 \quad \forall k \in \mathbb{N}\,,\ \forall x \in M$,

b) $(x_{k+1} - x_k, x - x_k) > 0 \quad \forall k \in \mathbb{N} : x_k \neq x_{k+1}\,,\ \forall x \in M$,

c) $\|x_{k+1} - x_k\| \leq 2\,\|x_k - x\| \quad \forall k \in \mathbb{N}\,,\ \forall x \in M$,

d) $\|x_{k+1} - x_k\| \leq 2\,\rho(x_k, M) \quad \forall k \in \mathbb{N}$.

Proof: Let (x_k) be M-Fejer monotone. Then we have in view of Lemma 2.5 for all k and all x in M

$$2(x_k - x_{k+1}, x_k - x) \geq \|x_k - x_{k+1}\|^2\,.$$

This inequality leads immediately to the assertions a) and b). For $x_k = x_{k+1}$ the assertion c) holds trivially. In the other case we get by Schwarz's inequality

$$2\|x_k - x_{k+1}\|\|x_k - x\| \geq \|x_k - x_{k+1}\|^2$$

and therefore the assertion c). Putting $x = P_M x_k$ in assertion c) assertion d) follows. ■

Remark 3.5 The relations a) and b) in Lemma 3.4 show that the angle between the elements $x_{k+1} - x_k$ and $x - x_k$ is not obtuse and for $x_k \neq x_{k+1}$ even acute considering any of the elements x in M. The relations c) and d) can be expressed as ball conditions

$$x_{k+1} \in \bigcap_{x \in M} B(x_k, 2\,\|x_k - x\|) \subseteq B(x_k, 2\,\rho(x_k(M))\,.$$

Obviously, they are weaker than the ball conditions in Lemma 3.1. But c) and d) show that for iterates x_k in a small neighbourhood of any $x \in M$ or of M itself the new iterates x_{k+1} are nearby the old.

4 Uniformly Fejer monotone sequences

Definition 4.1 *Let M be a nonempty subset of H and (τ_k) a sequence of nonnegative numbers. The sequence (x_k) is said to be* **uniformly M-Fejer monotone relative to** *(r.t.) (τ_k) iff*

$$d_k(x) = \|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \tau_k \quad \forall k \in \mathbb{N}\,,\ \forall x \in M$$

and

$$\tau_k > 0 \quad \forall k \in \mathbb{N} : x_k \notin M.$$

It is said to be **uniformly Fejer monotone** *iff it is uniformly M-Fejer monotone r.t.* (τ_k) *for any admissible sequence* (τ_k) *and for any subset* M*. The set*

$$C(x_k, \tau_k) := \{x \in H : d_k(x) \geq \tau_k \quad \forall k \in \mathbb{N}\}$$

is called the **Fejer carrier** *of* (x_k) **r.t.** (τ_k)*.*

Lemma 4.2 *Let* (x_k) *be uniformly M-Fejer monotone r.t.* (τ_k)*. Then holds*

a) (x_k) *is strictly M-Fejer monotone,*

b) (x_k) *is* $C(x_k, \tau_k)$*-Fejer monotone r.t.* (τ_k)*,*

c) $\lim_k \tau_k = 0$.

Proof: a) Let (x_k) be uniformly M-Fejer monotone r.t. (τ_k). This means by definition

$$d_k(x) = \|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \tau_k \geq 0$$

and implies

$$\|x_k - x\| \geq \|x_{k+1} - x\|$$

for all k and all $x \in M$. Hence (x_k) is M-Fejer monotone. Assuming $x_k \notin M$ we know $\tau_k > 0$. This leads to $\|x_k - x\| > \|x_{k+1} - x\|$. So (x_k) is strictly M-Fejer monotone.
b) Assume that $x_k \notin C(x_k, \tau_k)$. Then also $x_k \notin M$ because of $M \subseteq C(x_k, \tau_k)$. Hence $\tau_k > 0$. Now assertion b) is obvious.
c) Considering the first relation in part a) the assertion follows by Theorem 2.7e). ■

Lemma 4.2 shows that we can also put $M = C(x_k, \tau_k)$ in assertion a). The comparision of the corresponding types of Fejer carriers leads to the relation

$$C(x_k, \tau_k) \subseteq C(x_k) \ . \tag{7}$$

Already simple examples show that the equality need not to hold. The relative carrier $C(x_k, \tau_k)$ turns out to be convex and closed, too. This can be shown directly in an analogue way as in Theorem 2.6 for $C(x_k)$. But it is also a consequence of geometric facts given below (see Corollary 4.6).

