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Estimates for the resolvent of the Stokes operator with periodic boundary conditions in a layer of R^3

ABSTRACT. In this paper, we study the Stokes operator between two parallel planes of $\mathbb{R}^3$. We impose Dirichlet homogeneous boundary-conditions, and we consider vector fields, with fixed period . We show that the semi group, generated by the Stokes operator, is analytic with respect to the L^p norm under these boundary conditions. This property is closely related to an estimate of the operator's resolvent. For $p = 2$, we prove that this resolvent exists, then that it satisfies the desired estimate. To do this, we use a decomposition principle of [1], splitting the solenoidal vector fields of $\mathbb{R}^3$ into poloidal and toroidal fields and the mean flow. For larger values of p, we show that the estimate, obtained for $p = 2$, extends to larger values of p, in view of a result of [2], valid in a half-space for vectors with prescribed divergence.

Introduction

Analyticity of the semi-group, generated by a linear unbounded operator, can be seen as a consequence of an estimate of its resolvent (see [3]). The Stokes operator with Dirichlet homogeneous boundary-conditions has been proved to satisfy such an estimate with respect to the L^p norm, in a smooth bounded domain [4], and in a half-space [5]. Here we use the work of [2] to show that the Stokes operator is analytic with respect to the L^p norm, in a three-dimensional layer between two planes of $\mathbb{R}^3$, parallel to x_1Ox_2, where homogeneous boundary conditions are prescribed, while periodicity is imposed in the Ox_1 and Ox_2 directions .

We first recall the setting of the problem we want to solve, and then we show that it admits a solution with the help of a decomposition principle of [1]. Then, we derive an estimate on the resolvent of the Stokes operator, with respect to the L^2 norm. We use this result and a similar estimate established in [2] for vectors with prescribed divergence in a half-space to show, via localization, that this property remains true in L^p with $p > 2$. Finally, we consider the problem which is obtained when we ask the flowrate to vanish.

1 Setting of the problem

For $\lambda \in \{\lambda \in \mathbb{C}, |\arg\lambda| \leq \pi - \varepsilon\}$, and $F = (F_1, F_2, F_3) \in (L^p(\Omega_h))^3$ with $h = (h_1, h_2)$ and $\Omega_h =]0, h_1[\times]0, h_2[\times]0, 1[$, let $U = (U_1, U_2, U_3) \in (W^{2,p}_{\#}(\Omega_h))^3$ be such that there exist $q \in W^{1,p}_{\#}(\Omega_h)$ such that

$$\Delta U - \lambda U + \nabla q = F, \qquad \nabla \cdot U = 0, \qquad \text{in } \Omega_h \tag{1.a}$$

$$U = 0, \quad \text{on the planes} \quad x_3 = 0 \quad \text{and} \quad x_3 = 1. \tag{1.b}$$

Here, for $p > 1$, $W^{m,p}_{\#}(\Omega_h)$ (endowed with the $W^{m,p}(\Omega_h)$ norm) denotes the closure of $\mathcal{D}_{\#}(\Omega_h)$ in $W^{m,p}(\Omega_h)$, where $\mathcal{D}_{\#}(\Omega_h)$ is the subset of $\mathcal{C}^\infty(\mathbb{R}^2 \times [0,1])$ made of functions which are $h_i$periodic in the direction of x_i for $i = 1, 2$.

The resolvent $R_{\lambda,p}$ of the Stokes operator in $\Omega = \mathbb{R}^2 \times [0,1]$ with h_1 and h_2 periodic boundary-conditions in the directions of x_1 and x_2 will be defined in $\mathcal{L}((L^p(\Omega_h))^3, (W^{2,p}_{\#}(\Omega_h))^3)$ as soon as the problem (1) will have exactly one solution U, which will be denoted by $R_{\lambda,p}F$. Analycity of the semi-group, generated by the Stokes operator in $(L^p(\Omega_h))^3$ will then be [3] a consequence of the estimate

$$\|\nabla q\|_{(L^p(\Omega_h))} + \|U\|_{(W^{2,p}(\Omega_h))^3} + |\lambda|\|U\|_{(L^p(\Omega_h))^3} \leq K_{\eta,h,p}\|F\|_{(L^p(\Omega_h))^3}, \tag{2}$$

for

$$\lambda \in \Sigma_\eta = \Big\{\Gamma \in \mathbb{C}, \quad \arg\Gamma \in]\arctan\frac{1-\eta}{\eta} - \pi, \arctan\frac{1-\eta}{\eta} + \pi[\Big\}, \tag{3}$$

with $\eta \in]0; 1[$.

Our main result is the theorem 1:

Theorem 1 *With $\Omega_h = [0, h_1] \times [0, h_2] \times [0, 1]$ the resolvent $R_{\lambda,p}$ of the Stokes operator in Ω_h with homogeneous Dirichlet boundary-conditions at $x_3 = 0$ or 1 and prescribed periods h_1 and h_2 along x_1 and x_2 is continuous from $(L^p(\Omega_h))^3$ towards $(W^{2,p}_{\#}(\Omega_h))^3$ for $p \geq 6$, with periodic q. The estimate*

$$\Big(|\lambda|\|R_{\lambda,p}F\|_{(L^p(\Omega_h))^3} + \|R_{\lambda,p}F\|_{(W^{2,p}(\Omega_h))^3} + \|\nabla q\|_{(L^p(\Omega_h))^3}\Big) \leq K_{\eta,h,p}\|F\|_{(L^p(\Omega_h))^3}$$

is satisfied for $\lambda \in \Sigma_\eta$.

