Rostock. Math. Kolloq. **53**, 51–59 (1999)

Subject Classification (AMS)
34A37

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Stability of Linear Impulse Differential Equations with Unbounded Operator

ABSTRACT. This paper marks the beginning of the consideration of linear impulse differential equations with unbounded operators. Sufficient conditions for existence of some stabilities are founded.

KEY WORDS. Impulse differential equations with unbounded operators - theory and stability.

1 Introduction

The papers [1], [8], [9], [2] and [6] mark the beginning of the qualitative theory of impulse differential equations in an arbitrary Banach space. The present paper initiates the study of linear impulse differential equations with unbounded operators.

2 Problem statement

Let X be a complex Banach space with norm $|\cdot|$ and identity operator I. We denote by $\mathfrak{L}(X)$ the space of all linear bounded operators acting on X.

Let $T = \{t_n\}$ be a sequence of points satisfying the condition $t_n < t_{n+1}$.

We shall consider the linear impulse differential equation:

$$x' = Ax \text{ for } t \neq t_n \tag{1}$$

$$x(t_n^+) = Q_n\, x(t_n) \text{ for } t = t_n\,. \tag{2}$$

We shall assume that one of following two conditions holds:

H1 $A : D(A) \to X$ is a generating operator of the strongly continuous semi-group $U(t)$ $(U(\cdot) : [0, \infty) \to \mathfrak{L}(X))$, Q_n $(n = 1, 2, ...)$ are linear operators and $\lim\limits_{n\to\infty} t_n = \infty$;

H2 $A : D(A) \to X$ is a generating operator of the group $U(t)$ $(U(\cdot) : (-\infty, \infty) \to \mathfrak{L}(X))$, Q_n $(n = \pm1, \pm2, ...)$ are linear operators, $\lim\limits_{n\to\pm\infty} t_n = \pm\infty$.

We assume, that all considered functions are continuous from the left.

Remark 1 We shall note, that $U(t)$ is a group if for example X is a complex Hilbert space and the operator A is equal to iB, where B is a selfadjoint operator (see for example [7], 464). For other examples see [5], 829.

Definition 1 *The function* $x(t)$ *is said to be a solution of the impulse differential equation* (1), (2) *if the following conditions are satisfied:*

1. $x(t) \in D(A)$*;*
2. *For all* $t \neq t_n$ *there exists the strong derivative* $x'(t)$*;*
3. $x(t)$ *satisfies* (1) *for* $t \neq t_n$ *and* (2) *for* $t = t_n$*.*

We consider the Cauchy problem (1), (2) with initial condition

$$x(\tau) = x_\tau \tag{3}$$

at the point $t = \tau$.

Definition 2 *The Cauchy problem* (1)-(3) *is correct on* $[0, \infty)$ *if:*

1. *For all* $x_0 \in D(A)$ *there exists a unique solution of the impulse differential equation* (1), (2)*;*
2. *This solution depends continuously on the initial conditions, i.e.* $x_k(0) \to 0$ $(k \to \infty)$ $(x_k(0) \in D(A))$ *implies* $x_k(t) \to 0$ $(k \to \infty)$ *for all* $t \in [0, \infty)$*.*

In the present paper we find a formula for the solution of problem (1)-(3) and sufficient conditions for some types of stability. We shall extend some results, when A is a sectorial operator (see [4], 376).

3 Main results

We introduce the conditions:

H3 For any n we have either $Q_nU(t) : D(A) \to D(A)$ or $U(t)Q_n : D(A) \to D(A)$;

H4 For any n we have either $Q_nU(t) \in \mathfrak{L}(X)$ or $U(t)Q_n \in \mathfrak{L}(X)$.

Theorem 1 *Let conditions* H3 *and* H4 *be satisfied for* $n = 1, 2, \ldots$.

Then the solution of problem (1)-(3) *on* $[0, \infty\ )$ *is given by the formula* $x(t) = V(t,\tau)x_\tau$ $(\tau < t)$, *where*

$$V(t,\tau) = U(t - t_n) \prod_{j=n}^{p+1} Q_jU(t_j - t_{j-1})Q_pU(t_p - \tau) \tag{4}$$

for $t_{p-1} < \tau \le t_p < t_n < t \le t_{n+1}$.

Proof: From conditions H3 and H4 it follows, that the operator $V(t,\tau)$ is in $\mathfrak{L}(X)$. It is immediately verified that $V(t,\tau)$ satisfies the operator equations

$$V'(t,\tau) = AV(t,\tau) \text{ for } t \ne t_n\,,\ \tau < t\,;$$

$$V(t_n^+,\tau) = Q_nV(t_n,\tau) \text{ for } t = t_n\,;$$

$$V(\tau,\tau) = I\,.$$

□

Example 1 Let the powers of A - A^{α_n} exist $(0 < \alpha_n < 1)$ and let $Q_n = A^{\alpha_n}$ $(n = 1,\ 2, \ldots)$. Then conditions H3 and H4 hold (see [7], 464).

