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Embedding of general martingales into a Brownian motion

Introduction

In 1960 Skorohod proved [10] that for any mean zero, square integrable random variable X there is a Brownian motion and a stopping time τ such that X and B_τ have the same distribution. Meanwhile there exist a series of different, elegant proofs of this so-called Skorohod embedding (cf. [1], [2], [4], [11] and [5]). This embedding gives the possibility to measure the closeness of the distribution of X to a normal distribution $\nu_{0,s}$. Let us shortly indicate the basic idea in the most simple case. Suppose that g is a two times continuously differentiable function. Then an easy application of the Ito-formula gives the first order expansion

$$\mathbb{E}\, g(X) - \mathbb{E}\, g(B_s) = \mathbb{E} g\,(B_\tau) - \mathbb{E}\, g(B_s) = \frac{1}{2}\,\mathbb{E}\, g''(X)(\tau - s) + R\,,$$

with some remainder term R. This and also higher expansions are of course only useful, if the stopping time τ can be handled. For example, if $X = M_t$, where $M = (M_t)_{t\geq 0}$ is a continuous, square integrable martingale, then the theorem of Dubins and Schwartz (cf. [3]) gives the nice stopping time $\tau = \tau_t = [M]_t$.

In [8], P. A. Mykland has obtained a new construction for a Skorohod embedding. He uses this construction to embed general right continuous martingales into a Brownian motion, such that a kind of the Dubins and Schwartz result holds.

The present paper is based on the ideas of Mykland in [8] and had its origin in some gaps in the proofs of [8]. The efforts of filling these gaps finally lead to a slightly different type of embedding.

1 The Skorohod embedding of P. A. Mykland

Suppose that $[a,b]$ is an interval containing 0. Then it is well known that there exists a continuous martingale $(L_t, \mathcal{G}_t)_{0\le t\le 1}$ such that L_1 has the distribution $\frac{b}{b-a}\delta_a + \frac{-a}{b-a}\delta_b$. Just take a Brownian motion $(B_t, \mathcal{F}_t)_{t\ge 0}$ starting at 0 and let τ denote the hitting time of (B_t) of the set $\{a,b\}$. If $\varphi : [0,1] \to \overline{\mathbb{R}}_+$ is a continuous, strictly increasing function with $\varphi(0)=0$ and $\varphi(1)=\infty$, then the process $(L_t, \mathcal{G}_t)_{0\le t\le 1}$, defined by $L_t = B_{\varphi(t)\wedge\tau}$ and $\mathcal{G}_t = \mathcal{F}_{\varphi(t)}$, has the desired properties. We will call a martingale of this type a <u>martingale ending in</u> $\{a,b\}$. As e.g. in [2] such martingales will play a central role in the following Skorohod imbedding of a single random variable.

Theorem 1.1 *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with the properties $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$. Then there exists an extension $(\Omega', \mathcal{F}', P')$ of $(\Omega, \mathcal{F}, P)$ (cf. eg.* [11], 3.4.1 *for the usual definition of an extension) and a continuous, square integrable martingale $(M_t(X), \mathcal{G}_t)_{0\le t\le 1}$ on Ω' such that $M_1(X) = X$.*

Proof: Let $p \in]0,1[$ be given. For the construction of the martingale $M(X)$ we take an independent sequence $(L^k)_{k\ge 0}$ of continuous martingales $(L_t^k, \mathcal{G}_t^k)_{0\le t\le 1}$ ending in $\{-p, 1-p\}$, which is also independent of X.

We remark that every martingale L^k as a stopped Brownian motion is an L^q-martingale for every $q \ge 1$. For later use we define with the aid of the sequence (L^k) the following sequences of random variables: We set for $k \ge 0$

$$\begin{aligned}
\eta_k &:= p + L_1^k\,,\\
\zeta_k &:= p\eta_k + (1-p)(1-\eta_k)\,,\\
\xi_k &:= p(1-\eta_k) + (1-p)\eta_k\,,\\
\sigma_k &:= [L^k]_1 \quad \text{(the quadratic variation of } L^k \text{ at time } 1)\,.
\end{aligned} \tag{1.1.1}$$

Then obviously η_k takes the values 0 and 1 with probability $1-p$ and p resp.. Now we start with the construction of $M(X)$. Set $X_1 = X$ and denote by Y_1 an independent copy of X_1, which is also independent of $(L^k)_{k\ge 0}$. Then define

$$\begin{aligned}
X_1^{(1)} &= \eta_0 X_1 + (1-\eta_0)Y_1 \quad \text{and}\\
X_1^{(2)} &= (1-\eta_0)X_1 + \eta_0 Y_1\,,
\end{aligned}$$

and

$$X_{2^{-1}} = pX_1^{(1)} + (1-p)X_1^{(2)}\,.$$

If $X_{2^{-n}}$ $(n \geq 0)$ is already defined, we proceed similary to define $X_{2^{-(n+1)}}$. First we take an independent copy $Y_{2^{-n}}$ of $X_{2^{-n}}$, which is also independent of all random variables which are already introduced. Then we set

$$\begin{aligned} X^{(1)}_{2^{-n}} &= \eta_n X_{2^{-n}} + (1-\eta_n) Y_{2^{-n}}\,, \text{ and} \\ X^{(2)}_{2^{-n}} &= (1-\eta_n) X_{2^{-n}} + \eta_n Y_{2^{-n}}\,, \end{aligned} \tag{1.1.2}$$

and

$$X_{2^{-(n+1)}} = pX^{(1)}_{2^{-n}} + (1-p) X^{(2)}_{2^{-n}}\,. \tag{1.1.3}$$

Now we define a filtration $(\overline{\mathcal{F}}_{2^{-n}})_{n\geq 0}$ such that $(X_{2^{-n}})_{n\geq 0}$ is $(\overline{\mathcal{F}}_{2^{-n}})-$adapted. We set

$$\begin{aligned} \overline{\mathcal{F}}_1 &= \sigma\left(X,\, X^{(1)}_{2^{-k}},\, X^{(2)}_{2^{-k}}\ (k\geq 0)\right)\,, \quad \text{and} \\ \overline{\mathcal{F}}_{2^{-(n+1)}} &= \sigma\left(X^{(1)}_{2^{-k}},\, X^{(2)}_{2^{-k}}\ (k\geq n)\right)\ (n\geq 0)\,. \end{aligned} \tag{1.1.4}$$

Then $(X_{2^{-n}}, \overline{\mathcal{F}}_{2^{-n}})_{n\geq 0}$ is a martingale.

Proof: Essential for the martingale property of $(X_{2^{-n}}, \overline{\mathcal{F}}_{2^{-n}})$ is the observation that for every $n \geq 0$ the process L^n and the $\sigma-$algebra $\overline{\mathcal{F}}_{2^{-(n+1)}}$ are independent. [Proof: First one gets easily from the definitions that the variables $X^{(1)}_{2^{-n}}$ and $X^{(2)}_{2^{-n}}$ have the same distribution as $X_{2^{-n}}$ and $Y_{2^{-n}}$. For $0 \leq t_1 < \cdots < t_k \leq 1$, $A \in \mathcal{B}(\mathbb{R}^k)$, $C_1 \in \mathcal{B}(\mathbb{R})$, $C_2 \in \mathcal{B}(\mathbb{R})$ $(\mathcal{B}(\mathbb{R}^k) :=$ Borel field of $\mathbb{R}^k)$ then one computes

$$\begin{aligned} & P\left[(L^n_{t_1},\dots,L^n_{t_k}) \in A,\, X^{(1)}_{2^{-n}} \in C_1,\, X^{(2)}_{2^{-n}} \in C_2\right] \\ &= P\left[(L^n_{t_1},\dots,L^n_{t_k}) \in A,\, \eta_n = 0,\, X^{(1)}_{2^{-n}} \in C_1,\, X^{(2)}_{2^{-n}} \in C_2\right] \\ &\quad + P\left((L^n_{t_1},\dots,L^n_{t_k}) \in A,\, \eta_n = 1,\, X^{(1)}_{2^{-n}} \in C_1,\, X^{(2)}_{2^{-n}} \in C_2\right] \\ &= P\left[(L^n_{t_1},\dots,L^n_{t_k}) \in A,\, \eta_n = 0\right] P[Y_{2^{-n}} \in C_1] P[X_{2^{-n}} \in C_2] \\ &\quad + P\left[(L^n_{t_1},\dots,L^n_{t_k}) \in A,\, \eta_n = 1\right] P[X_{2^{-n}} \in C_1] P[Y_{2^{-n}} \in C_2] \\ &= P\left[(L^n_{t_1},\dots,L^n_{t_k}) \in A\right] P[X^{(1)}_{2^{-n}} \in C_1] P[X^{(2)}_{2^{-n}} \in C_2]\,, \end{aligned}$$

because of (1.1.2). This gives that L^n and $\left\{X^{(1)}_{2^{-n}},\, X^{(2)}_{2^{-n}}\right\}$ are independent. It follows that L^n and $\sigma\left(X^{(1)}_{2^{-n}},\, X^{(2)}_{2^{-n}},\, L^m(m>n), Y_{2^{-m}}(m>n)\right)$ are independent. Since the latter σ-algebra contains $\overline{\mathcal{F}}_{2^{-(n+1)}}$ by 1.1.3, the independence of L^n and $\overline{\mathcal{F}}_{2^{-(n+1)}}$ is proved.] From this independence property we obtain for every $n \geq 0$,

$$\begin{aligned} & \mathbb{E}\left[X_{2^{-n}} \,|\, \overline{\mathcal{F}}_{2^{-(n+1)}}\right] \\ &= \mathbb{E}\left[X_{2^{-n}} 1_{[\eta_n=1]} + X_{2^{-n}} 1_{[\eta_n=0]} \,|\, \overline{\mathcal{F}}_{2^{-(n+1)}}\right] \\ &= \mathbb{E}\left[X^{(1)}_{2^{-n}} 1_{[\eta_n=1]} + X^{(2)}_{2^{-n}} 1_{[\eta_n=0]} \,|\, \overline{\mathcal{F}}_{2^{-(n+1)}}\right] \qquad \text{(by (1.1.2))} \\ &= pX^{(1)}_{2^{-n}} + (1-p) X^{(2)}_{2^{-n}} = X_{2^{-(n+1)}}\,, \end{aligned}$$

i.e. $(X_{2^{-n}})_{n\geq 0}$ is an $(\overline{\mathcal{F}}_{2^{-n}})$ – martingale.

Next we observe that (1.1.3) implies

$$\begin{aligned} X^{(1)}_{2^{-n}} - X_{2^{-(n+1)}} &= (1-p)\left(X^{(1)}_{2^{-n}} - X^{(2)}_{2^{-n}}\right), \text{ and} \\ X^{(2)}_{2^{-n}} - X_{2^{-(n+1)}} &= (-p)\left(X^{(1)}_{2^{-n}} - X^{(2)}_{2^{-n}}\right), \end{aligned} \tag{1.1.5}$$

and it follows from the definition of η_n and (1.1.2) that

$$X_{2^{-(n+1)}} + \left(X^{(1)}_{2^{-n}} - X^{(2)}_{2^{-n}}\right) L^n_1 = X_{2^{-n}}\,. \tag{1.1.6}$$

Now we set for every $n \geq 0$ and $2^{-(n+1)} < t \leq 2^{-n}$,

$$M_t(X) = X_{2^{-(n+1)}} + \left(X^{(1)}_{2^{-n}} - X^{(2)}_{2^{-n}}\right) L^n_{2^{n+1}t-1}\,. \tag{1.1.7}$$

Then the process $(M_t(X))_{0<t\leq 1}$ is obviously a continuous process such that

$$M_{2^{-n}}(X) = X_{2^{-n}}$$

for every $n \geq 0$.

Since $\mathbb{E}X_{2^{-n}} = 0$, (1.1.2) implies $\mathbb{E}X^{(1)}_{2^{-n}}X^{(2)}_{2^{-n}} = \mathbb{E}X_{2^{-n}}Y_{2^{-n}} = 0$, and we obtain

$$\begin{aligned} \mathbb{E}(X_{2^{-n}})^2 &= (p^2 + (1-p)^2)\mathbb{E}(X_{2^{-(n+1)}})^2 \\ &= \cdots = (p^2 + (1-p)^2)^n \mathbb{E}X^2\,, \end{aligned}$$

which implies that $\lim_{t\to 0} M_t(X) = 0$ in L^2. Hence the definition $M_0(X) = 0$ gives the (a.s.) continuous process $(M_t(X))_{0\leq t\leq 1}$. We now define the following filtration associated to this process $M(X)$. We put

$$\begin{aligned} &\mathcal{G}_0 = \{\emptyset, \Omega'\} \\ &\quad \text{(where } \Omega' = \Omega\times\cdots \text{ denotes the extension of } \Omega \text{, such} \\ &\quad \text{that all random variables introduced above are defined} \\ &\quad \text{on } \Omega'.) \\ &\mathcal{G}_t = \sigma\left(\overline{\mathcal{F}}_{2^{-(n+1)}},\, \mathcal{G}^k_s\ (k>n,\ 0\leq s\leq 1),\, \mathcal{G}^n_s\ (0\leq s\leq 2^{n+1}t-1)\right) \\ &\quad \text{for } 2^{-(n+1)} \leq t < 2^{-n}\ (n\geq 0), \text{ and} \\ &\mathcal{G}_1 = \sigma\left(\overline{\mathcal{F}}_1, \mathcal{G}^n_s\ (n\geq 0,\ 0\leq s\leq 1)\right). \end{aligned} \tag{1.1.8}$$

Then $(M_t(X), \mathcal{G}_t)_{0\leq t\leq 1}$ is the asserted martingale.

Remarks 1.2 (1) Since the processes L^n in the construction of the martingale $M(X)$ are (as stopped Brownian motions) L^q – martingales for every $q \geq 1$, the martingale $M(X)$ is an L^q – martingale if and only if X is q – integrable. (2) By the theorem of Dubins and

Schwartz (cf. [3]) there exists a Brownian motion $(B_t, \mathcal{F}_t)_{0 \le t \le 1}$ on Ω' such that $\tau_t = [M(X)]_t$ $(0 \le t \le 1)$ is an $(\mathcal{F}_t)$ – stopping time, and such that $B_{\tau_t} = M_t(X)$ a.s. for every t. Especially, $B_{\tau_1} = X$, which means that theorem 1.1 gives a so – called Skorohod embedding of X. The essential feature of the Skorohod embedding of 1.1 is the special structure of $[M(X)]_1$, the quadratic variation of $M(X)$ at time 1. This is described in the following result.