Because $\mathcal{D}_{\#}(\Omega_h)$ is dense in $W^{m,p}_{\#}(\Omega_h)$, and in view of the inclusions $W^{m,p}(\Omega_h) \subset W^{m-1,p}(\Omega_h)$, we only have to consider $F \in \mathcal{D}_{\#}(\Omega_h)$. The proof of this theorem is based upon the construction of a solution U of (1), following ideas of [1]. Estimate (2) for $p = 2$ follows, and will later on be extended to other values of p. To do this, we will need properties of the solution u to (4):

$$\Delta^2 u - \lambda \Delta u = f \tag{4.a}$$

with the boundary-condition

$$u(x_1, x_2, 0) = u(x_1, x_2, 1) = 0 = \partial_{x_3} u(x_1, x_2, 0) = \partial_{x_3} u(x_1, x_2, 1). \tag{4.b}$$

In a first step, we solve (4) and derive estimates for u when $\mathrm{Re}\lambda > 0$. We pay special attention to the case

$$f = \Delta_2 F_3 - \partial^2_{x_1 x_3} F_1 - \partial^2_{x_2 x_3} F_2$$

where $\Delta_2 = \partial^2_{x_1^2} + \partial^2_{x_2^2}$. Then, we construct the solution U to (1), and arrive to (2) for $p = 2$ and $\mathrm{Re}\lambda > 0$. After that, we extend this property to a larger sector of $\mathbb{C}$. Then, we will be ready to use a result of [2] to establish (2) for $p \geq 6$ via localization.

2 Existence of the solution to (4) in $L^2(\Omega_h)$ when the real part of λ is positive

Let us show, via a variational method, that (4) admits a solution.

2.1 Variational problem

Let f be an element of $(\mathcal{D}_{\#}(\Omega_h))^3$. Consider the variational problem: find v in $\mathcal{V}$ such that for every ψ in $\mathcal{V}$,

$$(\Delta v | \Delta \psi) + \lambda((\nabla v | \nabla \psi)) = (f | \psi). \tag{5}$$

Here $(\ \ |\ \ )$ and $((\ \ |\ \ ))$ denote the scalar products of $L^2(\Omega_h)$ and $(L^2(\Omega_h))^3$, while

$$\mathcal{V} = \Big\{ v \in W^{2,p}_{\#}(\Omega_h) / v(x_1, x_2, 0) = v(x_1, x_2, 1) = 0 = \partial_{x_3} v(x_1, x_2, 0) = \partial_{x_3} v(x_1, x_2, 1) \Big\}. \tag{6}$$

The bilinear form on the left hand-side of (5) is continuous and coercive on $\mathcal{V}$, which itself is dense in $L^2(\Omega_h)$.The Lax-Milgram theorem implies there exists $v \in \mathcal{V}$ satisfying (5). Moreover, the mapping $b_v : \psi \mapsto (\Delta v | \Delta \psi) + \lambda((\nabla v | \nabla \psi))$ is continuous on $\mathcal{V}$ endowed with the $L^2(\Omega_h)$ norm. In view of the Weyl lemma, $v \in \mathcal{C}^\infty(\Omega_h)$. These properties imply more regularity for v.

2.2 Regularity of v

We have:

Lemma 1 *When $f \in \mathcal{D}_\#(\Omega_h)$ and with* $\mathrm{Re}(\lambda) > 0$, *the solution v of* (5) *satisfies*

$$\Delta v(0, x_2, x_3) - \Delta v(h_1, x_2, x_3) = 0 = \Delta v(x_1, 0, x_3) - \Delta v(x_1, h_2, x_3) \qquad (7.a)$$

$$\partial_{x_1} \Delta v(0, x_2, x_3) - \partial_{x_1} \Delta v(h_1, x_2, x_3) = 0 = \partial_{x_2} \Delta v(x_1, 0, x_3) - \partial_{x_2} \Delta v(h_2, h_2, x_3). \qquad (7.b)$$

This lemma is proved in Appendix A.

Integration by parts of (5) is now possible, and gives that v solves (4) with $|\lambda| \|v\|_{L^2(\Omega_h)} + \|\Delta v\|_{L^2(\Omega_h)} \leq \|f\|_{L^2(\Omega_h)}$.

Definition *Let X be a Banach space, made of functions of $x_3 \in]0,1[$. With $\Omega'_h =]0, h_1[\times]0, h_2[$, let us denote by $W^{m,p}_\#(\Omega'_h, X)$ the closure in $W^{m,p}(\Omega'_h, X)$ of the subset of the smooth functions, which are h_i periodic with respect to x_i, for $i = 1, 2$.*

When $f = \Delta_2 F_3 - \partial^2_{x_1 x_3} F_1 - \partial^2_{x_2 x_3} F_2$, with $(F_1, F_2, F_3) \in (\mathcal{D}_\#(\Omega_h))^3$, we have the following lemma.

Lemma 2 *When $f \in \mathcal{D}_\#(\Omega_h)$ and with* $\mathrm{Re}(\lambda) > 0$, *the problem* (4) *has one solution u in $W^{2,2}_\#(\Omega_h) \cap \mathcal{C}^\infty(\Omega_h)$. The solution satisfies*

$$\int_0^{h_1} \int_0^{h_2} u(x_1, x_2, x_3) dx_1 dx_2 \equiv < u >_{x_3} = 0, \qquad (8.a)$$

$$\Delta u \in W^{2,2}_\#(\Omega'_h, \mathcal{C}^\infty(]0,1[)), \qquad (8.b)$$

$$< \Delta u >_{x_3} = 0. \qquad (8.c)$$

Proof: The boundary conditions for $u = v$, and (7), imply that $\partial^4_{x_3^4} < u >_{x_3} - \lambda \partial^2_{x_3^2} < u >_{x_3} = 0$ in $]0,1[$ with $\partial_{x_3} < u >_{x_3} = < u >_{x_3} = 0$ at $x_3 = 0$ or 1. The scalar product of this o.d.e with $< u >_{x_3}$ in $[0,1]$ gives $(8.a)$. Then $(8.b)$ is a consequence of (7) and of the regularity of v, while the boundary conditions give $(8.c)$.