Remark 2 If the operator A is sectorial (see for example [3], 895), then the semi-group $U(t)$ is analytic and condition H4 implies H3.

The following theorem is proved analogously to Theorem 1.

Theorem 2 *Let conditions* H3 *and* H4 *hold.*

Then for $t > \tau$ *the solution of problem* (1)-(3) *on* $(-\infty, \infty)$ *is given by the formula* (4).

If there exist operators Q_n^{-1} $(n = \pm 1, \pm 2, \ldots)$ *and the following conditions*

H3$'$ *For any* $n = \pm 1, \pm 2, \ldots$ *we have either* $Q_n^{-1}U(t) : D(A) \to D(A)$ *or* $U(t)Q_n^{-1} : D(A) \to D(A)$;

H4 ′ *For any* $n = \pm 1, \pm 2, ...$ *we have either* $Q_n^{-1}U(t) \in \mathfrak{L}(X)$ *or* $U(t)Q_n^{-1} \in \mathfrak{L}(X)$

hold, then for $\tau > t$ *the solution of problem* (1)-(3) *is given by the formula*

$$V(t,\tau) = U(t-t_p)\prod_{j=p}^{n-1} Q_j^{-1}U(t_{j-1}-t_j)Q_n^{-1}U(t_n-\tau) \tag{5}$$

for $t_{p-1} < t \le t_p < t_n < \tau \le t_{n+1}$.

Definition 3 *The function* $x(t) = V(t,\tau)x_\tau$ $(x_\tau \in X)$ *is called a generalized solution of* (1)-(3).

Remark 3 If in Theorem 1 only condition H4 holds, then $x(t) = V(t,\tau)x_\tau$ is a generalized solution of (1)-(3).

Further we will consider only the case $t \in [0,\infty)$, i.e. the case if $U(t)(t \ge 0)$ is a semi-group.

Theorem 3 *If condition* H4 *holds, then the generalized solution* $x(t)$ *of the Cauchy problem* (1)-(3) *on the interval* $(t_n,\ t_{n+1}]$ *is an uniform limit of the solutions of* (1)-(3).

Proof: The function $x(t)$ on $(t_n,\ t_{n+1}]$ is equal to the solution of (1) with initial condition $x(t_n^+) = Q_n x(t_n)$ for all n for which $Q_n x(t_n) \in D(A)$. If $Q_n x(t_n) \notin D(A)$, then there exists a sequence of elements $\{x^{(k)}(t_n)\} \subset D(A)$ such that $x^{(k)}(t_n) \to x(t_n)$ $(k \to \infty)$. From condition H4 it follows that $x^{(k)}(t) = U(t-t_n)Q_n x^{(k)}(t_n) \to U(t-t_n)Q_n x(t_n) = x(t)$ $(k \to \infty)$ for $t \in (t_n,\ t_{n+1}]$, i.e. if $Q_n x(t_n) \notin D(A)$, then $x(t)$ is the limit of the sequence of solutions of (1) on $(t_n,\ t_{n+1}]$. □

Definition 4 *The impulse equation* (1), (2) *is stable* (*asymptotically stable; uniformly asymptotically stable; exponentially stable*) *if:*

$$\begin{gathered}\sup_{0<t<\infty} \|V(t,0)\| < \infty \ (\lim_{t\to\infty} |V(t,0)x_0| = 0 \ (x_0 \in X)\,; \\ \sup_{0<\tau<t<\infty} \|V(t,\tau)\| < \infty; \\ \textit{there exist numbers } M,\ \theta > 0 \textit{ such that} \\ \|V(t,\tau)\| \le Me^{-\theta(t-\tau)}\ (\tau < t)\ ).\end{gathered} \tag{6}$$

Let $n(t) = n$ for $t_n < t \le t_{n+1}$. We shall give sufficient conditions for stability of (1), (2).