Proposition 1.3 *With the notations introduced in the proof of* 1.1 *the following formula holds for* $[M(X)]_1$*:*

$$[M(X)]_1 = \sum_{n \ge 0} (X_{2^{-n}} - Y_{2^{-n}})^2 \, \sigma_n \quad \text{a.s.}, \tag{1.3.1}$$

and $X_{2^{-n}}$ *has the structure*

$$\begin{aligned} X_{2^{-n}} &= Z_n X + V_n\,, \text{ where } Z_0 = 1,\ V_0 = 0\,, \\ &\quad \text{and for } n \ge 1 \\ Z_n &= \zeta_{n-1}\zeta_{n-2}\cdots\zeta_0\,, \\ V_n &= \sum_{j=0}^{n-1} Z_{n,j} Y_{2^{-j}} \quad \text{with} \\ Z_{n,j} &= \zeta_{n-1}\cdots\zeta_{j+1}\xi_j\,. \end{aligned} \tag{1.3.2}$$

Proof: From the definition of $M(X)$ it follows that

$$[M(X)]_1 = \sum_{n \ge 0} \left(X^{(1)}_{2^{-n}} - X^{(2)}_{2^{-n}} \right)^2 \sigma_n = \sum_{n \ge 0} (X_{2^{-n}} - Y_{2^{-n}})^2 \, \sigma_n\,,$$

i.e. (1.3.1) holds. The assertion (1.3.2) is easily proved by induction, using that (1.1.2) implies (see (1.1.1))

$$X_{2^{-(n+1)}} = \zeta_n X_{2^{-n}} + \xi_n Y_{2^{-n}} \tag{1.3.3}$$

for all $n \ge 0$. [Since $X_{2^{-n}} = X$ and $X_{2^{-1}} = \zeta_0 X + \xi_0 Y_{2^{-0}}$ (by 1.3.3), the assertion (1.3.2) is true for $n = 0, 1$. Now suppose that (1.3.2) holds for $n \ge 1$. Then we get from (1.1.3)

$$\begin{aligned} X_{2^{-(n+1)}} &= \zeta_n Z_n X + \zeta_n V_n + \xi_n Y_{2^{-n}} \\ &= Z_{n+1} X + \sum_{j=0}^{n-1} \zeta_n Z_{n,j} Y_{2^{-j}} + \xi_n Y_{2^{-n}} \\ &= Z_{n+1} X + \sum_{j=0}^{n} Z_{n+1,j} Y_{2^{-j}}\,.] \end{aligned}$$

By (1.3.1) we have the representation

$$[M(X)]_1 = \sum_{n \ge 0} (Z_n X + V_n - Y_{2^{-n}})^2 \sigma_n, \text{ and we know that} \tag{1.3.4}$$

(i) σ_n is independent of $Z_n X + V_n - Y_{2^{-n}}$ (and of X), and

(ii) $V_n - Y_{2^{-n}}$ and Z_n are independent of X.

Furthermore, the construction of $X_{2^{-n}}$ shows that

$$\begin{aligned} \mathbb{E}Y_{2^{-n}}^2 &= \mathbb{E}X_{2^{-n}}^2 = a_n \mathbb{E}X^2 \qquad \text{and} \\ \mathbb{E}Y_{2^{-n}}^4 &= \mathbb{E}X_{2^{-n}}^4 = b_n \mathbb{E}X^4 + c_n (\mathbb{E}X^2)^2 \end{aligned}$$

for some constants a_n, b_n, c_n (which can be easily computed).
These observations lead to the following corollary of (1.3).

Corollary 1.4 *The following formulas hold for the conditional expectation* $\mathbb{E}([M(X)]_1|X)$ *and the conditional variance* $Var\,([M(X)]_1|X) = \mathbb{E}([M(X)]_1^2|X) - \{\mathbb{E}([M(X)]_1|X)\}^2$.
There exists constants $\alpha_1(p), \alpha_2(p), \beta_1(p), \dots, \beta_5(p)$ *such that*

$$\mathbb{E}([M(X)]_1|X) = \alpha_1(p) X^2 + \alpha_2(p)\mathbb{E}X^2, \quad \text{and} \tag{1.4.1}$$

$$\begin{aligned} Var\,([M(X)]_1|X) = {} & \beta_1(p) X^4 + \beta_2(p)\mathbb{E}X^4 + \beta_3(p)(\mathbb{E}X^2)^2 \\ & + \beta_4(p) X^2 \mathbb{E}X^2 + \beta_5(p) X \mathbb{E}X^3. \end{aligned} \tag{1.4.2}$$

Remark 1.5 It will turn out (cf. [8]) that most of the constants do not depend on p. More precisely, we will show in the third paragraph that

$$\begin{aligned} &\alpha_1(p) = \frac{1}{3}, &&\alpha_2(p) = \frac{2}{3}, \\ &\beta_1(p) = \frac{2}{45}, &&\beta_2(p) = \frac{8}{45}, \\ &\beta_3(p) = \frac{4}{9} - c(p), &&\beta_4(p) = c(p), \quad \beta_5(p) = -\,c(p), \end{aligned}$$

where only $c(p)$ depends on p. If we work in the following with the formulas (1.4.1) and (1.4.2), we will already take the above values of the yet unknown constants $\alpha_1(p), \dots, \beta_5(p)$.

For the following results we have to generalize slightly the notion of an extension of a given probability space. A probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ is henceforth called an <u>extension</u> of the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if there exists a map $\pi : \widetilde{\Omega} \to \Omega$ such that $\pi^{-1}(\mathcal{F}) \subset \widetilde{\mathcal{F}}$ and $\pi(\widetilde{\mathbb{P}}) = \mathbb{P}$. If $(\mathcal{F}_t)$ is a filtration on Ω, a filtration $(\widetilde{\mathcal{F}}_t)$ on $\widetilde{\Omega}$ is called an <u>extension of</u> $(\mathcal{F}_t)$, if $\pi^{-1}(\mathcal{F}_t) \subset \widetilde{\mathcal{F}}_t$ for all t. For the extension $(\pi^{-1}(\mathcal{F}_t))$ we usually use again the notation $(\mathcal{F}_t)$. Similarly, if X is a random variable on Ω, the extended random variable $X \circ \pi$ is usually again denoted by X.

Theorem 1.6 (Skorohod embedding of a discrete time martingale) *Let $M = (M_k, \mathcal{F}_k)_{k\geq 0}$ be a square integrable martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with $M_0 = 0$. Then there exists an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of Ω and a continuous, square integrable martingale $\widetilde{M} = (\widetilde{M}_t, \widetilde{\mathcal{F}}_t)_{t\geq 0}$ on $\widetilde{\Omega}$ with the following properties:*

$$
\begin{aligned}
&(\widetilde{\mathcal{F}}_k)_{k\geq 0} \text{ is an extension of } (\mathcal{F}_k)_{k\geq 0}\,,\\
&\widetilde{M}_k = M_k \quad \text{for all } k \geq 0\,, \text{ and}\\
&\mathbb{E}\left([\widetilde{M}]_n|\mathcal{F}\right) = \frac{1}{3}\sum_{k=0}^{n-1}(M_{k+1}-M_k)^2 + \frac{2}{3}\sum_{k=0}^{n-1}\mathbb{E}\left((M_{k+1}-M_k)^2|\mathcal{F}_k\right)\\
&\text{for all } n \geq 1\,.
\end{aligned}
$$

If, in addition, M is an L^4-martingale, then $\widetilde{M}$ is an L^4-martingale, and

$$
\begin{aligned}
&Var\left([\widetilde{M}]_n|\mathcal{F}\right)\\
&= \frac{2}{45}\sum_{k=0}^{n-1}(M_{k+1}-M_k)^4 + \frac{8}{45}\sum_{k=0}^{n-1}\mathbb{E}\left((M_{k+1}-M_k)^4|\mathcal{F}_k\right)\\
&\quad + \left(\frac{4}{9} - c(p)\right)\sum_{k=0}^{n-1}\left(\mathbb{E}\left((M_{k+1}-M_k)^2|\mathcal{F}_k\right)\right)^2\\
&\quad + c(p)\sum_{k=0}^{n-1}(M_{k+1}-M_k)^2\,\mathbb{E}\left((M_{k+1}-M_k)^2|\mathcal{F}_k\right)\\
&\quad - c(p)\sum_{k=0}^{n-1}(M_{k+1}-M_k)\,\mathbb{E}\left((M_{k+1}-M_k)^3|\mathcal{F}_k\right)
\end{aligned}
$$

for all $n \geq 1$.

Proof: We set $X_0 = 0$ and $X_n = M_n - M_{n-1}$ for $n \geq 1$. From theorem 1.1 we get that there exists an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of Ω and a continuous, square integrable martingale $(M_t(X_1), \mathcal{G}_t)_{0\leq t\leq 1}$ such that $M_1(X_1) = X_1$. Now we set

$$
\begin{aligned}
&\widetilde{\mathcal{F}}_t := \mathcal{G}_t \text{ for } 0 \leq t \leq 1\,, \quad \widetilde{\mathcal{F}}_1 := \mathcal{G}_1 \vee \mathcal{F}_1\,, \quad \widetilde{M}_t := M_t(X_1)\\
&\text{for } t \in [0,1] \text{ and } \left(\widetilde{\Omega}^{(1)}, \widetilde{\mathcal{F}}^{(1)}, \widetilde{\mathbb{P}}^{(1)}\right) := (\Omega', \mathcal{F}', \mathbb{P}')\,.
\end{aligned}
$$

Then it follows from theorem 1.1 that for $\mathbb{E}\left([\widetilde{M}]_1|\mathcal{F}\right)$ and $Var\left([\widetilde{M}]_1|\mathcal{F}\right)$ the asserted formulas of the theorem hold.

The structure of the proof is now as follows. First we define by induction a sequence $\left(\widetilde{\Omega}^{(n)}\right)_{n\geq 1}$ of probability spaces with the following properties:

1. For every $n \geq 1$ the probability space $\widetilde{\Omega}^{(n+1)}$ is an extension of $\widetilde{\Omega}^{(n)}$, i.e. there exists a measurable map $\Psi_{n+1} : \widetilde{\Omega}^{(n+1)} \longrightarrow \widetilde{\Omega}^{(n)}$ such that $\Psi_{n+1}\left(\widetilde{\mathbb{P}}^{(n+1)}\right) = \widetilde{\mathbb{P}}^{(n)}$.

2. On every $\widetilde{\Omega}^{(n)}$ there is a filtration $\left(\widetilde{\mathcal{F}}_t\right)_{0\leq t\leq n}$ such that for $0 \leq t \leq n-1$ the σ-algebra $\widetilde{\mathcal{F}}_t$ is just the canonical extension of the σ-algebra $\widetilde{\mathcal{F}}_t$ defined before on $\widetilde{\Omega}^{(n-1)}$.

3. On $\widetilde{\Omega}^{(n)}$ there exists a continuous L^2- (resp. L^4-)martingale $\left(\widetilde{M}_t\right)_{0\leq t\leq n}$ relative to the filtration $\left(\widetilde{\mathcal{F}}_t\right)$, such that for $0 \leq t \leq n-1$ the $\widetilde{M}_t$ is the canonical extension of the $\widetilde{M}_t$ defined before on $\widetilde{\Omega}^{(n-1)}$.

4. $\widetilde{M}_k = M_k$ for $0 \leq k \leq n$ and the asserted formulas for $\mathbb{E}\left([\widetilde{M}]_m|\mathcal{F}\right)$ and $Var\left([\widetilde{M}]_m|\mathcal{F}\right)$ hold for $1 \leq m \leq n$.

The final extension $\widetilde{\Omega}$ then is defined as the projective limit $\varprojlim \widetilde{\Omega}^{(n)}$, and the canonical extensions of the $\widetilde{M}_t$ and $\widetilde{\mathcal{F}}_t$ to $\widetilde{\Omega}$ then yield the martingale $\left(\widetilde{M}_t, \widetilde{\mathcal{F}}_t\right)_{t\geq 0}$ with the asserted properties.

First we construct the extensions $\widetilde{\Omega}^{(n)}$ under the additional hypothesis that for every $n \geq 1$ there exists a regular conditional probability of $\mathcal{F}_n$ relative to $\mathcal{F}_{n-1}$, i.e. a Markov kernel

$$\mathbb{K}^{(n)} : \Omega \times \mathcal{F}_n \longrightarrow [0,1]\,,$$

such that $\mathbb{K}^{(n)}(.,A)$ is $\mathcal{F}_{n-1}$-measurable for every $A \in \mathcal{F}_n$ and such that $\mathbb{K}^{(n)}(\omega,.)$ is a probability measure on $(\Omega, \mathcal{F}_n)$ for every ω, which we also denote by $\mathbb{K}^{(n)}_\omega$. At the end of the proof we will indicate how to dispense from this assumption.

If $\overline{\Omega}$ is an extension of Ω, and $\pi : \overline{\Omega} \longrightarrow \Omega$ the associated extension map, then for every $\overline{\omega} \in \overline{\Omega}$ we write $\mathbb{K}^{(n)}_{\overline{\omega}}$ for the probability measure $\mathbb{K}^{(n)}_{\pi(\overline{\omega})}$.

Now suppose that $\widetilde{\Omega}^{(n-1)}$ is already constructed. Let (S, Σ, Q) be a probability space, on which a martingale $(L_t, \mathcal{G}_t)_{0\leq t\leq 1}$ ending in $\{-p, 1-p\}$ is defined. Then we set

$$\widetilde{\Omega}^{(n)} := \prod_{k\geq 0} \widetilde{\Omega}_k \times \prod_{k\geq 0} S_k$$

with $\widetilde{\Omega}_k = \widetilde{\Omega}^{(n-1)}$ and $S_k = S$ for all $k \geq 0$, and we denote by π^1_k resp. π^2_k the canonical projections from $\widetilde{\Omega}^{(n)}$ onto $\widetilde{\Omega}_k$ resp. S_k. We identify $\widetilde{\Omega}^{(n-1)}$ with $\widetilde{\Omega}_0$.