2.3 Resolvent of the Laplace operator

We will need similar results for the problem

$$(\lambda Id - \Delta) g = \partial_{x_2} F_1 - \partial_{x_1} F_2, \qquad (9.a)$$

$$g(x_1, x_2, 0) = g(x_1, x_2, 1) = 0. \qquad (9.b)$$

Lemma 3 *When the real part of λ is positive, with $F \in (\mathcal{D}_{\#}(\Omega_h))^3$, the problem (9) has one solution $g \in W^{2,2}_{\#}(\Omega_h)$ with $< g >_{x_3} = 0$. Moreover, $g \in \mathcal{C}^\infty(\Omega_h)$.*

The proof follows from arguments, similar to the ones for the lemmas 1 and 2.

3 Construction of a solution to (1) with $p = 2$ when the real part of λ is positive

Following an idea of [1], let us set $U = (U_1, U_2, U_3)$

$$U_1 = S_1(x_3) - \partial_{x_2}\psi + \partial^2_{x_1 x_3}\varphi \tag{10.a}$$

$$U_2 = S_2(x_3) + \partial_{x_1}\psi + \partial^2_{x_2 x_3}\varphi \tag{10.b}$$

$$U_3 = -\Delta_2 \varphi \tag{10.c}$$

where ψ, φ, S_1 and S_2 will be defined later on such that U is a solution to $(1.a)$. In order to establish that ψ and φ are smooth enough, we will use the property 1.1 of [1]. It states that for given $\chi \in W^{m,2}_{\#}(\Omega'_h)$ with $< \chi >_{x_3} \equiv \int\int_{\Omega'_h} \chi(x_1, x_2, x_3) dx_1 dx_2 = 0$, there exist $\zeta \in W^{m+2,2}_{\#}(\Omega'_h)$ such that $\Delta_2 \zeta \equiv (\partial^2_{x_1^2} + \partial^2_{x_2^2})\zeta = \chi$ and $< \zeta >_{x_3} = 0$. Moreover, $\|\zeta\|_{W^{m+2,2}_{\#}(\Omega'_h)} \leq \mathcal{K}(m,h)\|\chi\|_{W^{m,2}_{\#}(\Omega'_h)}$. Assume χ is a $\mathcal{C}^1$ function of $x_3 \in]0,1[$, with vanishing average over Ω'_h for every x_3: this proposition implies $\zeta \in W^{m+2,2}_{\#}(\Omega'_h, \mathcal{C}^1]0,1[)$.

3.1 Construction of ψ

For given $F \in \mathcal{D}_{\#}(\Omega_h)$, take $g \in W^{2,2}_{\#}(\Omega_h)$ such that (9) is satisfied: $g \in W^{2,2}_{\#}(\Omega'_h, \mathcal{C}^\infty(]0,1[)$, according to lemma 3. The proposition 1.1 of [1] implies there is one $\psi \in W^{4,2}_{\#}(\Omega'_h, \mathcal{C}^\infty(]0,1[)$ defined by

$$\Delta_2 \psi = g \tag{11.a}$$

$$< \psi >_{x_3} = 0 \quad \forall x_3 \in [0,1]. \tag{11.b}$$

Recall that $g \in \mathcal{C}^\infty(\Omega_h)$: $\psi \in \mathcal{C}^\infty(\Omega_h)$, therefore $\Delta_2\Delta\psi = \Delta g$. The boundary-conditions for ψ imply $< \Delta\psi >_{x_3} = 0$ for every x_3, and $\Delta\psi \in W^{2,2}_{\#}(\Omega'_h, \mathcal{C}^\infty(]0,1[)$.

3.2 Construction of φ

Similarly, take u defined in $W^{2,2}_{\#}(\Omega_h)$ by (4), with $f = \Delta_2 F_3 - \partial^2_{x_1x_3}F_1 - \partial^2_{x_2x_3}F_2$. Recall $< u >_{x_3} = 0$ in $[0,1]$: the proposition 1.1 of [1] allows to define $\varphi \in W^{4,2}_{\#}(\Omega'_h, \mathcal{C}^\infty(]0,1[)$ by

$$\Delta_2\varphi = -u \tag{12.a}$$

$$< \varphi >_{x_3} = 0 \quad \forall x_3 \in]0,1[. \tag{12.b}$$

Recall that $u \in \mathcal{C}^\infty(\Omega_h)$: $\varphi \in \mathcal{C}^\infty(\Omega_h)$, therefore $\Delta_2\Delta\varphi = \Delta u$. The boundary-conditions for φ imply $< \Delta\varphi >_{x_3} = 0$ for every x_3, hence $\Delta\varphi \in W^{4,2}_{\#}(\Omega'_h, \mathcal{C}^\infty(]0,1[)$.

3.3 U solves (1)

Now let S_1 and S_2 be functions of x_3 such that

$$\partial^2_{x_3^2}S_j - \lambda S_j = < F_j >_{x_3} \quad \text{for} \quad j = 1,2, \tag{13.a}$$

$$S_j(0) = S_j(1) = 0. \tag{13.b}$$

In view of the regularity of F, ψ and φ, $U \in (\mathcal{C}^\infty(\Omega_h))^3$.

Let us set

$$w_1 = F_1 - \partial^2_{x_3}S_1 + \lambda S_1 + \partial_{x_2}\Delta\psi - \lambda\partial_{x_2}\psi + \lambda\partial^2_{x_1x_3}\varphi - \partial^2_{x_1x_3}\Delta\varphi \tag{14.a}$$

$$w_2 = F_2 - \partial^2_{x_3}S_2 + \lambda S_2 - \partial_{x_1}\Delta\psi + \lambda\partial_{x_1}\psi + \lambda\partial^2_{x_2x_3}\varphi - \partial^2_{x_2x_3}\Delta\varphi \tag{14.b}$$

$$w_3 = F_3 + (\Delta - \lambda Id)\Delta_2\varphi. \tag{14.c}$$

Then $w = (w_1, w_2, w_3)$ belongs to $(\mathcal{C}^\infty(\Omega_h))^3$ and for every vector field $\mathcal{V} \in (\mathcal{D}(\Omega_h))^3$ such that $\nabla \cdot \mathcal{V} = 0$, $\mathcal{V}$ is equal in Ω_h to a periodic vector field and can be put into the form