Lemma 4.3 *Let* (μ_k) *and* (ν_k) *be two number sequences satisfying* $\mu_k \geq \nu_k \geq 0$*. Let* (x_k) *be a sequence with* $\nu_k > 0$ *for* $x_k \notin M$*. Then there holds*

a) (x_k) *uniformly M-Fejer monotone r.t.* $(\mu_k) \Longrightarrow (x_k)$ *uniformly M-Fejer monotone r.t.* (ν_k)*,*

b) $C(x_k, \mu_k) \subseteq C(x_k, \nu_k)$.

Proof: The sequences (μ_k) and (ν_k) satisfy the conditions of Definition 4.1. Assertion a) follows immediately by the relations

$$\|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \mu_k \geq \nu_k \ ,$$

which are fulfilled for all k and all $x \in M$. Since the weaker estimate with ν_k on the right-hand side admits possibly more elements x than the estimate with μ_k, assertion b) is also obvious. ∎

If (x_k) is uniformly M-Fejer monotone r.t. (τ_k), then (τ_k) is not uniquely determined. The maximal choice is given by

$$\tau_k = \tau_k^m := \inf\nolimits_{x \in M} \ d_k(x).$$

Now we turn again to geometrical aspects. First we are interested in restriction sets of x_{k+1} for given M and x_k.

Lemma 4.4 *Let (x_k) be uniformly M-Fejer monotone r.t. (τ_k). Then*

$$x_{k+1} \in \bigcap_{x \in M} B(x, \sqrt{\|x_k - x\|^2 - \tau_k}) \subseteq B(x, \sqrt{\rho^2(x_k, M) - \tau_k}) \quad \forall k \in \mathbb{N} \ .$$

Proof: Observing Definition 4.1 we get for uniformly M-Fejer monotone sequences r.t. (τ_k) the ball condition

$$x_{k+1} \in B\left(x, \sqrt{\|x_k - x\|^2 - \tau_k}\right)$$

for all k and all $x \in M$. Hence x_{k+1} lies also in the intersection of these balls taken over all $x \in M$. A special ball is obtained for $x = P_M(x_k)$, namely $B(x, \sqrt{\rho^2(x_k, M) - \tau_k})$. ∎

The above stated ball conditions for x_{k+1} have a crucial disadvantage if τ_k depends for its part on x_{k+1} as it is the case for strongly Fejer monotone sequences (see section 5). Then x_{k+1} occurs on both sides of the element relation causing a reflexivity. This effect can be avoided for the mentioned special case by a rearrangement of the defining estimate (see Lemma 5.3 and Remark 5.5).

Now we assume that the given sequence (x_k) is uniformly M-Fejer monotone r.t. (τ_k) and look for the generation of M. We consider the sets

$$\begin{aligned} Q'_k &:= \{x \in H : \|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \tau_k\} \ , \\ H'_k &:= \{x \in H : \|x_k - x\|^2 - \|x_{k+1} - x\|^2 = \tau_k\} \ . \end{aligned}$$

Consulting the definition of $C(x_k, \tau_k)$ we can conclude

$$M \subseteq \bigcap_k Q'_k = C(x_k, \tau_k) \ .$$

Theorem 4.5 *Let (x_k) be uniformly M-Fejer monotone r.t. (τ_k). Then there holds for all k with $x_k \notin M$: The set Q'_k is a nonempty halfspace contained in Q_k with the boundary hyperplane H'_k parallel to H_k. The orthogonal projection y'_k of x_k onto H'_k is given by*

$$y'_k = \frac{1}{2}\left(1 - \frac{\tau_k}{\|x_k - x_{k+1}\|^2}\right)\, x_k + \frac{1}{2}\left(1 + \frac{\tau_k}{\|x_k - x_{k+1}\|^2}\right)\, x_{k+1} .$$