We proceed as in the proof of the theorem 1.1. For every $k \geq 0$ let $\left(L^k_t, \mathcal{G}^k_t\right)_{0\leq t\leq 1}$ be the copy of $(L_t, \mathcal{G}_t)_{0\leq t\leq 1}$ on $\widetilde{\Omega}^{(n)}$ defined by

$$L^k_t := L_t \circ \pi^2_k \text{ and } \mathcal{G}^k_t := \left(\pi^2_k\right)^{-1}(\mathcal{G}_t)$$

and set $\eta_k := p + L_1^k$. Similarly, we set

$$(X_n)_1 := X_n \circ \pi_0^1\,,\ (Y_n)_1 := X_n \circ \pi_1^1\,,$$

and define $(X_n)_1^{(1)}$, $(X_n)_1^{(2)}$ and $(X_n)_{2^{-1}}$ from $(X_n)_1$, $(Y_n)_1$, η_0 as in the proof of theorem 1.1. Suppose now that $(X_n)_{2^{-m}}$ is already defined on $\widetilde{\Omega}^{(n)}$ in such a way that

$$(X_n)_{2^{-m}} = Z_m \circ \left(\pi_0^1, \ldots, \pi_{2^m-1}^1\right)\,,$$

where Z_m is a random variable on $\widetilde{\Omega}_0 \times \ldots \times \widetilde{\Omega}_{2^m-1}$. Then we set

$$(Y_n)_{2^{-m}} := Z_m \circ \left(\pi_{2^m}^1, \ldots, \pi_{2^{m+1}-1}^1\right)\,,$$

and define

$$(X_n)_{2^{-m}}^{(1)}\,,\ (X_n)_{2^{-m}}^{(2)}\,,\ \text{and}\ (X_n)_{2^{-(m+1)}}$$

again from $(X_n)_{2^{-m}}$, $(Y_n)_{2^{-m}}$ and η_m as in the proof of theorem 1.1. Then we set for every $m \geq 0$ and $t \in]2^{-(m+1)}, 2^{-m}]$,

$$\begin{aligned}
M_t(X_n) &:= (X_n)_{2^{-(m+1)}} + \left((X_n)_{2^{-m}}^{(1)} - (X_n)_{2^{-m}}^{(2)}\right) L_{2^{m+1}t-1}^m\,,\\
\overline{\mathcal{F}}_1^{(n)} &:= \sigma\left(X_n, (X_n)_{2^{-k}}^{(1)}, (X_n)_{2^{-k}}^{(2)}\ (k \geq 0)\right)\,,\\
\overline{\mathcal{F}}_{2^{-(m+1)}}^{(n)} &:= \sigma\left((X_n)_{2^{-k}}^{(1)}, (X_n)_{2^{-k}}^{(2)}\ (k \geq m)\right) \quad (\text{for } m \geq 0)\,,\\
\mathcal{G}_0^{(n)} &:= \{\emptyset, \widetilde{\Omega}^{(n)}\}\,,\\
\mathcal{G}_t^{(n)} &:= \sigma\left(\overline{\mathcal{F}}_{2^{-(m+1)}}^{(n)}, \mathcal{G}_1^k\ (k > m), \mathcal{G}_t^m\right) \quad \text{for } t \in [2^{-(m+1)}, 2^{-m}[\,,\\
\mathcal{G}_1^{(n)} &:= \sigma\left(\overline{\mathcal{F}}_1^{(n)}, \mathcal{G}_1^k\ (k \geq 0)\right)\,.
\end{aligned}$$

Then it follows from theorem 1.1 that $\left(M_t(X_n), \mathcal{G}_t^{(n)}\right)_{0 \leq t \leq 1}$ is an L^2- (resp. L^4-)martingale on $\widetilde{\Omega}^{(n)}$ relative to every probability measure

$$\mathbb{R}_{\widetilde{\omega}}^{(n)} := \left(\mathbb{K}_{\widetilde{\omega}}^{(n)}\right)^{\otimes \mathbb{Z}_+} \otimes Q^{\otimes \mathbb{Z}_+} \quad \left(\widetilde{\omega} \in \widetilde{\Omega}^{(n-1)}\right)$$

on $\widetilde{\Omega}^{(n)}$. Finally, we set $\widetilde{\mathcal{F}}^{(n)} := \widetilde{\mathcal{F}}_{n-1} \vee \mathcal{G}_1^{(n)} \vee \mathcal{F}$ and

$$\widetilde{\mathcal{F}}_t = \begin{cases} \widetilde{\mathcal{F}}_t & \text{for } 0 \leq t \leq n-1 \\ & (\text{canonical extension from } \widetilde{\Omega}^{(n-1)} \text{ to } \widetilde{\Omega}^{(n)})\,, \\ \widetilde{\mathcal{F}}_{n-1} \vee \mathcal{G}_t^{(n)} & \text{for } n-1 < t < n\,, \\ \widetilde{\mathcal{F}}_{n-1} \vee \mathcal{G}_1^{(n)} \vee \mathcal{F}_n\,. & \end{cases}$$

Then we define on $\left(\widetilde{\Omega}^{(n)}, \widetilde{\mathcal{F}}^{(n)}\right)$ the probability measure $\widetilde{\mathbb{P}}^{(n)}$ by

$$\widetilde{\mathbb{P}}^{(n)}(A) := \int\limits_{\widetilde{\Omega}^{(n-1)}} \mathbb{R}^{(n)}_{\widetilde{\omega}}(A)\widetilde{\mathbb{P}}^{(n-1)}(d\widetilde{\omega})$$

for $A \in \widetilde{\mathcal{F}}^{(n)}$, and set

$$\widetilde{M}_t = \begin{cases} \widetilde{M}_t & \text{for } 0 \leq t \leq n-1 \\ & (\text{canonical extension from } \widetilde{\Omega}^{(n-1)} \text{ to } \widetilde{\Omega}^{(n)}), \\ \widetilde{M}_{n-1} + \widetilde{M}_{t-(n-1)}(X_n) & \text{for } n-1 < t \leq n\,. \end{cases}$$

Then we claim that $\left(\widetilde{M}_t, \widetilde{\mathcal{F}}_t\right)_{0\leq t\leq n}$ is an L^2-(resp. L^4-)martingale on $\left(\widetilde{\Omega}^{(n)}, \widetilde{\mathcal{F}}^{(n)}, \widetilde{\mathbb{P}}^{(n)}\right)$ such that the asserted formulas for $\mathbb{E}\left([\widetilde{M}]_k|\mathcal{F}\right)$ and $Var\left([\widetilde{M}]_k|\mathcal{F}\right)$ hold for $k = 1, \dots, n$. As already remarked, this is proved by induction. On $\widetilde{\Omega}^{(1)}$ the assertion holds by the theorem 1.1. Now assume that the assertion is proved on $\widetilde{\Omega}^{(n-1)}$. Then we have to show that

(i) $\left(\widetilde{M}_t, \widetilde{\mathcal{F}}_t\right)_{n-1\leq t\leq n}$ is a martingale,

(ii) the asserted formula holds for $\mathbb{E}\left([\widetilde{M}]_n|\mathcal{F}\right)$,

(iii) the asserted formula holds for $Var\left([\widetilde{M}]_n|\mathcal{F}\right)$.

Proof of (i): It is easily proved by induction that for every $n \geq 1$ we have the inclusion

$$\mathcal{F}_n \subset \widetilde{\mathcal{F}}_n \subset \mathcal{F}_n \vee \mathcal{H}_n\,,$$

where $\mathcal{H}_n$ is a σ-algebra independent of $\mathcal{F}_n$ (even independent of the whole σ-algebra $\mathcal{F}$ on Ω). This implies that for every $A \in \mathcal{F}_n$ we have

$$\widetilde{\mathbb{P}}^{(n-1)}\left(A|\widetilde{\mathcal{F}}_{n-1}\right) = \mathbb{P}\left(A|\mathcal{F}_{n-1}\right) = \mathbb{K}^{(n)}_{\bullet}(A)\,,$$

i.e. $\mathbb{K}^{(n)}$ is also a regular conditional probability of $\mathcal{F}_n$ given $\widetilde{\mathcal{F}}_{n-1}$.

For (i) we have to prove that

$$\mathbb{E}\left(M_t(X_n) - M_s(X_n)|\widetilde{\mathcal{F}}_s\right) = 0 \quad \widetilde{\mathbb{P}}^{(n)} - a.s.$$

for $0 \leq s < t \leq 1$. Because of $\widetilde{\mathcal{F}}_s = \widetilde{\mathcal{F}}_{n-1} \vee \mathcal{G}_s^{(n)}$ we choose an $A \in \widetilde{\mathcal{F}}_{n-1}$ and a $B \in \mathcal{G}_s^{(n)}$. Then we get by the above remark

$$
\begin{aligned}
&\int\limits_{A\cap B} \left(M_t(X_n) - M_s(X_n)\right) d\widetilde{\mathbb{P}}^{(n)} \\
&= \int\limits_{\widetilde{\Omega}^{(n-1)}} \int\limits_{\widetilde{\Omega}^{(n)}} 1_A\left(\widetilde{\omega}^{(n)}\right) 1_B\left(\widetilde{\omega}^{(n)}\right)\left(M_t(X_n)\left(\widetilde{\omega}^{(n)}\right)\right. \\
&\qquad\qquad \left. - M_s(X_n)\left(\widetilde{\omega}^{(n)}\right)\right) \mathbb{R}^{(n)}_{\widetilde{\omega}^{(n-1)}}(d\widetilde{\omega}^{(n)}) \mathbb{P}^{(n)}(d\widetilde{\omega}^{(n-1)}) \\
&= \int\limits_{\widetilde{\Omega}^{(n-1)}} 1_A\left(\widetilde{\omega}^{(n-1)}\right) \int\limits_B \left(M_t(X_n) - M_s(X_n)\right) d\,\mathbb{R}^{(n)}_{\widetilde{\omega}^{(n-1)}} \widetilde{\mathbb{P}}^{(n)}\left(d\widetilde{\omega}^{(n-1)}\right) = 0\,,
\end{aligned}
$$

since $\int_B \left(M_t(X_n) - M_s(X_n)\right) d\,\mathbb{R}^{(n)}_{\widetilde{\omega}^{(n-1)}} = 0$ for all $\widetilde{\omega}^{(n-1)} \in \widetilde{\Omega}^{(n-1)}$ by theorem 1.1. Hence (i) is proved.

Proof of (ii): Since

$$
\mathbb{E}\left([\widetilde{M}]_n|\mathcal{F}\right) = \mathbb{E}\left([\widetilde{M}]_{n-1}|\mathcal{F}\right) + \mathbb{E}\left([\widetilde{M} - \widetilde{M}_{n-1}]_n|\mathcal{F}\right)
$$

it remains to prove that

$$
\mathbb{E}\left([\widetilde{M} - \widetilde{M}_{n-1}]_n|\mathcal{F}\right) = \frac{1}{3}X_n^2 + \frac{2}{3}\mathbb{E}\left(X_n^2|\mathcal{F}_{n-1}\right)\,.
$$

Now for every $A \in \mathcal{F}$ we obtain

$$
\begin{aligned}
&\int\limits_A [\widetilde{M} - \widetilde{M}_{n-1}]_n d\,\widetilde{\mathbb{P}}^{(n)} \\
&= \int\limits_A [M(X_n)]_1 d\,\widetilde{\mathbb{P}}^{(n)} \\
&= \int\limits_{\widetilde{\Omega}^{(n-1)}} \int\limits_A [M(X_n)]_1 d\,\mathbb{R}^{(n)}_{\widetilde{\omega}^{n-1}} \widetilde{\mathbb{P}}^{(n-1)}(d\widetilde{\omega}^{n-1})\,.
\end{aligned}
$$

If we denote by $\mathbb{E}^{(n)}_{\widetilde{\omega}^{n-1}}$ the expectation relative to the probability measure $\mathbb{R}^{(n)}_{\widetilde{\omega}^{n-1}}$ on $\widetilde{\Omega}^{(n)}$, then we get from theorem 1.1

$$
\begin{aligned}
&\int\limits_A [M(X_n)]_1 \mathbb{R}^{(n)}_{\widetilde{\omega}^{n-1}} \\
&= \int\limits_A \mathbb{E}^{(n)}_{\widetilde{\omega}^{n-1}}\left([M(X_n)]_1|\mathcal{F}\right) d\,\mathbb{R}^{(n)}_{\widetilde{\omega}^{n-1}} \\
&= \int\limits_A \left(\frac{1}{3}X_n^2 + \frac{2}{3}\mathbb{E}^{(n)}_{\widetilde{\omega}^{n-1}}(X_n^2)\right) d\,\mathbb{R}^{(n)}_{\widetilde{\omega}^{n-1}}\,,
\end{aligned}
$$

and hence

$$\int_A [M(X_n)]_1 d\widetilde{\mathbb{P}}^{(n)} = \int_A \left(\frac{1}{3}X_n^2 + \frac{2}{3}\mathbb{E}(X_n^2|\mathcal{F}_{n-1})\right) d\widetilde{\mathbb{P}}^{(n)}.$$

Proof of (iii): We have

$$\begin{aligned}
& Var\left([\widetilde{M}]_n|\mathcal{F}\right) \\
&= \mathbb{E}\left(\left([\widetilde{M}]_{n-1} + [\widetilde{M} - \widetilde{M}_{n-1}]_n - \mathbb{E}\left([\widetilde{M}]_{n-1}|\mathcal{F}\right) - \mathbb{E}\left([\widetilde{M} - \widetilde{M}_{n-1}]_n|\mathcal{F}\right)\right)^2 |\mathcal{F}\right) \\
&= Var\left([\widetilde{M}]_{n-1}|\mathcal{F}\right) + Var\left([\widetilde{M} - \widetilde{M}_{n-1}]_n|\mathcal{F}\right) \\
&\quad + 2\mathbb{E}\left(\left([\widetilde{M}]_{n-1} - \mathbb{E}\left([\widetilde{M}]_{n-1}|\mathcal{F}\right)\right)\left([\widetilde{M} - \widetilde{M}_{n-1}]_n - \mathbb{E}\left([\widetilde{M} - \widetilde{M}_{n-1}]_n|\mathcal{F}\right)\right)|\mathcal{F}\right).
\end{aligned}$$