$$(-\partial_{x_2}\Psi + \partial^2_{x_1x_3}\Phi + \mathcal{F}_1(x_3), \partial_{x_1}\Psi + \partial^2_{x_2x_3}\Phi + \mathcal{F}_2(x_3), -\Delta_2\Phi) \tag{14}$$

with periodic Ψ and Φ such that $< \Psi >_{x_3} = < \Phi >_{x_3} = 0$, in view of [1, theorem 1.4]. Moreover, the boundary-conditions for $\mathcal{V}$ imply $\Phi = \partial_{x_3}\Phi = \Psi = 0$ at $x_3 = 0,1$. Therefore, integrating by parts gives

$$\int\int\int_{\Omega_h} w\overline{\mathcal{V}}dx_1dx_2dx_3 = 0. \tag{15}$$

In view of [6, prop. 1.1 p. 14] the existence of $q \in \mathcal{C}^\infty(\Omega_h)$ such that $w = \nabla q$ follows. In view of [1, lemma 1.3 p. 296] curl$(w) = 0$. Moreover, $< w_1 >_{x_3} = < w_2 >_{x_3} = 0$. Therefore using Fourier series, one gets $q \in W^{1,2}_{\#}(\Omega_h)$. Therefore U and q solve (1.a).

Moreover, $U = 0$ on the planes $x_3 = 0$, respectively 1, because $\Delta_2\partial_{x_3}\varphi = \partial_{x_3}\Delta_2\varphi = \partial_{x_3}u$ in the distributional sense, and then $\partial_{x_3}\varphi = 0$, while $g = 0$ implies $\psi = 0$ when $x_3 = 0$ or 1. Unicity of the solution of (1) such that $q \in W^{1,2}_{\#}(\Omega_h)$ is given by the L^2 scalar product of (1.a) by U.

3.4 Estimate for (U, q), when $\mathrm{Re}(\lambda) > 0$

The scalar product of (1) by U gives

$$\sqrt{|\lambda|}\|\nabla U\|_{L^2(\Omega_h)^3} + |\lambda|\|U\|_{L^2(\Omega_h)^3} < 2\|F\|_{L^2(\Omega_h)^3}.$$

Therefore, $\|\Delta U\|_{W^{-1,2}(\Omega_h)} \leq \frac{2}{\sqrt{\lambda}}\|F\|_{L^2(\Omega_h)^3}$, which implies $\|\nabla q\|_{W^{-1,2}(\Omega_h)} \leq 2/\sqrt{|\lambda|}+2\|F\|_{L^2(\Omega_h)^3}$. In view of [6, prop. 1.2 p. 14], $\|q\| \leq C'(h,\eta)\|F\|_{L^2(\Omega_h)^3}$.

The regularity of U implies that $\nabla \cdot \Delta U = \Delta(\nabla \cdot U) = 0$. Therefore, the scalar product of (1.a) by ΔU gives

$$\|\Delta U\|_{L^2(\Omega_h)^3} < \|F\|_{L^2(\Omega_h)^3},$$

and the lemma follows.

Lemma 4 *For every λ with positive real part, and every $F \in (\mathcal{D}_{\#}(\Omega_h))^3$,the problem* (1) *has one solution $(U, q) \in (W^{2,2}_{\#}(\Omega_h))^3 \times W^{1,2}_{\#}(\Omega_h)$, with*

$$\|U\|_{W^{2,2}(\Omega_h)^3} + |\lambda|\|U\|_{L^2(\Omega_h)^3} + \|\nabla q\|_{L^2(\Omega_h)^3} < C(h,\eta)\|F\|_{L^2(\Omega_h)^3}. \tag{16}$$

This result extends to $F \in (L^2(\Omega_h))^3$, because $\mathcal{D}_{\#}(\Omega_h)$ is dense in the $W^{p,2}_{\#}(\Omega_h)$. Later on, $\mathcal{R}_{\lambda,2}$ will denote the linear mapping $F \mapsto U$ of $(L^2(\Omega_h))^3$ such that (1) is satisfied with $U \in (W^{2,2}_{\#}(\Omega_h))^3$ and $q \in (W^{1,2}_{\#}(\Omega_h))^3$. The estimate (16) of the resolvent of the Stokes operator will now be extended to $\lambda \in \Sigma_\eta$.

4 Estimate of U in $(L^2(\Omega_h))^3$, for other values of λ

Suppose first that $\lambda \in \mathbb{C}$ is such that $0 \leq -\mathrm{Re}(\lambda) \leq |\mathrm{Im}(\lambda)|\eta$, $\eta \in]0,1[$. Set $\lambda = \lambda_1 + a$ with $\mathrm{Re}(\lambda_1) > 0$ and real negative a such that $|a| < \mathrm{Im}(\lambda)$. Then (1) is equivalent to

$$U - a\mathcal{R}_{\lambda_1,2}U = \mathcal{R}_{\lambda_1,2}F. \tag{17}$$

Equation (17) admits a solution (Neumann series) in $(L^2(\Omega_h))^3$ in view of the estimate (16) for $\mathcal{R}_{\lambda_1,2}$. The solution satisfies:

$$\|U\|_{(L^2(\Omega_h))^3} \leq \frac{1}{|\mathrm{Im}\lambda_1|}\|F\|_{(L^2(\Omega_h))^3} \tag{18.a}$$

$$\frac{1}{|\mathrm{Im}\lambda_1|}\|F\|_{(L^2(\Omega_h))^3} \leq \frac{\sqrt{2}}{|\lambda|}\|F\|_{(L^2(\Omega_h))^3}\ , \tag{18.b}$$

as can be seen from the imaginary part of the $(L^2(\Omega_h)^3$ scalar product of (1) by U. To obtain (18.b), we use

$$|\lambda|^2 = |\lambda_1 + a|^2 = |\mathrm{Im}(\lambda_1)|^2 + (\mathrm{Re}(\lambda_1) + a)^2 = |\mathrm{Im}(\lambda_1)|^2 + \mathrm{Re}(\lambda)^2 \leq 2(\mathrm{Im}(\lambda))^2.$$

