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$\mathcal{H}-$commutative $\Delta-$semigroups

Dedicated to the professors of mathematics
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ABSTRACT. A $\Delta-$semigroup is a semigroup whose lattice of congruences is a chain with respect to inclusion. Schein [9] and Tamura [11] have investigated commutative $\Delta-$semigroups, Trotter [13] exponential, Nagy [4] weakly exponential and Bonzini and Cherubini Spoletini [2] finite $\Delta-$semigroups. The pupose of the present paper is to investigate archimedean weakly commutative and $\mathcal{H}-$commutative $\Delta-$semigroups.

KEY WORDS. weakly commutative semigroups, archimedean semigroups

1 Weakly commutative semigroups

The theory of congruences on semigroups is fundamental and the study of semigroups, whose lattice of congruences is simple (in some sense), is therefore relevant. Important semigroups such as the full transformation semigroup, symmetric inverse semigroups and symmetric groups, on finite sets, are all $\Delta-$semigroups (see [3], vol. II, page 247 and [12], examples 1 and 2).

A semigroup S is weakly commutative if, for each $a, b \in S$, there are $x, y \in S$ and an integer $k > 0$ such that $(ab)^k = xa = by$ ([6]). A semigroup S is $\mathcal{H}-$commutative if, for each $a, b \in S$, there are $w, z \in S^1$ with $ab = baw = zba$ [14].

Remark: A $\mathcal{H}-$commutative semigroup is weakly commutative. Groups are $\mathcal{H}-$commutative semigroups and nilsemigroups are weakly commutative.
A semigroup S is archimedean if for each $a, b \in S$ there is an integer $n > 0$, such that $a^n \in SbS$. An archimedean semigroup is semilattice indecomposable. A semigroup S is right archimedean [8] if for each $a, b \in S$ there is an element $u \in S$ and an integer i such that $a^i = bu$. Left archimedean semigroups are defined analogously. Clearly, every right archimedean semigroup is archimedean.

Lemma 1 *S is a weakly commutative archimedean semigroup iff S is right and left archimedean.*

Proof: See [1], exercise 4, page 34.

Lemma 2 *A weakly commutative archimedean semigroup S has at most one idempotent.*

Proof: Let e and f be idempotents in S. By Lemma 1 there is an element x with $e = fx$ and therefore $e = fx = ffx = fe$. By Lemma 1, S is left archimedean, hence there is an element y with $ye = f$. It follows that $f = ye = yee = fe$ and $e = f$.

Theorem 3 *A semigroup S is a weakly commutative archimedean semigroup with an idempotent iff S is an ideal extension of a group G by a nilsemigroup N.*

Proof: 1) Suppose that S is a weakly commutative archimedean semigroup with an idempotent e. Then, by Lemma 2, S has only one idempotent e. Let G be the maximal subgroup of S. For each $g \in G$ and $a \in S$ we have $ag = age$. Since S is weakly commutative, there is an element $w \in S$ and an integer $n > 0$ such that $(ag)^n = gw$. Then $e(ag)^n = egw = gw = (ag)^n$. By Lemma 1, S is right archimedean, hence there is an element u such that $agu = e$. By multiplikation on the right by u it follows from $e(ag)^n = (ag)^n$ and $age = ag$ that $e(ag)^{n-1} = (ag)^{n-1}$. Repeating the argument gives the equation $e(ag) = ag$. We have that eue is the right inverse of ag. Since S ist left archimedean, a similar argument gives that eue is the left inverse of ag and $ag \in G$, then G is a left ideal of S. In a similar way it follows that G is a right ideal. By the archimedean property and Lemma 1 for each element $a \in S$ there is a $v \in S$ and an integer $n > 0$ such that $a^n = ev \in G$. Thus G is an ideal, therefore the Rees factor semigroup S/G is a nilsemigroup.