The new iterate x_{k+1} has the representation

$$x_{k+1} = \frac{\tau_k - \|x_k - x_{k+1}\|^2}{\tau_k + \|x_k - x_{k+1}\|^2}\, x_k \,+\, 2\,\frac{\|x_k - x_{k+1}\|^2}{\tau_k + \|x_k - x_{k+1}\|^2}\, y'_k \;.$$

Proof: We suppose a sequence (x_k) which is uniformly M-Fejer monotone r.t. (τ_k). Then the nonempty set M is contained in the intersection of all sets Q'_k. Hence these sets are nonempty. Using the abbreviation (1) we can write

$$Q'_k = \{x \in H : d_k(x) - \tau_k \geq 0\}\;,\; H'_k = \{x \in H : d_k(x) - \tau_k = 0\}\;.$$

By relation (2) in Lemma 2.5 we get

$$d_k(x) - \tau_k = (x_k - x_{k+1}, x_k + x_{k+1} - 2x) - \tau_k \geq 0$$

and by rearrangement

$$(x_k - x_{k+1}, x) \leq \frac{1}{2}\,(x_k - x_{k+1}, x_k + x_{k+1}) - \frac{1}{2}\,\tau_k \leq (x_k - x_{k+1}, \frac{1}{2}\,(x_k + x_{k+1})) \qquad (8)$$

for all elements x in Q'_k. As (x_k) is regularly M-Fejer monotone, we know $x_k - x_{k+1} \neq 0$. Consequently, the sets Q'_k are halfspaces whose boundary hyperplanes H'_k possess the normal direction $x_k - x_{k+1}$. A comparision with (5), (6) shows $Q'_k \subseteq Q_k$ and $H'_k \parallel H_k$ if (4) is taken into account. The projection y'_k of x_k onto H'_k satisfies

$$y'_k = x_k - \beta\,(x_k - x_{k+1}) \in H'_k \;.$$

If we substitute $x = y'_k$ in the equation for H'_k corresponding to (8), we obtain

$$(x_k - x_{k+1}, x_k) - \beta\,\|x_k - x_{k+1}\|^2 = \frac{1}{2}\,(x_k - x_{k+1}, x_k + x_{k+1}) - \frac{1}{2}\,\tau_k$$

and

$$\beta\,\|x_k - x_{k+1}\|^2 = \frac{1}{2}(\|x_k - x_{k+1}\|^2 + \tau_k)\;.$$

This leads to

$$\beta = \frac{1}{2}(1 + \frac{\tau_k}{\|x_k - x_{k+1}\|^2})\;.$$

Since y'_k has the equivalent representation $y'_k = (1-\beta)x_k + \beta x_{k+1}$, the asserted form of the projection arises. If we separate x_{k+1} in the last equation, we win

$$x_{k+1} = (1-\lambda)x_k + \lambda y'_k \ , \tag{9}$$

where

$$\lambda = \frac{1}{\beta} = 2\,\frac{\|x_k - x_{k+1}\|^2}{\tau_k + \|x_k - x_{k+1}\|^2} \ . \tag{10}$$

The calculation of $1-\lambda$ completes the proof for the asserted representation of x_{k+1}. ■

Denoting the orthogonal projector onto H'_k by P'_k the equation (9) has the form

$$x_{k+1} = (1-\lambda)x_k + \lambda P'_k x_k \ ,$$

where λ lies in the open interval $(0,2)$ because of (10). This kind of iteration is called **relaxation** or more precisely **relaxed projection** of x_k onto H'_k. The value $\lambda = 1$ supplies just the projection while $\lambda < 1$ and $\lambda > 1$ lead to **underrelaxation** and **overrelaxation**, respectively. In the first case the new iterate stops before the hyperplane H'_k, in the second case it passes H'_k and stops before the mirror point.

