# The complexity of minimizing the number of shape matrices subject to minimal beam-on time in multileaf collimator field decomposition with bounded fluence 

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#### Abstract

The use of multileaf collimators (MLCs) is a modern way to realize intensity modulated fields in radiotherapy. An important step in the treatment planning is the shape matrix decomposition: The desired fluence distribution, given by an integer matrix, has to be decomposed into a small number shape matrices, i.e. $(0,1)-$ matrices corresponding to the field shapes that can be delivered by the used MLC. The two main objectives are to minimize the total irradiation time and the number of shape matrices. Assuming that the entries of the fluence matrix are bounded by a constant, we prove that a shape matrix decomposition with minimal number of shape matrices under the condition that the total irradiation time is minimal, can be determined in time polynomial in the matrix dimensions. The results of our algorithm are compared with Engel's [8] heuristic for the reduction of the number of shape matrices.


Key words: leaf sequencing, radiation therapy optimization, intensity modulation, multileaf collimator, IMRT

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## 1 Introduction

In recent years intensity modulated radiation therapy (IMRT) has become an important method in cancer therapy. The objective in the treatment planning is to irradiate the tumor as efficiently as possible without damaging the organs near to it. A modern way to realize intensity modulated radiation fields is the use of a multileaf collimator (MLC). An MLC consists of two opposite banks of metal leaves which can be shifted towards each other and so open or close certain parts of the irradiated area. In this paper we assume that the desired fluence distribution is already determined. After discretization the fluence can be considered as an $m \times n$ matrix $A$ with nonnegative integer entries. We consider the problem to realize this fluence with an MLC in the static mode (step and shoot). This means that the radiation is switched off when the leaves of the collimator are moving. In other words we have to determine a (finite) set of leaf positions with corresponding irradiation times such that the superposition of the homogeneous fields yields the given fluence matrix. This principle is illustrated in Figure 1. The leaf posi-


Figure 1: Intensity modulation by superimposing 3 beams of different shapes. In each step the left figure shows a leaf position and in the right figure the grey scale indicates the total fluence.
tions can be described by certain $0-1$-matrices of size $m \times n$ called shape matrices, where a 0 -entry means the radiation is blocked and a 1 -entry means that the radiation goes through. For example the first leaf position in Fig. 1 corresponds to the shape matrix

$$
\left(\begin{array}{llll}
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1
\end{array}\right) .
$$

Clearly, the superposition of differently shaped beams corresponds to positive linear combinations of shape matrices, where the coefficient of a shape matrix measures for how long the corresponding field is applied. So any representation of the given fluence matrix $A$ as a positive integer linear combination of shape matrices is a feasible solution to our decomposition problem. For instance:

$$
A=\left(\begin{array}{llll}
1 & 3 & 3 & 0  \tag{1}\\
0 & 2 & 4 & 1 \\
1 & 1 & 4 & 4 \\
3 & 3 & 1 & 0
\end{array}\right)=2 \cdot\left(\begin{array}{llll}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0
\end{array}\right)+\left(\begin{array}{llll}
0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{array}\right)+\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0
\end{array}\right) .
$$

There are two quantities influencing the quality of a decomposition

$$
\begin{equation*}
A=\sum_{i=1}^{k} u_{i} S_{i} \tag{2}
\end{equation*}
$$

the total irradiation time (proportional to the sum of the coefficients) and the number of necessary beams (the number of nonzero coefficients). Thus the quality of (2) is measured by the decomposition time (DT) and the decomposition cardinality (DC):

$$
D T=\sum_{i=1}^{k} u_{i}, \quad D C=k
$$

In general, it is not possible to minimize both parameters simultaneously (see [12] for a counterexample). Instead we first determine the minimal DT and among all decompositions with this DT we search for one with minimal DC. In the literature there are several decomposition algorithms ( $[2,5,6,7$, $8,9,13,16,17,18])$. The most common approach is to find the minimal DT and use heuristic methods to reduce the DC. The algorithms differ in the extent to which they include additional machine-dependent constraints like the interleaf collision constraint. In principle both, DT and DC, can be optimized by integer programming [14], but the known IP-formulations do not lead to algorithms that can solve instances of practically relevant sizes. See [12] for a survey and a comparison of the different decomposition algorithms. In this paper we neglect machine-dependent constraints and focus on the complexity of the DC-minimization.