2) Conversely, suppose that S is an ideal extension of a group G by a nilsemigroup N. Then for each $a, b \in S$ there are intergers $k, l, m \geq 2$ such that $a^k = g_1$, $b^l = g_2$, $(ab)^m = g_3$, $g_1, g_2, g_3 \in G$. Note that $b^l = g_2 = g_1 g_1^{-1} g_2 = a(a^{k-1}) g_1^{-1} b^l$, therefore S is right archimedean. We can prove, in a similar way, that S is left archimedean. Note that $(ab)^m = g_3 = g_2 g_2^{-1} g_3 = b^l g_2^{-1} g_3 = b(b^{l-1} g_2^{-1} g_3)$ and $(ab)^m = g_3 = g_3 g_1^{-1} g_1 = (g_3 g_1^{-1} a^{k-1})a$, therefore S is weakly commutative.

Theorem 4 *A semigroup S is a $\mathcal{H}-$commutative archimedean semigroup with an idempotent iff S is an ideal extension of a group G by a commutative nilsemigroup.*

Proof: Suppose that S is a $\mathcal{H}-$commutative archimedean semigroup with idempotent e. Then for each $a, b \in S$ there are $w, z \in S^1$ such that $ab = baw$, $ba = abz$. Hence $ab = baw = abzw$ and $ab = ab(zw)^n$ for any n. Since the Rees factor semigroup S/G is a nilsemigroup, by Theorem 3, we have $ab = ba$ or ab and $ba \in G$, so S/G is commutative.

Conversely, suppose S is an ideal extension of a commutative nilsemigroup by a group G. Then, by Theorem 3, we have to prove only that S is $\mathcal{H}-$commutative. If $ab \notin G$ for $a, b \in S$ then $ab = ba$. If $ab \in G$ then $ba \in G$ and we have $ab = b(a(ba)^{-1}a)b = (ab(ba)^{-1}b)a$ and S is $\mathcal{H}-$commutative.

Proposition 5 *A weakly commutative archimedean $\Delta-$semigroup with idempotent e is either a $\Delta-$nilsemigroup or a $\Delta-$group.*

Proof: By Theorem 3, S is an ideal extension of a group G by a nilsemigroup N. Assume S is not a group, then $|N| > 1$. Define a relation ρ by $a\rho b$ iff $ae = be$. Then ρ is an equivalence relation which is left compatible and $a\rho b$ implies $ag = bg$ for all $g \in G$. For all $x \in S$, $xe = g \in G$ and therefore $a\rho b$ implies $axe = bxe$, thus ρ is right compatible. The restriction of ρ to G is the equality. By definition of a $\Delta-$semigroup the Rees congruence ρ_G determined by G and the relation ρ are comparable. Assume $|G| > 1$. Then ρ is contained in ρ_G and so ρ is the equality on S/G and therefore ρ is the equality. For $s \notin G$ we have $se = h \in G$. Therefore $se = he$, $s\rho h$, $s = h$, in contradiction to $s \notin G$, $h \in G$. Therefore $|G| = 1$, $G = \{e\}$ and S is a nilsemigroup.

Lemma 6 *If a weakly commutative archimedean semigroup S contains an element a and elements $x, y \in S^1$ with $xy \in S$ such that $a = xay$, then S contains an idempotent.*

Proof: Assume $x, y \in S$. The proof is similar if one of x or y is 1. If $a = xay$ then $a = x^n a y^n$ for all integers $n > 0$. By Lemma 1, S is right and left archimedean. Hence there are elements z, w with $az = x^n$ and $wa = y^n$ for a suitable n. We have $a = azawa$, a is a regular element and $azaw$ is idempotent.

Corollary 7 *A $\mathcal{H}-$commutative archimedean semigroup is commutative or contains an idempotent.*

Proof: If $ab \neq ba$ then there are $z_1, z_2 \in S$ such that $ab = baz_1$, $ba = abz_2$. This implies $ab = abz_2z_1$ and by Lemma 6, S contains an idempotent.