Corollary 4.6 *The relative carrier $C(x_k, \tau_k)$ is convex and closed.*

Proof: The set $C(x_k, \tau_k)$ is the intersection of all sets Q'_k which are by Theorem 4.5 half-spaces or for $x_k = x_{k+1}$ the whole space H. Hence $C(x_k, \tau_k)$ is convex and closed because the sets Q'_k are convex and closed. ■

5 Strongly Fejer monotone sequences

We introduce an important special case of uniformly Fejer monotone sequences.

Definition 5.1 *Let M be a nonempty subset of H and α a positive number. The sequence (x_k) is said to be α-**strongly M-Fejer monotone** iff*

$$d_k(x) = \|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \alpha \, \|x_k - x_{k+1}\|^2 \quad \forall k \in \mathbb{N} \ , \ \forall x \in M$$

and

$$x_{k+1} \neq x_k \quad \forall k \in \mathbb{N} : x_k \notin M \ .$$

It is called **strongly Fejer monotone** *iff it is α-strongly Fejer monotone for any α and any M. The set*

$$C^\alpha(x_k) := \{x \in H : d_k(x) \geq \alpha \, \|x_k - x_{k+1}\|^2 \quad \forall k \in \mathbb{N}\}$$

*is called the α-**strong Fejer carrier** of (x_k).*

Remark 5.2 Evidently, α-strongly M-Fejer monotone sequences are uniformly M-Fejer monotone r.t. the special sequences

$$\tau_k = \tau_k^s(\alpha) := \alpha \|x_k - x_{k+1}\|^2 \ . \tag{11}$$

Consequently, we get $C^\alpha(x_k) = C(x_k, \tau_k^s(\alpha))$. This carrier is also convex and closed by Corollary 4.6. Additionally there holds

$$C^\alpha(x_k) \subseteq C^\beta(x_k) \subseteq C(x_k) \quad \text{for} \quad \alpha \geq \beta > 0 \ .$$

The first relation follows by Lemma 4.3b). The second relation is a special case of (7). The maximal choice for the parameter α is given by

$$\alpha = \alpha^m := \inf_{k: x_k \notin M, x \in M} \frac{d_k(x)}{\|x_k - x_{k+1}\|^2} \ .$$

For α-strongly M-Fejer monotone sequences (x_k) Lemma 4.2c) leads with $\tau_k = \tau_k^s(\alpha)$ to

$$\lim_k \|x_k - x_{k+1}\| = 0 \ .$$

Now we turn to results which can not be obtained by specialization of statements in section 4.

Lemma 5.3 *Let (x_k) be α-strongly M-Fejer monotone. Then there holds*

a) $(1+\alpha)\|x_k - x_{k+1}\|^2 \leq 2\,(x_k - x_{k+1}, x_k - x) \quad \forall k \in \mathbb{N}\ , \ \forall x \in M\ ,$

b) $\|x_{k+1} - x_k + \dfrac{1}{1+\alpha}\,(x_k - x)\| \leq \dfrac{1}{1+\alpha}\,\|x_k - x\| \quad \forall k \in \mathbb{N}\ , \ \forall x \in M\ .$

Proof: We consider the expression

$$f_k(x) := d_k(x) - \tau_k^s(\alpha) \ . \tag{12}$$

In view of Definition 5.1 and (11) this expression is nonnegative for α-strongly M-Fejer monotone sequences (x_k). Using (2) and (11) this means explicitely

$$f_k(x) = 2\,(x_k - x_{k+1}, x_k - x) - \|x_k - x_{k+1}\|^2 - \alpha \|x_k - x_{k+1}\|^2 \geq 0$$

and consequently

$$(1+\alpha)\|x_k - x_{k+1}\|^2 \leq 2\,(x_k - x_{k+1}, x_k - x)$$

for all k and all $x \in M$. Hence assertion a) is fulfilled. Finally we obtain by rearrangement

$$\|x_{k+1} - x_k\|^2 + 2\,(x_{k+1} - x_k, \frac{1}{1+\alpha}\,(x_k - x)) \leq 0$$

and quadratic completion

$$\begin{aligned}
&\|x_{k+1} - x_k + \frac{1}{1+\alpha}\,(x_k - x)\|^2 \\
&\quad = \|x_{k+1} - x_k\|^2 + 2\,(x_{k+1} - x_k, \frac{1}{1+\alpha}(x_k - x)) + \frac{1}{(1+\alpha)^2}\,\|x_k - x\|^2 \\
&\quad \leq \frac{1}{(1+\alpha)^2}\,\|x_k - x\|^2 \,.
\end{aligned}$$