Consider now the case $\mathrm{Re}(\lambda) < 0$ with $|\mathrm{Re}(\lambda)| \leq (\eta/(1-\eta))|\mathrm{Im}(\lambda)|$. Following the same procedure as in the previous case, and using (18.a), we see that (1) admits a solution when $\lambda \in \Sigma_\eta$. Indeed, with $\lambda = \lambda_2 - ie$ where the real number e has the same sign as $\mathrm{Im}(\lambda)$ with $|\mathrm{Im}(\lambda) + e| > |\mathrm{Re}(\lambda)|/\eta$, the problem (1) can be written:

$$U + ie\mathcal{R}_{\lambda_2,2}U = \mathcal{R}_{\lambda_2,2}F.$$

It admits a solution because $e < |\mathrm{Im}(\lambda_2)|$. Then, imaginary part of the L^2 scalar product of (1.a) by U gives:

$$|\lambda|\frac{1-\eta}{\eta}||U||_{(L^2(\Omega_h))^3} \leq \mathrm{Im}(\lambda)||U||_{(L^2(\Omega_h)^3} \leq ||F||_{(L^2(\Omega_h)^3)}, \qquad (19.a)$$

hence

$$|\lambda|\|U\|_{(L^2(\Omega_h))^3} \leq C''(h,\eta)\|F\|_{(L^2(\Omega_h)^3)},$$

$$\|\nabla U\|_{(L^2(\Omega_h))^3} \leq C''(h,\eta)\|F\|_{(L^2(\Omega_h)^3)},$$

and

$$\|q\|_{(L^2(\Omega_h)^3)} \leq C''(h,\eta)\|F\|.$$

The last estimates allow to use via localization, alike in the section 5.1, [6, prop. 2.2 p. 33], which gives

$$\|U\|_{(W^{2,2}(\Omega_h))^3} \leq C(h,\eta)\|F\|_{(L^2(\Omega_h))^3}. \qquad (19.b)$$

So (16) extends to every $\lambda \in \Sigma_\eta$. Notice that Σ_η becomes larger as η approaches 1 from the left. This implies that the conclusion of the lemma 4 is still valid when $\lambda \in \Sigma_\eta$.

5 Estimate of U in L^p, with $p \geq 6$

Now we have to prove that this result remains true in $(L^p(\Omega_h))^3$ with large values of p. Recall that [4] and [5] have established a similar property for bounded domains, and for a half-space. Localization is a method, allowing to use results like their's: it consists in considering a solution of (1), and building from it a solution to the same problem, but in a domain, for which the wanted estimate is already established. But it is then necessary to consider that $\nabla \cdot U$ can be different from zero in the Stokes system (1). This is not the case with [4] and [5]. Fortunately, [2] gives an estimate of the resolvent of the Stokes operator, which is true for vectors with prescribed divergence. Our L^2 estimate will enable us to use their theorem by allowing us to control the non homogeneous term, which the localization process introduces in the continuity equation.

5.1 Extension of U to a solution of the Stokes operator in a half-space

As soon as $F \in (\mathcal{D}_{\#}(\Omega_h))^3$, U is in $(\mathcal{C}^{\infty}(\Omega_h))^3$. For every $(m,n) \in \mathbb{Z}^2$, let us call $\tau_{n,m}$ the translation by nh_1 respectively mh_2 in the x_1, respectively x_2 directions.

Let $\tilde{P}U$ be the extension of U in $\mathbb{R}^2 \times [0,1]$ giving a function, with period h_1 in the x_1 direction, and period h_2 in the x_2 direction. Let $\mathcal{P}'q$ be the periodic extension of q in the directions of x_1 and x_2.

Let PU be the extension of $\tilde{P}U$ to $\mathbb{R}^2 \times [0,2]$ such that the first and second components are even with respect to $z-1$ while the third one is odd: the divergence of PU vanishes. Let us call $\mathcal{P}q$ the extension of $\mathcal{P}'q$ which is even with respect to $z-1$. Let $\chi \in \mathcal{D}(\mathbb{R}^3)$ take the value 1 in Ω_h but vanish everywhere $|x_1| \geq 2h_1$ or $|x_2| \geq 2h_2$ or $x_3 \geq 2$. Then, χPU and $\chi \mathcal{P}q$ are in $(W^{2,2}(\mathbb{R}^3_+))^3$ respectively $W^{1,2}(\mathbb{R}^3_+)$ and satisfy

$$\Delta(\chi PU) - \lambda \chi PU + \nabla(\chi \mathcal{P}q) = \chi PF + 2(\nabla\chi\nabla)PU) + \mathcal{P}q\nabla\chi \tag{20.a}$$

$$\nabla.\chi PU = -\nabla\chi.PU. \tag{20.b}$$

The linear mappings χP and $\chi\mathcal{P}$ are bounded in $\mathcal{L}((W^{2,p}_{\#}(\Omega_h))^3, (W^{2,p}(\mathbb{R}^3_+))^3)$, respectively $\mathcal{L}((W^{1,p}_{\#}(\Omega_h))^3, (W^{1,p}(\mathbb{R}^3_+))^3)$ for every p. And the lemma 4 implies that the $L^2(\mathbb{R}^3_+)$ norms of $\Delta(\chi PU)$, $\nabla(\chi PU)$, $\chi\mathcal{P}q$ and $\lambda\chi PU$ are dominated by the $(L^2(\Omega_h))^3$ norm of F. This allows to use a $L^p(\mathbb{R}^n_+)$ estimate for the solution of the inhomogeneous Stokes problem

$$\Delta V - \lambda V + \nabla\tilde{q} = \tilde{g} \tag{21.a}$$

$$\nabla \cdot V = \tilde{G}. \tag{21.b}$$

Indeed, [2, th. 3.1] states that (21) implies

$$\begin{aligned} &\|\lambda V\|_{(L^p(\mathbb{R}^3_+))^3} + \sqrt{|\lambda|}\|V\|_{(W^{1,p}(\mathbb{R}^3_+))^3} + \|V\|_{(W^{2,p}(\mathbb{R}^3_+))^3} + \|\nabla\tilde{q}\|_{(L^p(\mathbb{R}^3_+))^3} \leq \\ &\qquad \kappa(\|\tilde{g}\|_{(L^p(\mathbb{R}^3_+))^3} + \|\nabla\tilde{G}\|_{(L^p(\mathbb{R}^3_+))^3} + \|\lambda\tilde{G}\|_{(\hat{W}^{-1,p,*}(\mathbb{R}^3_+))^3}), \end{aligned} \tag{22}$$

where $\hat{W}^{-1,p,*}(\mathbb{R}^3_+)$ is defined in [2].