Since $\left([\widetilde{M}]_{n-1} - \mathbb{E}\left([\widetilde{M}]_{n-1}|\mathcal{F}\right)\right)$ is $\widetilde{\mathcal{F}}_{n-1}$-measurable by (ii), we get for every $A \in \mathcal{F}$,

$$\begin{aligned}
& \int_A \left([\widetilde{M}]_{n-1} - \mathbb{E}\left([\widetilde{M}]_{n-1}|\mathcal{F}\right)\right)\left([M(X_n)]_1 - \mathbb{E}([M(X_n)]_1|\mathcal{F})\right) d\widetilde{\mathbb{P}}^{(n)} \\
&= \int_{\widetilde{\Omega}^{(n-1)}} \left([\widetilde{M}]_{n-1} - \mathbb{E}\left([\widetilde{M}]_{n-1}|\mathcal{F}\right)\right) \int_A \left([M(X_n)]_1 - \mathbb{E}([M(X_n)]_1|\mathcal{F})\right) d\mathbb{R}^{(n)}_{\widetilde{\omega}^{n-1}} \widetilde{\mathbb{P}}^{(n-1)}(d\widetilde{\omega}^{n-1}) \\
&= 0, \text{ since obviously} \\
& \int_A \left([M(X_n)]_1 - \mathbb{E}([M(X_n)]_1|\mathcal{F})\right) d\mathbb{R}^{(n)}_{\widetilde{\omega}^{n-1}} = 0
\end{aligned}$$

for all $\widetilde{\omega}^{n-1} \in \widetilde{\Omega}^{(n-1)}$. Hence we have proved that

$$Var\left([\widetilde{M}]_n|\mathcal{F}\right) = Var\left([\widetilde{M}]_{n-1}|\mathcal{F}\right) + Var\left([M(X_n)]_1|\mathcal{F}\right).$$

Now the induction hypothesis for $Var\left([\widetilde{M}]_{n-1}|\mathcal{F}\right)$ and an application of theorem 1.1 to $Var\left([M(X_n)]_1|\mathcal{F}\right)$ similar as in the proof of (ii) above yields the assertion for $Var\left([\widetilde{M}]_n|\mathcal{F}\right)$. So we have proved the theorem under the additional assumption that for every $n \geq 1$ there exists a regular conditional probability of $\mathcal{F}_{n+1}$ given $\mathcal{F}_n$. We shortly indicate that the construction of $\widetilde{\Omega}$ is possible even without this assumption.

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is given and that $\mathcal{F}_0$ is a sub-σ-algebra of $\mathcal{F}$. Then there is always a unique probability measure $\overline{\mathbb{P}}$ on $(\overline{\Omega}, \overline{\mathcal{F}}) = \left(\Omega^{\mathbb{N}}, \mathcal{F}^{\otimes\mathbb{N}}\right)$ with the following property: For every σ-algebra $\mathcal{G} \subset \mathcal{F}$, for which a regular conditional probability $Q^{\mathcal{G}}(\omega_0, d\omega)$ of $\mathcal{G}$ given $\mathcal{F}_0$ exists, and for every sequence $(A_n)_{n\geq 1}$ in $\mathcal{G}$,

$$\overline{\mathbb{P}}\left(\prod_{n\geq 1} A_n\right) = \int_\Omega \prod_{n\geq 1} Q^{\mathcal{G}}(\omega_0, A_n)\mathbb{P}(d\omega_0).$$

Using this result, it is now possible to construct $\widetilde{\mathbb{P}}^{(n)}$ from $\widetilde{\mathbb{P}}^{(n-1)}$ without the assumption on the existence of regular conditional probabilities.

In the next paragraph we need the following immediate corollary.

Corollary 1.7 *Let $M = (M_k)_{k\geq 0}$ be a discrete time, square integrable martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with $M_0 = 0$. Then there exists an extension $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ of Ω and a continuous, square integrable martingale $\widetilde{M} = \left(\widetilde{M}_t, \widetilde{\mathcal{F}}_t\right)_{t\geq 0}$ on $\widetilde{\Omega}$ with the following properties:*

(1) $\widetilde{M}_k = M_k$ *for all* $k \geq 0$,

(2) $$\mathbb{E}\left([\widetilde{M}]_n | M_1, \dots, M_n\right) = \frac{1}{3}\sum_{k=0}^{n-1}(M_{k+1}-M_k)^2 + \frac{2}{3}\sum_{k=0}^{n-1}\mathbb{E}\left((M_{k+1}-M_k)^2 | M_1, \dots, M_k\right)$$
for all $n \geq 1$.

If, in addition, M is an L^4-martingale, then $\widetilde{M}$ is an L^4-martingale, and

(3) $$\begin{aligned} &Var\left([\widetilde{M}]_n | M_1, \dots, M_n\right) \\ &= \frac{2}{45}\sum_{k=0}^{n-1}(M_{k+1}-M_k)^4 + \frac{8}{45}\sum_{k=0}^{n-1}\mathbb{E}\left((M_{k+1}-M_k)^4 | M_1, \dots, M_k\right) \\ &+ \left(\frac{4}{9} - c(p)\right)\sum_{k=0}^{n-1}\left(\mathbb{E}\left((M_{k+1}-M_k)^2 | M_1, \dots, M_k\right)\right)^2 \\ &+ c(p)\sum_{k=0}^{n-1}(M_{k+1}-M_k)^2\mathbb{E}\left((M_{k+1}-M_k)^2 | M_1, \dots, M_k\right) \\ &- c(p)\sum_{k=0}^{n-1}(M_{k+1}-M_k)\mathbb{E}\left((M_{k+1}-M_k)^3 | M_1, \dots, M_k\right) \end{aligned}$$

for all $n \geq 1$.

2 Embedding of general right continuous martingales

On the basis of theorem 1.6 we are now going to prove that any right continuous, square integrable martingale can be imbedded into a Brownian motion in such a way that the corresponding family of stopping times has nice properties.

We denote by D the set of all non–negative, dyadic numbers and we write in addition

$$\begin{aligned} D^{(n)} &:= \{s \in D | s = \frac{k}{2^n},\ k \in \mathbb{Z}_+\}, \\ D_t &:= \{s \in D | s \leq t\} \text{ for } t \in \mathbb{R}_+, \text{ and} \\ D_t^{(n)} &:= \{s \in D^{(n)} | s \leq t\} \text{ for } t \in \mathbb{R}_+. \end{aligned}$$

Now let $M = (M_t, \mathcal{F}_t)_{t \geq 0}$ be a right continuous, square integrable martingale on a probability space $(\Omega, \mathcal{F}, P)$ with $M_0 = 0$. An immediate application of theorem 1.6 together with the theorem of Dubins and Schwartz (cf. [3]) gives the following situation which we summarize in a proposition.

Proposition 2.1 *For every $n \in \mathbb{N}$ there exist a probability space $(\Omega^n, \mathcal{F}^n, P^n)$, which is an extension of $(\Omega, \mathcal{F}, P)$, a continuous, square integrable martingale $M^n = (M_t^n, \mathcal{F}_t^n)_{t \geq 0}$ on Ω^n, a Brownian motion $B^n = (B_t^n, \mathcal{G}_t^n)$ on Ω^n such that $B^n_{\tau_t^n} = M_t^n$ P^n – a.s. for $\tau_t^n := [M^n]_t$, and such that in addition the following properties hold:*

$$M_t^n = M_t \qquad \text{for all } t \in D^{(n)}\ , \ \text{and} \tag{2.1.1}$$

$$\begin{aligned}
&\mathbb{E}(\tau_t^n | \sigma(M_s^n; s \in D_t^{(n)})) \qquad\qquad (2.1.2)\\
&= \frac{1}{3} \sum_{j=0}^{k-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 + \frac{2}{3} \sum_{j=0}^{k-1} \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right)
\end{aligned}$$

$$\text{for } t = \frac{k}{2^n},\ k \geq 1\,.$$

If M in addition is an L^4 - martingale, then we have also the property

$$\begin{aligned}
&Var\,(\tau_t^n | \sigma(M_s^n;\, s \in D_t^{(n)}) \qquad\qquad (2.1.3)\\
&= \frac{2}{45} \sum_{j=0}^{k-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^4 + \frac{8}{45} \sum_{j=0}^{k-1} \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^4 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right)\\
&\quad + \left(\frac{4}{9} - c(p) \right) \sum_{j=0}^{k-1} \left(\mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right) \right)^2\\
&\quad + c(p) \sum_{j=0}^{k-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right)\\
&\quad - c(p) \sum_{j=0}^{k-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right) \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^3 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right)
\end{aligned}$$

$$\text{for } t = \frac{k}{2^n},\ k \in \mathbb{Z}_+\,.$$

The problem is now to get from all the embeddings collected in the proposition a limit embedding such that at the same time the equations (2.1.2) and (2.1.3) converge in a reasonable sense. The possibility of such a limit embedding is based on an embedding theorem of Monroe (cf. [6]), theorem 1.1 and [7], theorem 2 with a new and correct proof of the former theorem.) Let us first introduce some convenient notations, which will make it possible to write the limit of (2.1.3) in a reasonable short way.

Let $t \geq 0$ be given and $k_n = \max\{j | \frac{j}{2^n} \leq t\}$. Then we write

$$[M^4]_t := \lim_{n\to\infty} \sum_{j=0}^{k_n-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}}\right)^4 , \tag{2.2}$$

$$[\mathbb{E}(M^4|.)]_t := \lim_{n\to\infty} \sum_{j=0}^{k_n-1} \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}}\right)^4 |M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}}\right) ,$$

$$[(\mathbb{E}(M^2|.))^2]_t := \lim_{n\to\infty} \sum_{j=0}^{k_n-1} \left(\mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}}\right)^2 |M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}}\right)\right)^2 ,$$

$$[M^2\mathbb{E}(M^2|.)]_t := \lim_{n\to\infty} \sum_{j=0}^{k_n-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}}\right)^2 \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}}\right)^2 |M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}}\right) ,$$

$$[M\mathbb{E}(M^3|.)]_t := \lim_{n\to\infty} \sum_{j=0}^{k_n-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}}\right) \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}}\right)^3 |M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}}\right) ,$$

provided that the limits on the right hand sides exist. Of course, these notations are a little bit inconsistent with the usual notation $[M]$ for the quadratic variation, which we keep up instead of the more consistent notation $[M^2]$, and we will also write as usual $\langle M\rangle$ for the predictable quadratic variation instead of $[\mathbb{E}(M^2|.)]$. Let us note that for a square integrable, continuous martingale all five limits above are zero, and it follows for a general square integrable cadlag – martingale from the decomposition into a continuous and a purely discontinuous part that the five limits above indeed exist for a topology, which is used in the proof of theorem 2.3 below.

Now we can state the main result of this paper.

Theorem 2.3 *Let $M = (M_t, \mathcal{F}_t)_{t\geq 0}$ $(M_0 = 0)$ be a right continuous, square integrable martingale on $(\Omega, \mathcal{F}, P)$. Then there exist*

- *a probability space (S, Σ, Q),*
- *a Brownian motion $(B_t, \mathcal{G}_t)_{t\geq 0}$ on S, and*
- *a right continuous, increasing family (τ_t) of $(\mathcal{G}_t)$ – stopping times,*

such that the following assertions hold:

2.3.1 *The process $Y = (Y_t, \mathcal{H}_t)_{t\geq 0}$, defined on S by $Y_t = B_{\tau_t}$ and $\mathcal{H}_t = \mathcal{G}_{\tau_t}$ for all $t \geq 0$, is a right continuous (even cadlag) L^2 – martingale with the same finite – dimensional distributions as M.*

2.3.2 *For every $t \geq 0$ the following equation holds:*

$$\mathbb{E}(\tau_t|\sigma(Y_s; s \leq t)) = \frac{1}{3}[Y]_t + \frac{2}{3}\langle Y\rangle_t .$$

2.3.3 *If M is an L^4-martingale, then also*

$$\begin{aligned} &Var(\tau_t|\sigma(Y_s; s \leq t)) \\ &= \frac{2}{45}[Y^4]_t + \frac{8}{45}[\mathbb{E}(Y^4|.)]_t + \left(\frac{4}{9} - c(p)\right)[(\mathbb{E}(Y^2|.))^2]_t + c(p)[Y^2\mathbb{E}(Y^2|.)]_t \\ &\quad - c(p)[Y\mathbb{E}(Y^3|.)]_t \end{aligned}$$

holds for every $t \geq 0$.

Proof: The proof consists of two parts. First we prove 2.3.1. This is essentially the proof of Monroe in [7] (theorem 2). For this reason we refer for some steps of the proof to [7]. In the second part then we prove 2.3.2 and 2.3.3.

(1) Let $C(\mathbb{R}_+)$ denote the space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. It is well known that $C(\mathbb{R}_+)$, provided with the topology of compact convergence, is a polish space. By T we denote the space of all increasing, right continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ with the local L^1-topology rel. to the Lebesgue measure. This just means that a sequence (φ_n) in T converges to $\varphi \in T$ if and only if $\lim\limits_{n\to\infty} \int_0^C |\varphi_n(s) - \varphi(s)|\, ds = 0$ for all $C > 0$. Again, T is a polish space under this topology. The essential property of T is then that for any sequence $(C_n)_{n\geq 1}$ of positive numbers the set $\{\varphi \in T | \forall n : \varphi(n) \leq c_n\}$ is relatively compact. Now we put $S := C(\mathbb{R}_+) \times T$. S is a polish space under the product topology and we denote by Σ the Borel field of S.

With the notations of proposition 2.1 we define for every $n \geq 1$ a function $F_n : \Omega^n \to S$ by

$$F_n(\omega) := (B^n_{\cdot}(\omega), \overline{\tau}^n_{\cdot}(\omega)) \quad \text{for every } \omega \in \Omega^n ,$$

where $\overline{\tau}^n$ is defined by

$$\overline{\tau}^n_t(\omega) := \sum_{k\geq 0} \tau^n_{\frac{k}{2^n}}(\omega) 1_{[\frac{k}{2^n}, \frac{k+1}{2^n}[}(t)$$

for $t \geq 0$ and $\omega \in \Omega^n$. It is easy to prove that every F_n is measurable.