Theorem 4 *Let the following conditions hold:*

1. *The Cauchy problem* (1)-(3) *is correct;*

2. *There exists a constant* $\gamma \in \mathbb{R}$ *such that* $\|U(t)\| \le e^{\gamma t}$, *for* $t > 0$;

3. *The operators* $Q_n \in \mathfrak{L}(X)$ $(n = 1,\ 2, ...)$*;*

4. *There exist constants* $L > 0$ *and* $\delta \in \mathbb{R}$ *such that* $q_1 q_2 ... q_{n(t)} \leq L e^{\delta t}$ *, where* $q_j = \|Q_j\|$*.*

Then the generalized solution $x(t)$ $(t > 0)$ *is bounded* (*asymptotically stable*) *if* $\gamma + \delta = 0$ $(\gamma + \delta < 0)$*.*

Proof: We shall estimate the generalized solution $x(t)$ of Cauchy problem (1)-(3):

$$\begin{aligned}
&|x(t)| = |U(t-t_n)Q_nU(t_n-t_{n-1})Q_{n-1}...Q_1U(t_1)x_0| \\
&\leq e^{\gamma(t-t_n)}q_n|U(t_n-t_{n-1})Q_{n-1}U(t_{n-1}-t_{n-2})Q_{n-2}...Q_1U(t_1)x_0| \\
&\leq e^{\gamma(t-t_{n-1})}q_nq_{n-1}|U(t_{n-1}-t_{n-2})Q_{n-2}...Q_1U(t_1)x_0| \\
&\leq ... \leq e^{\gamma t}q_nq_{n-1}...q_1|x_0|\,.
\end{aligned} \tag{7}$$

The statement of Theorem 4 is immediatly implied by (7). □

Corollary 1 *Let conditions* 1, 3 *and* 4 *of Theorem* 4 *hold and let constants* $C,\ a > 0$ *exist, such that*

$$\|U(t)\| \leq Ce^{-at}\,, \quad t > 0\,. \tag{8}$$

Then for

$$\ln C \lim_{t\to\infty} \frac{n(t)}{t} = a - \delta \ \ (\ \ln C \lim_{t\to\infty}\frac{n(t)}{t} < a - \delta\ )$$

the generalized solution is bounded (*asymptotically stable*)*.*

Proof: Analogously to the proof of Theorem 4 we obtain the estimate

$$|x(t)| \leq CLe^{\ln C^n}e^{(\delta-a)t}|x_0| = CLe^{\frac{n}{t}\ln C + (\delta - a)t}|x_0|\,.$$

□

Remark 4 Condition (8) is valid if the operator $-A$ is sectorial and $\operatorname{Re}\sigma(-A) > a > 0$, where $\sigma(-A)$ is the spectrum of $-A$.

Let $Q_n \in \mathfrak{L}(X)$. We set

$$\omega = \overline{\lim_{t\to\infty}}\frac{\ln\|V(t,0)\|}{t}\,, \qquad \tilde{\omega} = \overline{\lim_{\tau,t-\tau\to\infty}}\frac{\ln\|V(t,\tau)\|}{t-\tau}\,,$$

$$\omega_c = \overline{\lim_{t\to\infty}}\frac{\ln\|U(t)\|}{t}\,, \qquad \tilde{\omega}_c = \overline{\lim_{\tau,t-\tau\to\infty}}\frac{\ln\|U(t-\tau)\|}{t-\tau}\,,$$

$$\omega_q = \overline{\lim_{t\to\infty}}\frac{\sum\limits_{0\leq t_n<t}\ln\|Q_n\|}{t} \quad \text{and} \quad \tilde{\omega}_q = \overline{\lim_{\tau,t-\tau\to\infty}}\frac{\sum\limits_{\tau\leq t_n<t}\ln\|Q_n\|}{t-\tau}\,,$$

Lemma 1 *Let the Cauchy problem* (1)-(3) *is correct and the following conditions are satisfied:*

1. *There exists* $N \in \mathbb{N}$ *such that* $Q_j U(t) = U(t) Q_j$ *for* $j \geq N,\ t > 0$*;*
2. $Q_n \in \mathfrak{L}(X)$ *and there exist constants* $M > 0$ *and* $\beta \in \mathbb{R}$ *such that* $\| \prod\limits_{j=n(t)}^{N} Q_j \| \leq M e^{\beta t}$.

Then $\omega \leq \omega_c + \beta$*.*

Proof: The operator $V(t,0)$ in this case can be writen in the form

$$V(t,0) = U(t - t_N) \prod_{j=n(t)}^{N} Q_j U(t_N - t_{N-1}) Q_{N-1} ... Q_1 U(t_1)\,. \tag{9}$$

From (9) we obtain the estimate

$$\frac{\ln \|V(t,0)\|}{t} \leq$$

$$\leq \frac{\ln \|U(t - t_N)\|}{t} + \frac{\ln \| \prod\limits_{j=n(t)}^{N} Q_j \|}{t} + \frac{\ln \|U(t_N - t_{N-1}) Q_{N-1} ... Q_1 U(t_1)\|}{t}\,.$$