5.2 Estimates for U

In what follows, for a sake of simplicity, and except when more precise notations will be useful, all constants, depending only on h, η and p, will be denoted by κ.

In view of [2, remark 3.1], gives

$$\|\nabla\chi.PU\|_{(\hat{W}^{-1,6,*}(\mathbb{R}^3_+))^3} \leq \kappa\|\nabla\chi.PU\|_{L^2(\mathbb{R}^3_+)^3} \leq \kappa\|U\|_{(L^2(\Omega_h))^3}. \tag{23}$$

In view of Sobolev inequalities and of the already established L^2 estimates for ∇U and q, and because Ω_h is bounded, [2, th. 3.1] implies that

$$\|U\|_{(W^{2,6}(\Omega_h))^3} + \|\nabla q\|_{(L^6(\Omega_h))^3} + |\lambda|\|U\|_{(L^6(\Omega_h))^3} \leq \kappa(\|F\|_{(L^6(\Omega_h))^3}). \tag{24}$$

By the Holder inequality, for $r = 2\theta + (1-\theta)$ with θ between 0 and 1, $r \in]2, 6[$ and

$$\|\nabla\chi.PU\|_{(L^r(\mathbb{R}^3_+))^3} \leq \kappa\|U\|_{(L^r(\Omega_h))^3} \leq \kappa\|U\|^{2\theta/r}_{(L^2(\Omega_h))^3}\|U\|^{6(1-\theta)/r}_{(L^6(\Omega_h))^3} \leq \kappa\|U\|_{(L^6(\Omega_h))^3}. \tag{25}$$

After that, we use [2, remark 3.1] again, which states that for $r \in]2, 3[$, with $p = \frac{3r}{3-r}$,

$$\|\nabla\chi.PU\|_{(\hat{W}^{-1,p,*}(\mathbb{R}^3_+))^3} \leq \kappa\|\nabla\chi.PU\|_{L^r(\mathbb{R}^3_+)^3} \leq \kappa\|U\|_{(L^6(\Omega_h))^3}. \tag{26}$$

Consider that every $p > 6$ is in the form $3r/(3-r)$ with $r \in]2, 3[$. Recall that, because of Sobolev inequalities,

$$\|\nabla\chi.PU\|_{(W^{1,p,}(\mathbb{R}^3_+))^3} \leq \kappa\|U\|_{(W^{1,p}_{\#}(\Omega_h))^3} \leq \kappa\|U\|_{(\mathcal{C}^1(\Omega_h))^3} \leq \kappa\|U\|_{(W^{6,2}(\Omega_h))^3}. \tag{27}$$

In view of [2, th. 3.1],

$$\|U\|_{(W^{2,p}(\Omega_h))^3} + \|\nabla q\|_{(L^p(\Omega_h))^3} + |\lambda|\|U\|_{(L^p(\Omega_h))^3} \leq \kappa\|F\|_{(L^p(\Omega_h))^3} \tag{28}$$

for $p > 6$.

This estimate extends to $F \in (L^p(\Omega_h))^3$, alike in the section 3.4. Hence the theorem 1.

6 Further condition for the flowrate

Sometimes, it is useful to consider solutions of (1) with vanishing fluxes through the surfaces $x_1 = 0$ and $x_2 = 0$. This is the case, for instance, when natural thermal convection is studied.

From the solution (U, q) of (1) in $(W^{2,p}_{\#}(\Omega_h))^3 \times W^{1,p}_{\#}(\Omega_h)$, it is possible to build a solution (U', q') to (1) in $(W^{2,p}_{\#}(\Omega_h))^3 \times W^{1,p}(\Omega_h)$, satisfying the condition

$$\int_{x_2=0}^{h_2}\int_{x_3=0}^{1} U'_1(x_1, x_2, x_3)dx_2dx_3 = \int_{x_1=0}^{h_1}\int_{x_3=0}^{1} U'_2(x_1, x_2, x_3)dx_1dx_3 = 0. \tag{29}$$

With λ' in the half complex plane where $\mathrm{Re}(\lambda') > 0$, let us set $\lambda = \lambda'^2$,
$\mathcal{H}_\lambda(x_3) = (\frac{\sinh\lambda' z + \sinh\lambda'(1-z)}{\sinh\lambda'} - 1)/\lambda$, $\gamma(\lambda) = (2\frac{\cosh\lambda' - 1}{\lambda'\sinh\lambda'} - 1)/\lambda$,

$$\mathcal{W}(x_3) = \frac{\mathcal{H}_\lambda(x_3)}{\gamma(\lambda)}(h_1\int_0^1 S_1(x_3)dx_3, h_2\int_0^1 S_2(x_3)dx_3, 0) \tag{30}$$

and

$$Q(x_1, x_2) = -\frac{1}{\gamma(\lambda)}(h_1 x_1 \int_0^1 S_1(x_3)dx_3 + h_2 x_2 \int_0^1 S_2(x_3)dx_3). \tag{31}$$

Then, $U' = U - \mathcal{W}$ and $q' = q - Q$ satisfy (1). Notice that S_1 and S_2 depend continuously on $F \in (L^p(\Omega_h))^3$: therefore, (U', q') satisfies (28) for large enough $|\lambda|$ because $|\lambda\gamma(\lambda)| \to 1$ when $|\lambda| \to \infty$. Moreover, U' satisfies (29), but q' is no longer periodic. Hence an other version of the theorem 1.