Now we consider the sequence (Q_n) of probability measures on S given by $Q_n = F_n(P^n)$, and we show that (Q_n) is uniformly tight. For this it is sufficient to show that the two sequences $(\pi_1(Q_n))$ and $(\pi_2(Q_n))$ of the projections onto $C(\mathbb{R}_+)$ and T resp. are both uniformly tight. Now $(\pi_1(Q_n))$ is trivially uniformly tight as a constant sequence. For every n the measure $\pi_1(Q_n)$ is the Wiener measure on $C(\mathbb{R}_+)$. The uniform tightness of $(\pi_2(Q_n))$ is also easy to prove:

For every $\varepsilon > 0$ and every $k \in \mathbb{N}$ there exists a positive number d_k such that $\mathbb{E}M_k^2 < \frac{\varepsilon}{2^k} \cdot d_k$. We put $C_k := \{\varphi \in T | \varphi(k) \leq d_k\}$. Then $C_\varepsilon = \bigcap_{k \geq 1} C_k$ is a relatively compact subset of T. We get

$$\begin{aligned}\pi_2(Q_n)(C_k^c) &= P^n\left[\overline{\tau}_k^n > d_k\right] = P^n\left[\tau_k^n > d_k\right] \\ &\leq \frac{1}{d_k} \int \tau_k^n dP^n = \frac{1}{d_k} \int \left(B_{\tau_k^n}^n\right)^2 dP^n = \frac{1}{d_k} \mathbb{E}M_k^2 < \frac{\varepsilon}{2^k},\end{aligned}$$

and it follows that $\pi_2(Q_n)(C_\varepsilon^c) < \varepsilon$, hence $(\pi_2(Q_n))$ is uniformly tight.

Since (Q_n) is a uniformly tight sequence, it has an accumulation point Q, and w.l.o.g. we assume that the whole sequence (Q_n) converges weakly to Q.

On the probability space (S, Σ, Q) we now define

$$B_t(\omega) = f(t) \quad \text{and} \quad \tau_t(\omega) = \varphi(t)$$

for $\omega = (f, \varphi) \in S$.

With its natural filtration, (B_t) is clearly a Brownian motion on S, but for this filtration the family (τ_t) is in general not a family of stopping times. We put

$$\mathcal{G}_t := \sigma\left(\sigma(B_s; s \leq t) \cup \{[\tau_r \leq u] | r \in \mathbb{R}_+,\ u \leq t\}\right).$$

Then it is proved in [7] (not very clear but correct, whereas the first proof of this fact was false: the boundary of a set of the form $[\tau_s < u]$ is different from that stated in [6]) that $(B_t, \mathcal{G}_t)_{t \geq 0}$ is still a Brownian motion such that now every τ_t is a $(\mathcal{G}_t)$-stopping time.

For the proof that M and Y have the same finite-dimensional distributions we repeat the essential arguments of Monroe in [7], since our situation is a little bit different. Since M and Y are right continuous, it is sufficient to prove

$$(Y_{s_1}, \ldots, Y_{s_m})(Q) = (M_{s_1}, \ldots, M_{s_m})(P) \tag{2.3.4}$$

for $s_1 < \cdots < s_m$ with $s_i \in D$ for $i = 1, \ldots, m$. There exists an n_0 such that $s_i \in D^{(n_0)}$ for $i = 1, \ldots, m$ and it follows that for all Borel sets $B_i \subset \mathbb{R}$ $(i = 1, \ldots, m)$ we have the equality

$$P[M_{s_1} \in B_1, \ldots, M_{s_m} \in B_m] = Q_n\left\{(f, \varphi) \in S | f(\varphi(s_i)) \in B_i \text{ for } i = 1, \ldots, m\right\},$$

if $n \geq n_0$. For every $i = 1, \ldots, m$ we take a compact subset K_i of $\mathbb{R}$ and a larger compact set $L_i \subset \mathbb{R}$ such that $K_i \subset \mathring{L}_i$ and define the following subsets of S:

$$\Gamma_\circ = \{(f, \varphi) \in S | f(\varphi(s_i)) \in K_i \text{ for } i = 1, \ldots, m\}, \tag{2.3.5}$$

$$\Gamma_1 = \{(f, \varphi) \in S | f(\varphi(s_i)) \in L_i \text{ for } i = 1, \ldots, m\},$$

and for every δ with $0 < \delta < 1$,

$$\Gamma_\delta = \{(f, \varphi) \in S | f(\varphi(s)) \in L_i \text{ for } s_i \leq s < s_i + \delta, i = 1, \ldots, m\}.$$

Then every Γ_δ is a closed subset of S, and the right continuity of the elements of S implies

$$\Gamma_\circ \subset \bigcup_\delta \Gamma_\delta \subset \Gamma_1, \quad \text{and} \qquad (2.3.6)$$
$$\Gamma_{\delta'} \subset \Gamma_\delta \text{ for } \delta < \delta'.$$

We obtain from (2.3.6)

$$\begin{aligned} Q(\Gamma_1) &\geq Q(\Gamma_\delta) \geq \overline{\lim} Q_n(\Gamma_\delta) \\ &= \overline{\lim_n} P^n \left[M^n_{\frac{k}{2^n}} \in L_i \text{ for } s_i \leq \frac{k}{2^n} < s_i + \delta, \quad i = 1, \dots, m \right] \\ &= \overline{\lim_n} P \left[M_{\frac{k}{2^n}} \in L_i \text{ for } s_i \leq \frac{k}{2^n} < s_i + \delta, \quad i = 1, \dots, m \right] \\ &= P[M_s \in L_i \text{ for } s_i \leq s < s_i + \delta, \quad i = 1, \dots, m]. \end{aligned}$$

For $\delta \downarrow 0$ we get

$$Q(\Gamma_1) \geq P\left[M_{s_i} \in K_i \text{ for } \ i = 1, \dots m\right],$$

and this implies (for $L_i \downarrow K_i$, $i = 1, \dots, m$)

$$Q(\Gamma_0) \geq P\left[M_{s_i} \in K_i \text{ for } \ i = 1, \dots m\right].$$

Now is not difficult to extend this inequality to arbitrary Borel sets $B_i \subset \mathbb{R}$ ($i = 1, \dots, m$), and since Q and P as probability measures have the same total mass, it follows that (2.3.4) holds.

(2) Now we start with the proof of 2.3.2. We have to prove that for every $A \in \sigma(Y_s; s \leq t)$ the equation

$$\int_A \tau_t dQ = \int_A \left(\frac{1}{3}[Y]_t + \frac{2}{3}\langle Y \rangle_t \right) dQ \qquad (2.3.7)$$

holds. We know that for every $n \geq 1$ the equation (2.1.2) holds and that $\lim_{n\to\infty} Q_n = Q$ weakly. The main problem however is that the function $\tau(\omega) \to \tau_t(\omega)$ from T into $\mathbb{R}_+$ (and hence also the function $(B(\omega), \tau(\omega) \longmapsto B_{\tau_t}(\omega) = Y_t(\omega)$ from S into $\mathbb{R}$) is not a continuous function. For this reason we define for every $h > 0$ the function

$$\psi_{t,h} : T \to \mathbb{R}_+$$

by $\psi_{t,h}(\varphi) = \frac{1}{h} \int_h^{2h} \varphi(t+s) ds$. Then $\psi_{t,h}$ is a continuous, non–negative function from T into $\mathbb{R}$ and $\lim_{h\to\infty} \psi_{t,h}(\varphi) = \varphi(t)$ by the right-continuity of φ.

Now we define the following four measures on the σ–algebra $\sigma(Y_s; s \le t)$. For every $A \in \sigma(Y_s; s \le t)$ we put

$$\mu_t(A) = \int\limits_A \tau_t dQ,\ \ \nu_t(A) = \int\limits_A \left(\frac{1}{3}[Y]_t + \frac{2}{3}\langle Y\rangle_t\right) dQ, \text{ and} \tag{2.3.8}$$

$$\mu_{t,h}(A) = \int\limits_A \psi_{t,h}(\tau) dQ,\ \ \nu_{t,h}(A) = h^{-1}\int\limits_h^{2h}\int\limits_A \left(\frac{1}{3}[Y]_{t+s} + \frac{2}{3}\langle Y\rangle_{t+s}\right) dQ ds.$$

Then by right – continuity,

$$\lim_{h\to\infty}\mu_{t,h}(A) = \mu_t(A) \text{ and } \lim_{n\to\infty}\nu_{t,h}(A) = \nu_t(A)$$

for every $A \in \sigma(Y_s; s \le t)$. Hence for the proof of (2.3.7) it is sufficient to prove

$$\mu_{t,h} = \nu_{t,h} \quad \text{for every } h > 0. \tag{2.3.9}$$

We observe that $\mu_{t,h}$ and $\nu_{t,h}$ have the same total mass. This follows from

$$\int \tau_s dQ = \int B^2_{\tau_s} dQ = \int Y_s^2 dQ \qquad (s \ge 0).$$

For the proof of (2.3.9) we will show that for every finite sequence $(s_i)_{1\le i\le m}$ in D with $s_1 < \ldots < s_m \le t$ and every finite sequence $(B_i)_{1\le i\le m}$ of Borel subsets of $\mathbb{R}$ the following inequality holds:

$$\int_{[Y_{s_1}\in B_1,\ldots,Y_{s_m}\in B_m]^c} \psi_{t,h}(\tau) dQ \le h^{-1}\int\limits_h^{2h}\int_{[Y_{s_1}\in B_1,\ldots,Y_{s_m}\in B_m]^c} \left(\frac{1}{3}[Y]_{t+s} + \frac{2}{3}\langle Y\rangle_{t+s}\right) dQ ds\,. \tag{2.3.10}$$

The equality of the total mass of $\mu_{t,h}$ and $\nu_{t,h}$ then implies that (2.3.9) holds.
For the proof of (2.3.10) we will make use of following property of weak convergence.

Lemma 2.3.11 *Suppose that X is a polish space and that (μ_n) is a sequence of probability measures converging weakly towards μ. Then for every open subset G of X and every non – negative, continuous function $f : X \to \mathbb{R}$ with $\int f\, d\mu < \infty$ the inequality*

$$\int_G f\, d\mu \le \underline{\lim} \int_G f\, d\mu_n \quad \textit{holds.}$$

Proof: Let $g : X \to \mathbb{R}_+$ be a bounded, continuous function and let $\nu_n = g\mu_n$ and $\nu = g\mu$ be the measures with density g relative to μ_n and μ resp. Then it is immediate that $\nu_n \to \nu$ weakly for $n \to \infty$. It follows that

$$\nu(G) = \int_G g\, d\mu \le \underline{\lim}\nu_n(G) = \underline{\lim}\int_G g\, d\mu_n\,.$$

Hence we have for $g = f \wedge m$,

$$\int_G (f \wedge m)\, d\mu \leq \underline{\lim} \int_G (f \wedge m)\, d\mu_n \leq \underline{\lim} \int_G f\, d\mu_n\,,$$

and the assertion of the lemma follows.

Now we start with the proof of (2.3.10). The essential argument in the proof of (2.3.10) and also in the proof of (2.3.12) below will be the following observation. Since M and Y are two right continuous martingales with the same finite dimensional distributions, it follows that the random variables

$$\begin{aligned}
&\sum_{j=0}^{k-1} \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right)\,,\\
&\sum_{j=0}^{k-1} \mathbb{E}\left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^4 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right)\,,\ \text{etc.}
\end{aligned}$$

on Ω have the same distributions as the random variables

$$\begin{aligned}
&\sum_{j=0}^{k-1} \mathbb{E}\left(\left(Y_{\frac{j+1}{2^n}} - Y_{\frac{j}{2^n}} \right)^2 | Y_{\frac{1}{2^n}}, \ldots, Y_{\frac{j}{2^n}} \right)\,,\\
&\sum_{j=0}^{k-1} \mathbb{E}\left(\left(Y_{\frac{j+1}{2^n}} - Y_{\frac{j}{2^n}} \right)^4 | Y_{\frac{1}{2^n}}, \ldots, Y_{\frac{j}{2^n}} \right)\,,\ \text{etc.}
\end{aligned}$$

on (S, Σ, Q). Hence also the random variables $\langle M \rangle_t$, $[\mathbb{E}(M^4|.)]_t$ etc. have the same distributions as the random variables $\langle Y \rangle_t$, $[\mathbb{E}(Y^4|.)]_t$ etc..

Let Γ_0, Γ_1 , Γ_δ $(0 < \delta < 1)$ be the sets defined in (2.3.5), and suppose in addition that $t \in D$. Since $\psi_{t,h}$ is non-negative and continuous, and since Γ_δ^c is open, we obtain from lemma 2.3.11 for $\delta < h$,

$$
\begin{aligned}
&\int_{\Gamma_\delta^c} \psi_{t,h}(\tau)\, dQ \le \underline{\lim} \int_{\Gamma_\delta^c} \psi_{t,h}(\tau)\, dQ_n \\
&= \underline{\lim} \frac{1}{h} \int_h^{2h} \int_S \left(1_{[B_{\tau_s} \in L_i (s_i \le s < s_i + \delta,\ i=1,\dots,m)]^c} \tau_{t+r} \right) dQ_n\, dr \\
&= \underline{\lim} \frac{1}{h} \int_h^{2h} \int_{\Omega^n} (1_{[M_s^n \in L_i(s_i \le s < s_i+\delta, s \in D^n, i=1,\dots,m)]^c} \overline{\tau}_{t+r}^n\, dP^n\, dr \\
&= \underline{\lim} \frac{1}{h} \int_h^{2h} \int_{\Omega^n} \Bigg(1_{[M_s^n \in L_i(s_i \le s < s_i+\delta, s \in D^n, i=1,\dots,m)]^c} \\
&\qquad \times \left(\frac{1}{3} \sum_{j=0}^{k(r)-1} \left(M^n_{\frac{j+1}{2^n}} - M^n_{\frac{j}{2^n}} \right) + \frac{2}{3} \sum_{j=0}^{k(r)-1} \mathbb{E}\left(\left(M^n_{\frac{j+1}{2^n}} - M^n_{\frac{j}{2^n}} \right)^2 | M^n_{\frac{1}{2^n}}, \dots, M^n_{\frac{j}{2^n}} \right) \right) dP^n\, dr \\
&\quad \left(\text{by (2.3.2), where } \frac{k(r)}{2^n} \le t + r < \frac{k(r)+1}{2^n} \right) \\
&= \frac{1}{h} \int_h^{2h} \int_\Omega 1_{[M_s \in L_i(s_i \le s < s_i+\delta, s \in D, i=1,\dots,m)]^c} \left(\frac{1}{3}[M]_{t+r} + \frac{2}{3} \langle M \rangle_{t+r} \right) dP\, dr \\
&= \frac{1}{h} \int_h^{2h} \int_S 1_{\Gamma_\delta^c} \left(\frac{1}{3}[Y]_{t+r} + \frac{2}{3} \langle Y \rangle_{t+r} \right) dQ\, dr\,.
\end{aligned}
$$

Hence we proved

$$
\begin{aligned}
\mu_{t,h}(\Gamma_\delta^c) &\le \nu_{t,h}(\Gamma_\delta^c), \qquad \text{which implies} \\
\mu_{t,h}(\Gamma_1^c) &\le \nu_{t,h}(\Gamma_0^c), \\
\text{and for } L_i &\downarrow K_i (i = 1, \dots, m) \\
\mu_{t,h}(\Gamma_0^c) &\le \nu_{t,h}(\Gamma_0^c).
\end{aligned}
$$

By the same argument as in the proof (2.3.4) this finally implies (2.3.10) and hence (2.3.9).