Lemma 8 *In a weakly commutative archimedean semigroup S without idempotent,* Green*'s relation $\mathcal{J}$ is the equality.*

Proof: If $S^1aS^1 = S^1bS^1$ then $a = ubv$, $b = xab$ for some $x, y, u, v \in S^1$. Hence $a = uxayv$. Assume $a \neq b$ then $uv \in S$, $xy \in S$, therefore $uxyv \in S$. By Lemma 6, S contains an idempotent, which is a contradiction.

2 Weakly commutative and $\mathcal{H}-$commutative $\Delta-$semigroups

Proposition 9 *A weakly commutative archimedean $\Delta-$semigroup S without an idempotent is cancellative.*

Proof: In [7] it was shown that if the relation σ is defined by $a\sigma b$ iff $a^{n+1} = a^n b = ba^n$ and $b^{n+1} = b^n a = ab^n$ for some n, then σ is a congruence on a weakly commutative semigroup. We show that in our case σ is the equality. Denote by σ_x the Rees congruence modulo the ideal $I = S^1 x S^1$. σ_x is comparable with σ. Assume $\sigma_x \subseteq \sigma$. Since $x\sigma_x x^2$ we have $x\sigma x^2$ and $x^{n+1} = x^n x^2 = x^{n+2}$ for some n. Then x^{n+1} is idempotent, a contradiction. Therefore we have $\sigma \subset \sigma_x$ for all $x \in S$. If $x\sigma y$ and $x \neq y$ then $y \in S^1 x S^1$ and $x \in S^1 y S^1$, so $y = uxv$, $x = wyz$ and $y = uwyzv$, where $u, v, w, z \in S^1$ and $uvwz \in S$. By Lemma 6, S contains an idempotent, a contradiction, so σ is the equality. Therefore S is separative (in the sense of [6]), archimedean and by [6] cancellative.

Corollary 10 *Any $\mathcal{H}-$commutative archimedean $\Delta-$semigroup S has an idempotent.*

Proof: If S has no idempotent then S is commutative. By [9] or [11] then S has idempotents, a contradiction.

In [7] it was shown that a weakly commutative semigroup is a semilattice of archimedean semigroups, and in [11] it was shown that a $\Delta-$semigroup is either semilattice indecomposable or the set-theoretical union of two semilattice indecomposable semigroups S_0 and S_1 with $S_0 S_1 \subseteq S_0$, $S_1 S_0 \subseteq S_0$, $S_0 \cap S_1 = \emptyset$. In the following we denote by S a weakly commutative semilattice decomposable $\Delta-$semigroup and by S_0, S_1 the archimedean components as above.

Lemma 11

a) *S_1 is a weakly commutative archimedean $\Delta-$semigroup.*

b) *S_0 is weakly commutative and archimedean.*

Proof: a) S_1 is a $\Delta-$semigroup because the Rees factor semigroup S/S_0 is a $\Delta-$semigroup. b) For each $a, b \in S$ there is an integer n and an element $w \in S$ with $(ab)^n = bw$. By multiplikation it follows that $(ab)^{n+1} = bwab$, and S_0 is an ideal, therefore $wab \in S_0$.

Remark. S_0 is not necessary a $\Delta-$semigoup.

Proposition 12 *If S_1 is a nilsemigroup, then $|S_1| = 1$.*

Proof: Let e be the idempotent of S_1. Then $S_0 \cup \{e\}$ is an ideal I of S. The Rees congruence ρ modulo I is comparable with the semilattice congruence. This implies that ρ is the universal ralation and $S_1 = \{e\}$.

Corollary 13 *If S_1 has an idempotent e, then S_1 is a $\Delta-$group.*

Proof: This follows immediately from Proposition 5 , Lemma 11 and Proposition 12.