Theorem 2 *Let us consider the Stokes operator in Ω_h with homogeneous Dirichlet boundary - conditions at $x_3 = 0$ or 1, prescribed periods along x_1, x_2, and vanishing flowrates through the surfaces, parallel to x_1Ox_3 and x_2Ox_3. The resolvent $R^f_{\lambda,p}$ is continuous from $(L^p(\Omega_h))^3$ towards $(W^{2,p}_{\#}(\Omega_h))^3$ for $p \geq 6$, and $\lambda \in \Sigma_\eta$ for large enough $|\lambda|$. Moreover, the estimate*

$$|\lambda| \|R^f_{\lambda,p}F\|_{(L^p(\Omega_h))^3} + \|R^f_{\lambda,p}F\|_{(W^{2,p}(\Omega_h))^3} \leq K_{\eta,h,p}\|F\|_{(L^p(\Omega_h))^3}$$

is satisfied.

Appendix A

Proof of $(7.a)$: Suppose that for $x_2^0 \in]0; h_2[$ and $x_3^0 \in]0, 1[$, $(9.a)$ is not satisfied. For instance, $\Delta v(0, x_2^0, x_3^0) - \Delta v(h_1, x_2^0, x_3^0) > 0$. Let us construct functions $\psi_n \in \mathcal{V}$ such that $(\Delta v|\Delta\psi_n)$ is less than a negative number while $\psi_n \to 0$ in $W^{0,2}_{\#}(\Omega_h)$, and $((\nabla v|\nabla\psi_n)) \to 0$. This contradicts the continuity of b_v and gives $(7.a)$.

Let χ be a piecewise $\mathcal{C}^2(\mathbb{R})$ odd function, taking non negative values on $[0, h_1/4]$ with $\chi'(0) = 1$, and with support in $I = [-h_1/4, h_1/4]$; here χ' denotes the derivative of χ. Let ζ be a $\mathcal{C}^\infty(\mathbb{R})$ even function, taking non negative values on I, and with support in this interval. We have $\int_0^{h_1/4} \chi''(s)ds = -1$ and $\int_{-h_1/4}^{h_1/4} \zeta''(s)ds = 0$. Let ψ_n be defined in

$$R_{k,l}(n) = [-\frac{h_1}{4n} + kh_1, \frac{h_1}{4n} + kh_1] \times [x_2^0 - \frac{h_1}{4n} + lh_2, x_2^0 + \frac{h_1}{4n} + lh_2] \times [x_3^0 - \frac{h_1}{4n}, x_3^0 + \frac{h_1}{4n}]$$

by $\psi_n(x_1, x_2, x_3) = n\chi(n(x_1 - kh_1))\zeta(n(x_2 - lh_2 - x_2^0))\zeta(n(x_3 - x_3^0))$,

and vanishing out of $\cup_{(k,l)\in\mathbb{Z}^2} R_{k,l}(n)$.

For large n, $\psi_n \in \mathcal{V}$, and $\psi_n \to 0$ in $W^{0,2}_{\#}(\Omega_h)$ when $n \to \infty$. Integration by parts shows that $((\nabla v|\nabla\psi_n)) \to 0$. Moreover,

$$\Delta\psi_n(x_1, x_2, x_3) =$$

$$n^3\chi''(n(x_1 - kh_1))\zeta(n(x_2 - x_2^0))\zeta(n(x_3 - x_3^0)) + n^3\chi(n(x_1 - kh_1))\zeta''(n(x_2 - x_2^0))\zeta(n(x_3 - x_3^0))$$
$$+ n^3\chi(n(x_1 - kh_1))\zeta(n(x_2 - x_2^0))\zeta''(n(x_3 - x_3^0))$$

when $(x_1,x_2,x_3) \in R_{k,l}$, $\Delta\psi_n(x_1,x_2,x_3) = 0$ elsewhere. Hence with

$$R^+_{k,l}(n) = [0, \frac{h_1}{4n} + kh_1] \times [x_2^0 - \frac{h_1}{4n} + lh_2, x_2^0 + \frac{h_1}{4n} + lh_2] \times [x_3^0 - \frac{h_1}{4n}, x_3^0 + \frac{h_1}{4n}],$$

$$(\Delta v|\Delta\psi_n) = \int\int\int_{R^+_{0,0}(n)} [\Delta v]^{(x_1,x_2,x_3)}_{(h_1+x_1,x_2,x_3)} \Delta\psi_n(x_1,x_2,x_3)dx_1dx_2dx_3 =$$

$$\int\int\int_{R^+_{0,0}(n)} [\Delta v]^{(x_1,x_2,x_3)}_{(h_1+x_1,x_2,x_3)} n^3\chi''(nx_1)\zeta(n(x_2-x_2^0))\zeta(n(x_3-x_3^0))dx_1dx_2dx_3$$

$$+\int\int\int_{R^+_{0,0}(n)} [\Delta v]^{(x_1,x_2,x_3)}_{(h_1+x_1,x_2,x_3)} n^3\chi(nx_1))\zeta''(n(x_2-x_2^0))\zeta(n(x_3-x_3^0))dx_1dx_2dx_3$$

$$+\int\int\int_{R^+_{0,0}(n)} [\Delta v]^{(x_1,x_2,x_3)}_{(h_1+x_1,x_2,x_3)} n^3\chi(nx_1))\zeta(n(x_2-x_2^0))\zeta''(n(x_3-x_3^0))dx_1dx_2dx_3.$$