For the proof of 2.3.3 we use that we already know 2.3.2. Hence we have to prove

$$
\begin{aligned}
&\mathbb{E}\left(\tau_t^2 | \sigma(Y_s; s \le t) \right) \\
&= \left(\frac{1}{3}[Y]_t + \frac{2}{3} \langle Y \rangle_t \right)^2 \\
&\quad + \left(\frac{2}{45}[Y^4]_t + \dots + (-c(p))[Y \mathbb{E}(Y^3 |.)]_t \right)\ Q - a.s.
\end{aligned}
\tag{2.3.12}
$$

for every $t \ge 0$.

For the proof of (2.3.12) we proceed similarly as in the proof of (2.3.2). Let $G(Y,t)$ denote the right hand side of (2.3.12), and let $\varphi_{t,h} : T \to \mathbb{R}_+$ be the continuous function defined by

$$
\varphi_{t,h}(\tau) = \frac{1}{h} \int_h^{2h} \tau_{t+s}^2 ds. \tag{2.3.13}
$$

Let $\mu_t, \nu_t, \mu_{t,h}, \nu_{t,h}$ now denote the four measures on $\sigma(Y_s; s \leq t)$ defined by

$$\begin{aligned} \mu_t(A) &= \int_A \tau_t^2 dQ, \\ \nu_t(A) &= \int_A G(Y,t) dQ, \\ \mu_{t,h}(A) &= \int_A \varphi_{t,h}(\tau) dQ, \text{ and} \\ \nu_{t,h}(A) &= \frac{1}{h} \int_h^{2h} \int_A G(Y, t+s) dQ ds \end{aligned} \tag{2.3.14}$$

for $A \in \sigma(Y_s; s \leq t)$.
Again, by right continuity, $\mu_{t,h}(A) \to \mu_t(A)$ and $\nu_{t,h}(A) \to \nu_t(A)$ for $h \to 0$, and it is sufficient for the proof of (2.3.12) to prove

$$\mu_{t,h} = \nu_{t,h} \tag{2.3.15}$$

for every $h > 0$.
The main problem now is to prove that $\mu_{t,h}$ and $\nu_{t,h}$ have the same total mass. For the proof of this fact it is sufficient to prove

$$\int \tau_s^2 dQ = \int G(Y,s) dQ \tag{2.3.16}$$

for every $s \geq 0$. It is easy to see that

$$\lim_{n\to\infty} \int G(Y,s) dQ_n = \int G(Y,s) dQ$$

and from (2.1.3) we have

$$\int \tau_s^2 dQ_n = \int G(Y,s) dQ_n$$

for every $n \geq 1$. Hence it is sufficient to prove

$$\lim_{n\to\infty} \int \tau_s^2 dQ_n = \int \tau_s^2 dQ. \tag{2.3.17}$$

Proof of (2.3.17): We use the relation

$$\int \tau_s^2 dQ = 2 \int B_{\tau_s}^2 \tau_s dQ - \frac{1}{3} \int B_{\tau_s}^4 dQ \tag{2.3.18}$$

(cf: e. g. [3], p. 128) and suppose w. l. o. g. that $s \in D$.
Define

$$R_n = \int M_s^2 \left(\frac{1}{3} \left([M]_s - \sum_{j=0}^{k(s)-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 \right) \right.$$
$$\left. + \frac{2}{3} \left(\langle M \rangle_s - \sum_{j=0}^{k(s)-1} \mathbb{E} \left(\left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 | M_{\frac{1}{2^n}}, \ldots, M_{\frac{j}{2^n}} \right) \right) \right) dP$$

(with $s = \frac{k(s)}{2^n}$ for sufficiently large n).

Then we obtain from (2.3.18) together with (2.1.2)

$$\begin{aligned}
\int \tau_s^2 dQ &= 2 \int Y_s^2 \left(\frac{1}{3}[Y]_s + \frac{2}{3}\langle Y \rangle_s \right) dQ - \frac{1}{3} \int Y_s^4 dQ \\
&= 2 \int M_s^2 \left(\frac{1}{3}[M]_s + \frac{2}{3}\langle M \rangle_s \right) dP - \frac{1}{3} \int M_s^4 dP \\
&= 2 \int M_s^2 \left(\frac{1}{3} \sum_{j=0}^{k(s)-1} \left(M_{\frac{j+1}{2^n}} - M_{\frac{j}{2^n}} \right)^2 + \frac{2}{3}\langle M \rangle_s \right) dP - \frac{1}{3} \int M_s^4 dP + 2R_n \\
&= \int \tau_s^2 dQ_n + 2R_n;
\end{aligned}$$

and (2.3.17) follows from $\lim_{n \to \infty} R_n = 0$.
Now the proof of (2.3.15) is done essentially in the same way as the proof of (2.3.9), using lemma 2.3.11 and the sets $\Gamma_0, \Gamma_1, \Gamma_\delta$ defined in (2.3.5). This finishes the proof of theorem 2.3.

3 Evaluation of the constants

Suppose that $M = (M_t, \mathcal{F}_t)$ is a continuous, bounded martingale, which is embedded into a Brownian motion $B = (B_t, \mathcal{G}_t)$ by the theorem of Dubins and Schwartz, i.e. $M_t = B_{\tau_t}$ a.s. for the family (τ_t) of stopping times given by $\tau_t = [M]_t$. It is well known that for every θ the process Z^θ defined by

$$Z_t^\theta = \exp\left(\theta B_t - \frac{1}{2}\theta^2 t \right)$$

is a martingale. Since M is assumed to be bounded, it follows that also

$$Y_t^\theta = Z_{\tau_t}^\theta = \exp\left(\theta B_{\tau_t} - \frac{1}{2}\theta^2 \tau_t \right) = \exp\left(\theta M_t - \frac{1}{2}\theta^2 \tau_t \right)$$

is a martingale for every θ.

Now let $t \geq 0$ be fixed and let $\mathcal{H}$ be a $\sigma-$algebra for which M_t (but not necessarily τ_t) is measurable. It follows that $\mathbb{E}[Y_t^\theta|\mathcal{H}]$ has the obvious series expansion

$$\mathbb{E}[Y_t^\theta|\mathcal{H}] = e^{\theta M_t} \cdot \sum_{k\geq 0} \left(-\frac{1}{2}\right)^k \frac{\theta^{2k}}{k!}\mathbb{E}[\tau_t^k|\mathcal{H}]\,.$$

It follows (cf. [9]) that for small θ the function

$$L(\theta) := \ln \mathbb{E}[Y_t^\theta|\mathcal{H}] \qquad (t \text{ fixed}) \tag{3.1.1}$$

has a series expansion given by

$$L(\theta) = \theta M_t + \sum_{k\geq 1} \theta^{2k} \left(-\frac{1}{2}\right)^k \frac{1}{k!}\mathrm{cum}_k(\tau_t|\mathcal{H})\,, \tag{3.1.2}$$

where $\mathrm{cum}_k(\tau_t|\mathcal{H})$ denotes the kth conditional cumulant of τ_t. For $k = 1$ or 2 resp. we have $\mathrm{cum}_1(\tau_t|\mathcal{H}) = \mathbb{E}(\tau_t|\mathcal{H})$ and $\mathrm{cum}_2(\tau_t|\mathcal{H}) = Var(\tau_t|\mathcal{H})$. Moreover, since $(Y_t^\theta)_{t\geq 0}$ is a martingale, we have that

$$\mathbb{E}\exp(L(\theta)) = 1 \tag{3.1.3}$$

i.e. $L(\theta)$ is a likelihood function. This implies that $L(\theta)$ satisfies the so−called Bartlett identities, which are a trivial consequence of the Taylor expansion of $\mathbb{E}\exp(L(\theta))$ at 0. For our purposes we only need the first five Bartlett identities.

3.2 Bartlett identities Let L^I, L^{II}, L^{III}, L^{IV}, L^V denote the first five derivatives of L at $\theta = 0$. Then the following equations hold:

$$\mathbb{E}L^I = 0\,, \tag{3.2.1}$$

$$\mathbb{E}(L^{II} + (L^I)^2) = 0\,, \tag{3.2.2}$$

$$\mathbb{E}(L^{III} + 3L^{II}\cdot L^I + (L^I)^3) = 0\,, \tag{3.2.3}$$

$$\mathbb{E}(L^{IV} + 4L^{III}\cdot L^I + 3(L^{II})^2 + 6L^{II}(L^I)^2 + (L^I)^4) = 0\,, \text{ and} \tag{3.2.4}$$

$$\mathbb{E}(L^V + 5L^{IV}\cdot L^I + 10L^{III}\cdot L^{II} + 10L^{III}(L^I)^2 + 15(L^{II})^2L^I + 10L^{II}(L^I)^3 + (L^I)^5) = 0\,. \tag{3.2.5}$$

Now we can compute the constants of Corollary 1.4 as already stated in remark 1.5.

Proposition 3.3 *The constants $\alpha_1(p), \ldots, \beta_5(p)$ of corollary 1.4 are given as follows*

$$\alpha_1(p) = \frac{1}{3}\,, \qquad \alpha_2(p) = \frac{2}{3}\,,$$
$$\beta_1(p) = \frac{2}{45}\,, \quad \beta_2(p) = \frac{8}{45}\,,$$
$$\beta_3(p) = \frac{4}{9} - c(p)\,, \quad \beta_4(p) = c(p)\,, \quad \beta_5(p) = -c(p)\,,$$

where only $c(p)$ depends on p.

Proof: With the notations of §1 we know from proposition 1.3 that the constants in (1.4.1) and (1.4.2) only depend on the special embedding of theorem 1.1 and not on the embedded X. Hence we may assume that X is bounded, which implies that the martingale $(M_t(X))_{0\le t\le 1}$ is bounded. For $L(\theta)=\ln \mathbb{E}(Z^\theta_{\tau_1}|X)$ with $\tau_1=[M(X)]_1$ we have therefore the series expansion

$$L(\theta)=\theta M_1(X)+\sum_{k\ge 1}\theta^{2k}\left(-\frac{1}{2}\right)^k\frac{1}{k!}\mathrm{cum}_k(\tau_1|X)$$

from (3.1.2), and we know that the Bartlett identities 3.2 must hold. Since

$$L^I=M_1(X)=X\,,\quad L^{II}=-\mathbb{E}(\tau_1|X)\,,\quad L^{III}=0\,,$$
$$L^{IV}=3\,Var(\tau_1|X)\,,\quad L^V=0\,,$$

the first five Bartlett identities are now

(1) $\mathbb{E}X=0\,,$

(2) $\mathbb{E}X^2=\mathbb{E}(\mathbb{E}(\tau_1|X))\,,$

(3) $\mathbb{E}X^3=3\mathbb{E}(\mathbb{E}(\tau_1|X)\cdot X)\,,$

(4) $3\mathbb{E}(\,Var(\tau_1|X))+3\mathbb{E}(\mathbb{E}(\tau_1|X))^2-6\mathbb{E}(\mathbb{E}(\tau_1|X)\cdot X^2)+\mathbb{E}X^4=0\,,$ and

(5) $15\mathbb{E}(\,Var(\tau_1|X)\cdot X)+15\mathbb{E}((\mathbb{E}(\tau_1|X))^2\cdot X)-10\mathbb{E}(\mathbb{E}(\tau_1|X)\cdot X^3)+\mathbb{E}X^5=0\,.$

Substituting $\mathbb{E}(\tau_1|X)=\alpha_1(p)X^2+\alpha_2(p)\mathbb{E}X^2$ into (3) yields $\mathbb{E}X^3=3\alpha_1(p)\mathbb{E}X^3$, which implies $\alpha_1(p)=\frac{1}{3}$, since X is arbitrary (beside the assumption of boundedness). From (2) we get $\alpha_2(p)=\frac{2}{3}$.
Substitution of

$$Var(\tau_1|X)=\beta_1(p)X^4+\beta_2(p)\mathbb{E}X^4+\beta_3(p)(\mathbb{E}X^2)^2+\beta_4(p)X^2\mathbb{E}X^2+\beta_5(p)X\mathbb{E}X^3 \qquad (*)$$

into (4) yields

$$\left(3\left(\beta_1(p)+\beta_2(p)\right)-\frac{2}{3}\right)\mathbb{E}X^4+\left(3\left(\beta_3(p)+\beta_4(p)\right)-\frac{4}{3}\right)(\mathbb{E}X^2)^2=0,$$

which gives

$$\beta_1(p)+\beta_2(p)=\frac{2}{9}\text{ and }\beta_3(p)+\beta_4(p)=\frac{4}{9}. \qquad (6)$$

Substitution of $(*)$ into (5) yields

$$\left(15\beta_1(p)-\frac{2}{3}\right)\mathbb{E}X^5+15\left(\beta_4(p)+\beta_5(p)\right)\mathbb{E}X^3\cdot\mathbb{E}X^2=0,$$

which gives

$$\beta_1(p) = \frac{2}{45} \text{ and } \beta_5(p) = -\beta_4(p). \tag{7}$$

With $c(p) := \beta_4(p)$ we obtain from (6) and (7) that

$$\beta_2(p) = \frac{8}{45}, \quad \beta_3(p) = \frac{4}{9} - c(p), \quad \beta_4(p) = c(p), \quad \beta_5(p) = -c(p).$$

So it remains to compute the constant $c(p)$. For this we need some moments of the random variables involved in the Skorohod imbedding (see (1.1.1) and(1.3.2)).
In [10] Skorohod derived the Laplace transform of the exit time τ from an interval $[a, b]$ containing 0 of a Brownian motion B starting at 0. He obtained

$$\mathbb{E}e^{-t\tau} = \frac{\sinh b\sqrt{2t} - \sinh a\sqrt{2t}}{\sinh(b-a)\sqrt{2t}}.$$

This gives

$$\mathbb{E}\tau^2 = \frac{d^2}{dt^2}\left(\mathbb{E}e^{-t\tau}\right)|_{t=0} = -\frac{1}{3}ab(a^2 - 3ab + b^2).$$

The well-know equation (cf. e.g. [3], p. 128)

$$6\mathbb{E}(B_\tau^2 \cdot \tau) = \mathbb{E}B_\tau^4 + 3\mathbb{E}\tau^2$$

then also gives a formula for $\mathbb{E}(B_\tau^2 \cdot \tau)$. From this we derive immediately for the exit times $\sigma_k (k \geq 0)$ of $[-p, 1-p]$ the corresponding moments, which are listed in the proposition below.