Proposition 14 *If S_0 contains an idempotent, then S_0 is a nilsemigroup.*

Proof: By theorem 3 S_0 is an ideal extension of a group G_e by a nil semigroup (e is the unit of G_e). G_e is an ideal in S, since $G_e = (G_e)^2$. We denote the REES congruence of S modulo G_e by ρ and define the relation σ as follows

$$a\sigma b \quad \text{iff} \quad ae = be.$$

Then σ is an equivalence relation and is left compatible with multiplikation. For all $x \in S$, xe is an element from G_e, therefore $xe = xee = exe$ and $a\sigma b$ implies $axe = aexe = bexe = bxe$, so σ is right compatible. Suppose $\rho \subseteq \sigma$. For all $x \in S_1$, $xe = xee$ which means that $x\sigma xe$. Then for all $g, h \in G_e$ we have $ge = he = g = h$, $G_e = \{e\}$ and S_0 is a nilsemigroup.

Proposition 15 *If S is $\mathcal{H}-$commutative, then S_1 is a $\Delta-$group and his unit is the unit of S.*

Proof: By Corollaries 10 and 13 S_1 is a $\Delta-$group. Let e be the unit of S_1. Then eSe is an ideal I of S, because S is $\mathcal{H}-$commutative. $e \in S$ and the REES congruence is comparable with the semilattice congruence. Therefore $eSe = S$ and e is the unit of S.

Lemma 16 *If S is $\mathcal{H}-$commutative, then S_0 contains an idempotent.*

Proof: GREEN's relation $\mathcal{H}$ in a $\mathcal{H}-$commutative semigroup is a congruence [14] and S_1 is a group, hence S_1 has only one $\mathcal{H}-$class. If the REES congruence modulo S_0 is comparable with $\mathcal{H}$, then S_0 is a $\mathcal{H}-$class or $|S_1| = 1$. If S_0 is a $\mathcal{H}-$class, then $S^1a = S^1a^2$ for all $a \in S$. Then there exists $s \in S$ with $a = sa^2$ and $y \in S$ with $sa = ay = sa^2y = sasa$ and sa is idempotent. If $|S_1| = 1$ then S_0 is a $\mathcal{H}-$commutative archimedean $\Delta-$semigroup. By Corollary 10, S_0 has an idempotent.

Theorem 17 *Let S be a $\mathcal{H}-$commutative semilattice decomposable $\Delta-$semigroup. Then $S = G^0$, a $\Delta-$group with zero adjoined, or $S = N^1$, a commutative nilsemigroup with unit adjoined.*

Proof: By Proposition 15, S_1 is a group and the identity of S_1 is the identity of S and, by Lemma 16, S_0 contains an idempotent 0 and S_0 is a nilsemigroup [see Proposition 14]. Consider the relation

$$a\tau b \text{ iff there is a } g \in S_1 \text{ with } a = gb.$$

The relation τ is reflexive by Proposition 15, symmetric, transitive and right compatible. For every $x \in S$ there are $u, v \in S = S^1$ with $xg = ugx$, $gx = vgx$. Hence $xg = ugx = uvxg = (uv)^n xg$, and S_0 is nilpotent, therefore u, v can be chosen from S_1, and we have $xa = xgb = ugxb$ with $u \in S_1$, so τ is left compatible. S_1 is a congruence class modulo τ and τ is comparable with the REES congruence modulo S_0. Because S_0 is a nilsemigroup, we have $|S| = 1$ or $S_0 = N$ is a single class modulo τ. In the last case for every $s_0 \in S_0$ there is an $g \in S^1$ with $gs_0 = g^{-1}gs_0 = es_0 = s_0$. Therefore $|S_0| = 1$ and $S = G^0$.
If $S = N^1$ then for $a, b \in N$ there are x and x' with $ab = xba$ and $ba = x'ab$. This implies $ab = xx'ab$ and $ab = (xx')^n ab$. If x or $x' \in S_0$ then $(xx')^n = 0$ for suitable n, $ab = ba = 0$. If x and $x' \in S_1$, then $x = x' = e$ and S_0 is commutative.

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