But

$$\int\int\int_{R^+_{0,0}(n)} [\Delta v]^{(x_1,x_2,x_3)}_{(h_1+x_1,x_2,x_3)} n^3\chi''(nx_1)\zeta(n(x_2-x_2^0))\zeta(n(x_3-x_3^0))dx_1dx_2dx_3 =$$

$$[\Delta v]^{(0,x_2^0,x_3^0)}_{(h,x_2^0,x_3^0)} \int_0^{\frac{h_1}{4n}} n\chi''(nx)dx \int_{x_2^0-\frac{h_1}{4n}}^{x_2^0+\frac{h_1}{4n}} n\zeta(n(y-x_2^0))dy \int_{x_3^0-\frac{h_1}{4n}}^{x_3^0+\frac{h_1}{4n}} n\zeta(n(z-x_3^0))dz +$$

$$\int\int\int_{R^+_{0,0}(n)} \varepsilon_n(x_1,x_2,x_3)n^3\chi''(nx_1)\zeta(n(x_2-x_2^0))\zeta(n(x_3-x_3^0))dx_1dx_2dx_3,$$

with ε_n defined by

$$\varepsilon_n(x_1,x_2,x_3) = \Delta v(x_1,x_2,x_3) - \Delta v(x_1+h_1,x_2,x_3) - \Delta v(0,x_2^0,x_3^0) + \Delta v(h_1,x_2^0,x_3^0)$$

in $R^+_{0,0}(n)$. In view of the smoothness of v, $\varepsilon_n \to 0$ in $\mathcal{C}^0(R^+_{0,0}(n))$ when $n \to 0$. Recalling that $\int_0^{\frac{h_1}{4n}} n\chi''(nx_1)dx_1 = -1$ and $\int_{-\frac{h_1}{4n}}^{\frac{h_1}{4n}} n\zeta(n(z))dz = 1$, and that $\int_0^{\frac{h_1}{4n}} n \mid \chi''(nx) \mid dx$ and $\int_{-\frac{h_1}{4n}}^{\frac{h_1}{4n}} n \mid \zeta(n(z) \mid dz$ are bounded, we get

$$\int\int\int_{R^+_{0,0}(n)} (\Delta v(x_1,x_2,x_3) - \Delta v(h_1+x_1,x_2,x_3))n^3\chi''(nx_1)\zeta(n(x_2-x_2^0))$$
$$\zeta(n(x_3-x_3^0))dx_1dx_2dx_3 < -(\Delta v(0,x_2^0,x_3^0)) - \Delta v(h,x_2^0,x_3^0))/2$$

for large n.

Then

$$\int\int\int_{R^+_{0,0}(n)} [\Delta v]^{(x_1,x_2,x_3)}_{(h_1+x_1,x_2,x_3)} n^3\chi(nx_1))\zeta''(n(x_2-x_2^0))\zeta(n(x_3-x_3^0))dx_1dx_2dx_3 =$$

$$[\Delta v]_{(h_1,x_2^0,x_3^0)}^{(0,x_2^0,x_3^0)} \int_0^{\frac{h_1}{4n}} n\chi(nx)dx \int_{x_2^0-\frac{h_1}{4n}}^{x_2^0+\frac{h_1}{4n}} n\zeta''(n(y-x_2^0))dy \int_{x_3^0-\frac{h_1}{4n}}^{x_3^0+\frac{h_1}{4n}} n\zeta(n(z-x_3^0))dz$$

$$+\int\int\int_{R_{0,0}^+(n)} \varepsilon_n(x_1,x_2,x_3)n^3\chi(nx_1)\zeta''(n(x_2-x_2^0))\zeta(n(x_3-x_3^0))dx_1dx_2dx_3.$$

This expression tends towards 0 when $n \to +\infty$. Indeed, $\int_{x_2^0-\frac{h_1}{4n}}^{x_2^0+\frac{h_1}{4n}} n\zeta''(n(y-x_2^0))dy = 0$ while $\int_0^{\frac{h_1}{4n}} n \mid \chi(nx) \mid dx$ and $\int_{x_2^0-\frac{h_1}{4n}}^{x_2^0+\frac{h_1}{4n}} n \mid \zeta''(n(y-x_2^0) \mid dy$ are bounded. Hence $(\Delta v \mid \Delta\psi_n) < -(\Delta v(0,x_2^0,x_3^0) - \Delta v(h_1,x_2^0,x_3^0))/4$ for large n, and $(7.a)$ is proved.

We only sketch the proof of $(7.b)$, which is similar to the one of $(7.a)$. Assume $\partial_{x_1}\Delta v(0,x_2^0,x_3^0) - \partial_{x_1}\Delta v(h_1,x_2^0,x_3^0) < 0$. Let us still consider functions ψ_n, defined in $R_{k,l}(n)$ by

$$\psi_n(x_1,x_2,x_3) = \chi(n^3(x_1-kh_1))\zeta(n(x_2-x_2^0-lh_1))\zeta(n(x_3-x_3^0)),$$

and vanishing out of $\cup_{(k,l)\in\mathbb{Z}^2}R_{k,l}$. But now, χ and ζ are even $\mathcal{C}^\infty(\mathbb{R})$ functions, with supports in I: χ is negative on I, increasing on $[0,h_1/4]$ with $\chi(0)=-1$, while ζ takes non negative values with $\int_{-\frac{h_1}{4n}}^{\frac{h_1}{4n}} n\zeta(z)dz = 1$. Assuming $\partial_{x_1}\Delta v(0,x_2^0,x_3^0) - \partial_{x_1}\Delta v(h_1,x_2^0,x_3^0) > 0$ gives a contradiction. Indeed, integrating by parts gives $(\Delta v \mid \Delta\psi_n) = A_n + B_{2,n} + B_{3,n}$ where $A_n = -((\partial_{x_1}\Delta v - \partial_{x_1}\Delta v) \mid \partial_{x_1}\psi_n)$ and $B_{i,n} = (\Delta v \mid \partial^2_{x_i^2}\psi_n)$ for $i=2,3$, where $A_n > (\partial_{x_1}\Delta v(0,x_2^0,x_3^0) - \partial_{x_1}\Delta v(h_1,x_2^0,x_3^0))/2$ while $B_{i,n} \to 0$ when $n \to +\infty$. So the lemma is proved.

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