Proposition 3.4 *With the notations of §1 we have the following moment relations*

$$\begin{aligned}
\mathbb{E}\sigma_k &= \mathbb{E}\left(B_{\sigma_k}^k\right)^2 = \mathbb{E}\xi_k^2 = q := p(1-p)\,, && (3.4.1)\\
\mathbb{E}\sigma_k^2 &= \frac{1}{3}(q+q^2)\,, \\
\mathbb{E}\sigma_k\xi_k^2 &= \mathbb{E}\sigma_k\left(B_{\sigma_k}^k\right)^2 = \frac{1}{3}(q-q^2)\,, \\
\mathbb{E}\zeta_k &= (1-2q), \mathbb{E}\zeta_k^2 = (1-3q)\,, && (3.4.2)\\
\mathbb{E}X_{2^{-n}}^2 &= \mathbb{E}Y_{2^{-n}}^2 = (1-2q)^n\mathbb{E}X^2\,, \\
\mathbb{E}X_{2^{-n}}^3 &= (1-3q)^n\mathbb{E}X^3\,, \ \textit{and} \\
\zeta_k\xi_k &\equiv q\,.
\end{aligned}$$

Now we can compute the constant $c(p)$.

Proposition 3.5 *For every p with $0 < p < 1$ we have*

$$c(p) = \frac{16}{45}\frac{5-3q}{5-6q}.$$

Proof: We have already remarked that the constants of proposition 3.3 only depend on the Skorohod embedding, i.e. on moments of random variables connected with the sequence $(L^k)_{k\geq 0}$ of stopped Brownian motions introduced in the proof of Theorem 1.1, and not on the embedded random variable X. One special feature of the Skorohod embedding of theorem 1.1 is the property that

$$X_{2^{-n}} = M_{2^{-n}}(X)$$

can be viewed itself as the Skorohod embedding $M_1(X_{2^{-n}})$.
Hence we have also for

$$\tau_{2^{-n}} = [M(X)]_{2^{-n}} = \sum_{k\geq n} (X_{2^{-k}} - Y_{2^{-k}})^2 \sigma_k$$

the relations

$$\begin{aligned} \mathbb{E}(\tau_{2^{-n}}|X_{2^{-n}}) &= \frac{1}{3}X^2_{2^{-n}} + \frac{2}{3}\mathbb{E}X^2_{2^{-n}} \text{ and} \\ Var\,(\tau_{2^{-n}}|X_{2^{-n}}) &= \frac{2}{45}X^4_{2^{-n}} + \frac{8}{45}\mathbb{E}X^4_{2^{-n}} + \left(\frac{4}{9} - c(p)\right)(\mathbb{E}X^2_{2^{-n}})^2 \\ &\quad + c(p)X^2_{2^{-n}}\mathbb{E}X^2_{2^{-n}} + (-c(p))X_{2^{-n}}\mathbb{E}X^3_{2^{-n}}. \end{aligned} \tag{3.5.1}$$

We use (3.5.1) to compute $c(p)$. We have

$$\tau_1 = \tau_{2^{-1}} + A_0 \text{ with } A_0 = (X - Y_0)^2\sigma_0,$$

and we obtain

$$\begin{aligned} Var(\tau_1|X) &= \mathbb{E}((A_0 + \tau_{2^{-1}})^2|X) - \left(\frac{1}{3}X^2 + \frac{2}{3}\mathbb{E}X^2\right)^2 \\ &= \mathbb{E}(A_0^2|X) \\ &\quad + 2\mathbb{E}(A_0\tau_{2^{-1}}|X) \\ &\quad + \mathbb{E}\left[Var(\tau_{2^{-1}}|X_0^{(1)}, X_0^{(2)}, X)|X\right] \\ &\quad + \mathbb{E}\left[\mathbb{E}(\tau_{2^{-1}}|X_0^{(1)}, X_0^{(2)}, X)|X\right] \\ &\quad - \left(\frac{1}{3}X^2 + \frac{2}{3}\mathbb{E}X^2\right)^2. \end{aligned} \tag{3.5.2}$$

The coefficicent of $X\mathbb{E}X^3$ in the equation (3.5.1) for $n=0$ is just $(-c(p))$. We now compute the coefficient of $X\mathbb{E}X^3$ on the right hand side of (3.5.2), which gives the equation for $c(p)$.

$$\begin{aligned}\mathbb{E}(A_0^2|X) &= \mathbb{E}((X-Y_0)^4\sigma_0^2|X) \qquad (1)\\ &= \mathbb{E}(-4XY_0^3\sigma_0^2|X)+R_1\\ &= -4X\mathbb{E}X^3\mathbb{E}\sigma_0^2+R_1=\left\{-\frac{4}{3}(q+q^2)\right\}X\mathbb{E}X^3+R_1\end{aligned}$$

by (3.4.1), where R_1 (and in the following $R_i(i\geq 2)$) contains the remaining terms (with $X^4, \mathbb{E}X^4, (\mathbb{E}X^2)^2$, $X^2\mathbb{E}X^2$ as factors).

$$\begin{aligned}&\mathbb{E}(A_0\tau_{2^{-1}}|X) \qquad (2)\\ &= \mathbb{E}\left\{(X-Y_0)^2\sigma_0\sum_{n\geq 1}(Z_nX+V_n-Y_{2^{-n}})^2\sigma_n|X\right\}\end{aligned}$$

(cf. 1.3)

$$\begin{aligned}&= B_1+B_2+R_2, \quad \text{with}\\ &B_1 = -2\mathbb{E}\left\{XY_0\sigma_0\sum_{n\geq 1}(V_n-Y_{2^{-n}})^2\sigma_n|X\right\} \text{ and}\\ &B_2 = 2\mathbb{E}\left\{Y_0^2\sigma_0\sum_{n\geq 1}Z_nX(V_n-Y_{2^{-n}})\sigma_n|X\right\}.\end{aligned}$$

We get with the aid 1.3 and 3.4

$$\begin{aligned}B_1 &= -2X\mathbb{E}X^3\sum_{n\geq 1}\mathbb{E}(\sigma_0Z_{n,0}^2\sigma_n)\\ &= -2X\mathbb{E}X^3\sum_{n\geq 1}q\mathbb{E}((\sigma_0\xi_0^2)\zeta_1^2\dots\zeta_{n-1}^2)\\ &= -2X\mathbb{E}X^3\left\{q\left(\frac{1}{3}(q-q^2)\right)\sum_{n\geq 1}(1-3q)^{n-1}\right\}\\ &= \left\{-\frac{2}{9}(q-q^2)\right\}X\mathbb{E}X^3,\\ B_2 &= 2X\mathbb{E}X^3\cdot\sum_{n\geq 1}\mathbb{E}(\sigma_n\sigma_0Z_nZ_{n,0})\\ &= 2X\mathbb{E}X^3q\sum_{n\geq 1}\mathbb{E}(\sigma_0(\xi_0\zeta_0)\zeta_1^2\dots\zeta_{n-1}^2)\\ &= 2X\mathbb{E}X^3\cdot q^3\sum_{n\geq 1}(1-3q)^{n-1}\\ &= \left\{\frac{2}{3}q^2\right\}X\mathbb{E}X^3.\end{aligned}$$

So we have

$$\mathbb{E}(A_0\tau_{2^{-1}}|X) = \left\{\frac{2}{3}q^2 - \frac{2}{9}(q-q^2)\right\} X\mathbb{E}X^3 + R_2\,,$$

$$\begin{aligned}
&\mathbb{E}[\,Var\,(\tau_{2^{-1}}|X_0^{(1)}, X_0^{(2)}, X)|X] \qquad (3)\\
&= \mathbb{E}[\,Var\,(\tau_{2^{-1}}|X_{2^{-1}})|X] \qquad \text{(because of } X_{2^{-1}} = pX_0^{(1)} + (1-p)X_0^{(2)})\\
&= \mathbb{E}\left[\left(\frac{2}{45}X_{2^{-1}}^4 + c(p)X_{2^{-1}}^2\mathbb{E}X_{2^{-1}}^2 - c(p)X_{2^{-1}}\mathbb{E}X_{2^{-1}}^2\right)|X\right] + R_{3,1}\\
&\quad \text{(by(3.5.1))}\\
&= \mathbb{E}\left[\frac{2}{45}(\zeta_0X+\xi_0Y_0)^4 + c(p)(\zeta_0X+\xi_0Y_0)^2\mathbb{E}X_{2^{-1}}^2 - c(p)(\zeta_0X+\xi_0Y_0)\mathbb{E}X_{2^{-1}}^3|X\right] + R_{3,1}\\
&= \frac{8}{45}X\mathbb{E}X^3\mathbb{E}(\zeta_0\xi_0^3) - c(p)X\mathbb{E}\zeta_0\mathbb{E}X_{2^{-1}}^3 + R_3 \qquad \text{(because of } \mathbb{E}Y_0 = 0)\\
&= \left\{\frac{8}{45}q^2 - c(p)(1-2q)(1-3q)\right\} X\mathbb{E}X^3 + R_3 \qquad \text{(by 3.4)},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}[(\mathbb{E}(\tau_{2^{-1}}|X_0^{(1)}, X_0^{(2)}, X))^2|X] \qquad (4)\\
&= \mathbb{E}[(\mathbb{E}(\tau_{2^{-1}}|X_{2^{-1}}))^2|X]\\
&= \mathbb{E}\left[\left(\frac{1}{3}X_{2^{-1}}^2 + \frac{2}{3}\mathbb{E}X_{2^{-1}}^2\right)^2|X\right]\\
&= \mathbb{E}\left[\frac{1}{9}(\zeta_0X+\xi_0Y_0)^4 + \frac{4}{9}(\zeta_0X+\xi_0Y_0)^2\mathbb{E}X_{2^{-1}}^2|X\right] + R_{4,1}\\
&= (X\mathbb{E}X^3)\left(\frac{4}{9}\mathbb{E}\zeta_0\xi_0^3\right) + R_4\\
&= \left\{\frac{4}{9}q^2\right\} X\mathbb{E}X^3 + R_4.
\end{aligned}$$

From (1) to (4) we now have the following equation for $c(p)$:

$$-c(p) = -\frac{4}{3}(q+q^2) + 2\left\{\frac{2}{3}q^2 - \frac{2}{9}(q-q^2)\right\} + \frac{8}{45}q^2 + \frac{4}{9}q^2 - c(p)(1-5q-6q^2),$$

from which we obtain the asserted equation

$$c(p)(5q-6q^2) = \frac{16}{9}q - \frac{16}{15}q^2.$$

Remark 3.6 One can modify the embedding of a right continuous martingale M into a Brownian motion in theorem 2.3 in the following way. Suppose that the martingale M^n of proposition 2.1 are Skorohod embeddings of $(M_t)_{t\in D^{(n)}}$ for $p_n(0 < p_n < 1)$ such that $\lim_{n\to\infty} p_n = 0$ ($\lim_{n\to\infty} p_n = 1$ would do the same). Then

$$\lim_{n\to\infty} c(p_n) = \frac{16}{45},$$

and assertion 2.3.3 of theorem 2.3 would read

$$\begin{aligned}&Var(\tau_t|\sigma(Y_s; s \le t))\\&= \frac{2}{45}[Y^4]_t + \frac{8}{45}[\mathbb{E}(Y^4|\cdot)]_t + \frac{4}{45}[(\mathbb{E}(Y^2|\cdot))^2]_t + \frac{16}{45}[Y^2\mathbb{E}(Y^2|\cdot)]_t - \frac{16}{45}[Y\mathbb{E}(Y^3|\cdot)]_t.\end{aligned}$$

4 A series expansion for a right continuous martingale

For an application of the foregoing results we will prove a series expansion of second order for a right continuous martingale. This expansion is based on the following lemma, which is probably well-known, but in lack of a suitable reference we give a complete proof.