Moreover

$$\lim_{t \to \infty} \frac{\ln \|U(t - t_N)\|}{t} = \lim_{\tau \to \infty} \frac{\ln \|U(\tau)\|}{\tau + t_N} = \lim_{\tau \to \infty} \frac{\ln \|U(\tau)\|}{\tau (1 + \frac{t_N}{\tau})} =$$

$$= \lim_{\tau \to \infty} \frac{\ln \|U(\tau)\|}{\tau} = \omega_c\,.$$

□

Theorem 5 *Let the following conditions hold:*

1. *There exists* $N \in \mathbb{N}$ *such that* $Q_j U(t) = U(t) Q_j$ *for* $j \geq N,\ t > 0$*;*
2. $Q_n \in \mathfrak{L}(X),\ n = 1,\ 2, ...$*;*
3. $\omega_c + \omega_q = 0$ $(\omega_c + \omega_q < 0;\ \tilde{\omega}_c + \tilde{\omega}_q = 0;\ \tilde{\omega}_c + \tilde{\omega}_q < 0)$.

Then the impulse equation (1), (2) *is stable* (*asymptotically stable; uniformly asymptotically stable; exponentially stable*)*.*

Proof: The operators $V(t,0)$ and $V(t,\tau)$ can be written in the form (9) and

$$V(t,\tau) = U(t-t_N)\prod_{j=n(t)}^{N} Q_j U(t_N - t_{N-1})Q_{N-1}...Q_p U(t_p-\tau) \tag{10}$$

for $t_{p-1} < \tau \leq t_p < t_n < t \leq t_{n+1}$. From (9) and (10) we obtain the estimates

$$\omega \leq \omega_c + \omega_q\,, \tag{11}$$

$$\tilde{\omega} \leq \tilde{\omega}_c + \tilde{\omega}_q\,. \tag{12}$$

Then (11) and (12) imply the assertion of Theorem 5. □

At the end we shall consider the stability of the periodical impulse differential equations. Let there exist numbers p and s such that

$$t_{n+s} = t_n + p,\ Q_{n+s} = Q_n\,. \tag{13}$$

Impulse equations of this type will be called (p,s) - periodic.

Set

$$W = Q_s U(t_s - t_{s-1})Q_{s-1}...Q_1 U(t_1)\,. \tag{14}$$

The operator W is an analogue of the monodromy operator.

Theorem 6 *If the impulse equation* (1), (2) *is* (p,s) *- periodic,* $U(t)$ *is a semi-group and condition* H4 *holds, then the impulse equation* (1), (2) *is:*

a) *stable if and only if the sequence of iterations of the operator* W *is bounded;*

b) *asymptotically stable if and only if the sequence of iterations of the operator* W *strongly tends to zero for each element* $x_0 \in X$*;*

c) *uniformly asymptotically stable if and only if the spectral radius* $\rho(W)$ *of the operator* W *is smaller than one, i. e.* $\rho(W) < 1$*;*

d) *exponentially stable if and only if the sequence of iterations of the operator* W *is exponentially bounded, i.e. there exist numbers* $C,\ \theta > 0$ *such that* $\|W^n\| \leq Ce^{-\theta n}$.

Proof: From (13) and (14) it follows that the operators $V(t,0)$ and $V(t,\tau)$ can be written in the form:

$$V(t,0)=F(t)W^{\nu(t)}\,, \tag{15}$$

where $\nu(t)=[\frac{t}{p}]$ $(t>0)$ and

$$V(t,\tau)=F(t)W^{\mu(t,\tau)}P(\tau)\,, \tag{16}$$

where $\mu(t,\tau)=[\frac{t-\tau}{p}]$ $(0<\tau<t)$, $F(t)$ and $P(t)\in\mathfrak{L}(X)$.

The proof is implied immediatly by (15) and (16). □

Corollary 2 *If the impulse equation* (1), (2) *is* $(p,1)$ *- periodic, then the impulse equation* (1), (2) *is:*

a) *stable if and only if the sequence of iterations of the operator* $QU(p)$ $(Q_1=Q)$ *is bounded;*

b) *asymptotically stable if and only if the sequence of iterations of the operator* $QU(p)$ *strongly tends to zero for each element* $x_0\in X$*;*

c) *uniformly asymptotically stable if and only if the spectral radius* $\rho(W)$ *of the operator* $QU(p)$ *is smaller than one;*

d) *exponentially stable if and only if the sequence of iterations of the operator* $QU(p)$ *is exponentially bounded.*

Remark 5 We note that assertions analogous to Theorem 5 and Theorem 6 can be proved also when $U(t)$ $(t\in(-\infty,+\infty))$ is a group.

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received: March 3, 1997 **revised:** August 10, 1998

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