This leads to assertion b). ■

Corollary 5.4 *Let (x_k) be α-strongly M-Fejer monotone. Then there holds*

a) $(1+\alpha)\|x_k - x_{k+1}\| \leq 2\,\|x_k - x\| \quad \forall k \in \mathbb{N}\,,\ \forall x \in M\,,$

b) $(1+\alpha)\|x_k - x_{k+1}\| \leq 2\,\rho(x_k, M) \quad \forall k \in \mathbb{N}\,.$

Proof: Starting with the estimate a) of Lemma 5.3 and applying Schwarz's inequality to the right-hand side the assertion a) occurs if $\|x_k - x_{k+1}\|$ is cancelled in the case $x_k \neq x_{k+1}$. But assertion a) is trivially fulfilled for $x_k = x_{k+1}$. The special choice $x = P_M(x)$ in assertion a) supplies assertion b). ■

Remark 5.5 The statement b) of Lemma 5.3 corresponds to the ball condition

$$x_{k+1} \in \bigcap_{x \in M} B(x_k - \frac{1}{1+\alpha}(x_k - x), \frac{1}{1+\alpha}\,\|x_k - x\|) \quad \forall k \in \mathbb{N}\,,$$

whereas the ball with the minimal radius arises for $x = P_M x_k$. The midpoints of the balls can be written in the form

$$x' = \frac{\alpha}{1+\alpha}\,x_k + \frac{1}{1+\alpha}\,x\,.$$

Hence they are convex combinations of x_k and x. The statement a) of Corollary 5.4 is equivalent to the ball condition

$$x_{k+1} \in \bigcap_{x \in M} B(x_k, \frac{2}{1+\alpha}\,\|x_k - x\|) \quad \forall k \in \mathbb{N}\,.$$

Evidently, we have

$$B(x', \frac{1}{1+\alpha}\,\|x_k - x\|) \subseteq B(x_k, \frac{2}{1+\alpha}\,\|x_k - x\|)\,.$$

By the way, both compared balls have the same nearest point to x, namely

$$\begin{aligned}
x'' &= x' - \frac{1}{1+\alpha}(x_k - x) = x_k - \frac{2}{1+\alpha}(x_k - x) \\
&= \frac{\alpha - 1}{\alpha + 1}\,x_k + \frac{2}{\alpha + 1}\,x\,.
\end{aligned}$$

Hence x'' is as x' a convex combination of x_k and x.

Theorem 5.6 *Let (x_k) be α-strongly M-Fejer monotone. Then*

$$y_k^\alpha := \frac{1-\alpha}{2}\, x_k + \frac{1+\alpha}{2}\, x_{k+1}$$

is the projection of x_k onto the halfspace

$$Q_k^\alpha := \{x \in H : \|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \alpha\, \|x_k - x_{k+1}\|^2\}$$

and onto the corresponding boundary hyperplane H_k^α, respectively. The new iterate x_{k+1} satisfies the equation

$$x_{k+1} = \frac{\alpha - 1}{\alpha + 1}\, x_k + \frac{2}{\alpha + 1}\, y_k^\alpha\ .$$

It realizes with respect to H_k^α overrelaxation for $\alpha < 1$ and underrelaxation for $\alpha > 1$. Moreover, it is

$$M \subseteq \bigcap_k Q_k^\alpha = C^\alpha(x_k)\ .$$

Proof: The results follow from Theorem 4.5 and the preceding set relation if we put there $\tau_k = \tau_k^s(\alpha)$. Especially, we get $\lambda = \dfrac{2}{\alpha+1}$ and $\alpha = \dfrac{2-\lambda}{\lambda}$. This implies also the statements about overrelaxation and underrelaxation (see remarks at the end of section 4). ■

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