Lemma 4.1 *Let $(B_t, \mathcal{F}_t) = B$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, let further σ, τ be square integrable $(\mathcal{F}_t)$–stopping times such that $\sigma \vee \tau$ is $\mathcal{F}_\tau$–measurable, and $f : \mathbb{R} \to \mathbb{R}$ a bounded, four times continuously differntiable function with bounded derivatives. Then the following second order series expansion holds:*

$$\begin{aligned}&\mathbb{E}f(B_\tau) - \mathbb{E}f(B_\sigma)\\&= \frac{1}{2}\mathbb{E}\{f^{II}(B_\tau)(\tau - \sigma)\} - \frac{1}{8}\mathbb{E}\{f^{IV}(B\tau)(\tau - \sigma)^2\} + R(\sigma, \tau),\end{aligned} \tag{4.1.1}$$

where the remaider term $R(\sigma, \tau)$ is estimated by

$$\begin{aligned}|R(\sigma,\tau)| &\le \frac{1}{8}\mathbb{E}\{\rho(\sigma,\tau)(\tau - \sigma)^2\} \text{ with}\\ \rho(\sigma,\tau) &= \sup_{s\in[\sigma\wedge\tau, \sigma\vee\tau]} |f^{IV}(B_s) - f^{IV}(B_\tau)|.\end{aligned}$$

Proof: The proof is based on a twofold application of the Ito–formula. We have

$$\begin{aligned}&\mathbb{E}f(B_\tau) - \mathbb{E}f(B_\sigma)\\&= \ \mathbb{E}\{1_{[\tau<\sigma]}(f(B_\tau) - f(B_\sigma))\} + \mathbb{E}\{1_{[\tau>\sigma]}(f(B_\tau) - f(B_\sigma))\}\\&= -\mathbb{E}\{1_{[\tau<\sigma]}(f(B_{\sigma\vee\tau}) - f(B_\tau))\} + \mathbb{E}\{1_{[\tau>\sigma]}(f(B_{\sigma\vee\tau}) - f(B_\sigma))\}\\&= -\mathbb{E}\left\{1_{[\tau<\sigma]}\left(\frac{1}{2}\int_\tau^{\sigma\vee\tau} f^{II}(B_s)ds + \int_\tau^{\sigma\vee\tau} f^I(B_s)dB_s\right)\right\}\\&\quad + \mathbb{E}\left\{1_{[\tau>\sigma]}\left(\frac{1}{2}\int_\sigma^{\sigma\vee\tau} f^{II}(B_s)ds + \int_\sigma^{\sigma\vee\tau} f^I(B_s)dB_s\right)\right\}.\\&\qquad\qquad \text{(by the Ito–formula)}\end{aligned}$$

$$= -\frac{1}{2}\mathbb{E}\left\{1_{[\tau<\sigma]}\int\limits_{\tau}^{\sigma\vee\tau} f^{II}(B_s)ds\right\} + \frac{1}{2}\mathbb{E}\left\{1_{[\tau>\sigma]}\int\limits_{\sigma}^{\sigma\vee\tau} f^{II}(B_s)ds\right\}$$

$\Bigg($ since $[\tau<\sigma], [\tau>\sigma] \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau($ cf.[12]), and

$$\mathbb{E}\left[\int\limits_{\tau}^{\sigma\vee\tau} f^I(B_s)dB_s|\mathcal{G}_\tau\right] = 0, \mathbb{E}\left[\int\limits_{\sigma}^{\sigma\vee\tau} f^I(B_s)dB_s|\mathcal{G}_\tau\right] = 0$$

by Doob's stopping theorem applied to the martingale $(\int\limits_0^t f^I(B_s)dB_s)_{t\geq 0}\Bigg)$

$$= -\frac{1}{2}\mathbb{E}\{1_{[\tau<\sigma]}f^{II}(B_\tau)(\sigma\vee\tau-\tau)\} + \frac{1}{2}\mathbb{E}\{1_{[\tau>\sigma]}f^{II}(B_\tau)(\sigma\vee\tau-\sigma)\} + S(\sigma,\tau)$$

$$= \frac{1}{2}\mathbb{E}\{f^{II}(B_\tau)(\tau-\sigma)\} + S(\sigma,\tau), \quad \text{where}$$

$$S(\sigma,\tau) = -\frac{1}{2}\mathbb{E}\left\{1_{[\tau<\sigma]}\int\limits_{\tau}^{\sigma\vee\tau}(f^{II}(B_s)-f^{II}(B_\tau))ds\right\} + \frac{1}{2}\mathbb{E}\left\{1_{[\tau>\sigma]}\int\limits_{\sigma}^{\sigma\vee\tau}(f^{II}(B_s)-f^{II}(B_\tau))ds\right\}.$$

For $S(\sigma,\tau)$ we handle both terms on the right hand side separately.

(i) $\mathbb{E}\left\{1_{[\tau<\sigma]}\int\limits_{\tau}^{\sigma\vee\tau}(f^{II}(B_s)-f^{II}(B_\tau))ds\right\}$

$$= \mathbb{E}\left\{1_{[\tau<\sigma]}\int\limits_{\tau}^{\sigma\vee\tau}\left[\frac{1}{2}\int\limits_{\tau}^{s} f^{IV}(B_r)dr + \int\limits_{\tau}^{s} f^{III}(B_r)dB_r\right]ds\right\}$$

(from the Ito-formula).

Now it is not difficult to prove the formula

$$\int\limits_0^t\int\limits_0^s f^{III}(B_r)dB_r ds = t\int\limits_0^t f^{III}(B_r)dB_r - \int\limits_0^t (rf^{III}(B_r))dB_r.$$

[Consider the martingale $M_t = \int\limits_0^t f^{III}(B_r)dB_r$ and apply the Ito-formula with the function $f(t,x) = tx$.]

From this formula and the assumption that the stopping time $\sigma \vee \tau$ is $\mathcal{F}_\tau$–measurable, we get that

$$\mathbb{E}\left\{\int_\tau^{\sigma\vee\tau}\int_\tau^{s} f^{III}(B_r)dB_r ds|\mathcal{F}_\tau\right\}$$
$$= \mathbb{E}\left\{(\sigma\vee\tau)\int_\tau^{\sigma\vee\tau} f^{III}(B_s)dB_s - \int_\tau^{\sigma\vee\tau}(sf^{III}(B_s))dB_s|\mathcal{F}_\tau\right\} = 0\,.$$

Thus we have obtained that

$$\mathbb{E}\left\{1_{[\tau<\sigma]}\int_\tau^{\sigma\vee\tau}(f^{II}(B_s)-f^{II}(B_\tau))ds\right\}$$
$$= \frac{1}{2}\mathbb{E}\left\{1_{[\tau<\sigma]}\int_\tau^{\sigma\vee\tau}\int_\tau^{s} f^{IV}(B_r)dr\right\}.$$

(ii) $\mathbb{E}\left\{1_{[\tau>\sigma]}\int_\sigma^{\sigma\vee\tau}(f^{II}(B_s)-f^{II}(B_\tau))ds\right\}$
$$= \mathbb{E}\left\{1_{[\tau>\sigma]}\int_\sigma^{\sigma\vee\tau}\left[-\frac{1}{2}\int_s^{\sigma\vee\tau} f^{IV}(B_r)dr + \int_s^{\sigma\vee\tau} f^{III}(B_r)dB_r\right]ds\,.\right.$$

Now we put $N_t = \int_0^t\int_s^t f^{III}(B_r)dB_r ds$ and obtain as above $N_t = \int_0^t (rf^{III}(B_r))dB_r$, which shows that (N_t) is a martingale. By Doob's stopping theorem we get

$$\mathbb{E}\left[\int_\sigma^{\sigma\vee\tau}\int_s^{\sigma\vee\tau} f^{III}(B_r)dB_r ds|\mathcal{F}_\sigma\right]$$
$$= \mathbb{E}\left[\int_\sigma^{\sigma\vee\tau}(rf^{III}(B_r))dB_r|\mathcal{F}_\sigma\right] = 0\,,$$

and hence

$$\mathbb{E}\left\{1_{[\tau>\sigma]}\int_\sigma^{\sigma\vee\tau}(f^{II}(B_s)-f^{II}(B_\tau))ds\right\}$$
$$= -\frac{1}{2}\mathbb{E}\left\{1_{[\tau>\sigma]}\int_\sigma^{\sigma\vee\tau}\int_s^{\sigma\vee\tau} f^{IV}(B_r)ds\,dr\right\}.$$

Altogether we have proved

$$S(\sigma,\tau) = -\frac{1}{4}\mathbb{E}\left\{ 1_{[\tau<\sigma]} \int_{\tau}^{\sigma\vee\tau} \int_{\tau}^{s} f^{IV}(B_r)dr\,ds \right\} - \frac{1}{4}\mathbb{E}\left\{ 1_{[\tau>\sigma]} \int_{\sigma}^{\sigma\vee\tau} \int_{s}^{\sigma\vee\tau} f^{IV}(B_r)drds \right\}.$$

We compute further

$$\begin{aligned} S(\sigma,\tau) = &-\frac{1}{4}\mathbb{E}\left\{ 1_{[\tau<\sigma]} \int_{\tau}^{\sigma\vee\tau} \int_{\tau}^{s} f^{IV}(B_\tau)dr\;ds \right\} \\ &- \frac{1}{4}\mathbb{E}\left\{ 1_{[\tau>\sigma]} \int_{\sigma}^{\sigma\vee\tau} \int_{s}^{\sigma\vee\tau} f^{IV}(B_\tau)drds \right\} \\ &- \frac{1}{4}\mathbb{E}\left\{ 1_{[\tau<\sigma]} \int_{\tau}^{\sigma\vee\tau} \int_{\tau}^{s} (f^{IV}(B_r) - f^{IV}(B_\tau))dr\,ds \right\} \\ &- \frac{1}{4}\mathbb{E}\left\{ 1_{[\tau>\sigma]} \int_{\sigma}^{\sigma\vee\tau} \int_{s}^{\sigma\vee\tau} (f^{IV}(B_r) - f^{IV}(B_\tau))dr\,ds \right\} \\ = &-\frac{1}{8}\mathbb{E}\{1_{[\tau<\sigma]} f^{IV}(B_\tau)(\sigma\vee\tau-\tau)^2\} \\ &- \frac{1}{8}\mathbb{E}\{1_{[\tau>\sigma]} f^{IV}(B_\tau)(\sigma\vee\tau-\sigma)^2\} + R(\sigma,\tau) \\ = &-\frac{1}{8}\mathbb{E}\{f^{IV}(B_\tau)(\tau-\sigma)^2\} + R(\sigma,\tau)\,, \end{aligned}$$

where

$$R(\sigma,\tau) = -\frac{1}{4}\mathbb{E}\left\{ 1_{[\tau<\sigma]} \int_{\tau}^{\sigma\vee\tau} \int_{\tau}^{s} (f^{IV}(B_r) - f^{IV}(B_\tau))dr\;ds \right\} - \frac{1}{4}\mathbb{E}\left\{ 1_{[\tau>\sigma]} \int_{\sigma}^{\sigma\vee\tau} \int_{s}^{\sigma\vee\tau} (f^{IV}(B_r) - f^{IV}(B_\tau)drds \right\}.$$

With $\rho(\sigma,\tau)$ as defined in the lemma, we obviously get the asserted inequality

$$R(\sigma,\tau) \le \frac{1}{8}\mathbb{E}\{\rho(\sigma,\tau)(\tau-\sigma)^2\}$$

and the lemma is proved.

Now let $M = (M_t, \mathcal{F}_t)_{t\geq 0}$ be a right continuous L^4-martingale on the probability space $(\Omega, \mathcal{F}, P)$, and suppose that M is embedded into a Brownian motion $B = (B_t, \mathcal{G}_t)_{t\geq 0}$ on (S, Σ, Q) in the sense of theorem 2.3.
Then for any $s \geq 0, t \geq 0$ the stopping time $\tau_t \vee s$ is $\mathcal{G}_{\tau_t}$-measurable, and an application of the lemma gives the series expansion (cf. theorem 2.3)

$$
\begin{aligned}
&\mathbb{E}f(M_t) - \mathbb{E}f(B_s)\\
&= \mathbb{E}f(B_{\tau_t}) - \mathbb{E}f(B_s)\\
&= \frac{1}{2}\mathbb{E}\{f^{II}(B_{\tau_t})(\tau_t - s)\} - \frac{1}{8}\mathbb{E}f^{IV}(B_{\tau_t})(\tau_t - s)^2 + R(s,\tau_t)\\
&= \frac{1}{2}\mathbb{E}\{\mathbb{E}[f^{II}(Y_t)(\tau_t - s)|\sigma(Y_s; s \leq t)]\}\\
&\quad - \frac{1}{8}\mathbb{E}\{\mathbb{E}[f^{IV}(Y_t)(\tau_t - s)^2|\sigma(Y_s; s \leq t]\} + R(s,\tau_t)\\
&= \frac{1}{2}\mathbb{E}\{f^{II}(Y_t)\mathbb{E}[\tau_t - s|\sigma(Y_s; s \leq t]\}\\
&\quad - \frac{1}{8}\mathbb{E}\{f^{IV}(Y_t)\mathbb{E}[(\tau_t - s)^2|\sigma(Y_s; s \leq t)]\} + R(s,\tau_t)\\
&= \frac{1}{2}\mathbb{E}\left\{f^{II}(Y_t)\left[\frac{1}{3}[Y]_t + \frac{2}{3}\langle Y\rangle_t - s\right]\right\}\\
&\quad - \frac{1}{8}\mathbb{E}\{f^{IV}(Y_t)[Var(\tau_t|\sigma(Y_s; s \leq t) + (\mathbb{E}(\tau_t|\sigma(Y_s; s \leq t)) - s)^2]\} + R(s,\tau_t)\,.
\end{aligned}
$$

Since the processes M and Y have the same finite–dimensional distributions, we have obtained the following result:

Theorem 4.2 *Let M be a right continuous L^4-martingale embedded into a Brownian motion B in the sense of theorem 2.3.*
Then for any $t \geq 0, s \geq 0$, and any bounded four times continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with bounded derivatives the following series expansion holds:

$$
\begin{aligned}
&\mathbb{E}f(M_t) - \mathbb{E}f(B_s)\\
&= \frac{1}{2}\mathbb{E}\left\{f^{II}(M_t)\left[\frac{1}{3}[M]_t + \frac{2}{3}\langle M\rangle_t - s\right]\right\}\\
&\quad - \frac{1}{8}\mathbb{E}\left\{f^{IV}(M_t)\left[\left(\frac{2}{45}[M^4]_t + \frac{8}{45}[\mathbb{E}(M^4|.)]_t + \left(\frac{4}{9} - c(p)\right)[(\mathbb{E}(M^2|.))^2]_t\right.\right.\right.\\
&\quad \left.\left.\left. + c(p)[M^2\mathbb{E}(M^2|.)]_t - c(p)[M\mathbb{E}(M^3|.)]_t\right) + \left(\frac{1}{3}[M]_t + \frac{2}{3}\langle M\rangle_t - s\right)^2\right]\right\} + R(s,\tau_t)\,.
\end{aligned}
$$

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