Throughout the paper we use the notation $[n]:=\{1,2, \ldots, n\}$ for positive integers $n$. Let $A=\left(a_{i, j}\right)$ denote the given $m \times n$-fluence matrix. For brevity of notation we put $a_{i, 0}=a_{i, n+1}=0$ for $i \in[m]$. We start with a formal definition of a shape matrix, that is a $0-1$-matrix describing a leaf position of the MLC.

Definition 1. A shape matrix is an $m \times n$-matrix $S=\left(s_{i, j}\right)$, such that there exist integers $l_{i}, r_{i} \quad(i \in[m])$ with the following properties:

$$
\begin{align*}
& l_{i} \leq r_{i}+1  \tag{3}\\
& s_{i, j}= \begin{cases}1 & \text { if } l_{i} \leq j \leq r_{i} \\
0 & \text { otherwise }\end{cases}  \tag{4}\\
&(i \in[m]), \\
&
\end{align*}
$$

The interpretation is that $l_{i}-1$ and $r_{i}+1$ are the positions of the $i-$ th left and right leaf, respectively. A 1-entry in the shape matrix indicates that the corresponding region receives radiation while a 0 -entry indicates a region that is covered by a leaf. For a nonnegative integer matrix $A$, a shape matrix decomposition of $A$ is a representation of $A$ as a positive integer combination of shape matrices like (2) with shape matrices $S_{i}$ and positive integers $u_{i}$ $(i \in[k])$. In this paper we consider the following problem, which we call shape matrix decomposition problem.

Shape matrix decomposition problem: Given the nonnegative integer matrix $A$, find a shape matrix decomposition $A=\sum_{i=1}^{k} u_{i} S_{i}$ with in first instance minimal DT and in second instance minimal DC.

There are several efficient algorithms for determining DT-optimal decompositions $[5,8,13]$. In $[8]$ it is proved that the minimal DT equals

$$
\begin{equation*}
c(A):=\max _{i \in[m]} \sum_{j=1}^{n} \max \left\{0, a_{i, j}-a_{i, j-1}\right\} . \tag{5}
\end{equation*}
$$

The problem of minimizing DC is NP-complete in the strong sense even for single row matrices, as was shown in [2] by a reduction of 3 -Partition [10]. As it was observed in [3] this reduction yields even the APX-hardness of the problem for matrices with entries polynomially bounded in $n$. But the reduction essentially depends on the fact that the entries can become arbitrary large. In this paper we show that the DC-minimization problem can be solved in time polynomial in the matrix dimensions $m$ and $n$, provided the matrix entries are bounded by some constant $L$. This seems to be a reasonable assumption in practice: for instance the authors of [18] report, that they obtained matrices with 7 nonzero fluence levels when they applied a preliminary version of the CORVUS inverse treatment planning system (NOMOS corporation) to a very complex head and neck tumor case. The algorithm proposed here is an application of the dynamic programming principle (see [4]). The paper is organized as follows. The cases of single row and multiple row fluence matrices are treated separately in Sections 2 and 3, respectively.

For both cases we describe polynomial algorithms for the construction of decompositions with minimal DT and minimal DC. In Section 4 we test our algorithm with randomly generated matrices and with matrices from clinical practice, and we compare the results with the heuristic method from [8].

## 2 Single row intensity maps

First we give an exact formulation of the problem $L$-One Row-Min DTMin DC:

Instance: A vector $\boldsymbol{a}=\left(a_{1} a_{2} \ldots a_{n}\right)$ of integers with $0 \leq a_{i} \leq L(i=$ $1, \ldots, n)$.

Problem: Find a shape matrix decomposition with in first instance minimal DT and in second instance minimal DC.

We put $a_{0}=a_{n+1}=0$. Let

$$
\begin{aligned}
& P=\left\{i \in[n]: a_{i} \geq a_{i-1} \text { and } a_{i}>a_{i+1}\right\}, \\
& Q=\left\{i \in[n]: a_{i}<a_{i-1} \text { and } a_{i} \leq a_{i+1}\right\} .
\end{aligned}
$$

Clearly, $|P|=|Q|+1$ if $a_{n} \neq 0$ and $|P|=|Q|$ if $a_{n}=0$. If $a_{n} \neq 0$ denote the elements of $P$ and $Q$ by $p_{1}, \ldots, p_{t}$ and $q_{1}, \ldots, q_{t-1}$ such that

$$
p_{1}<q_{1}<p_{2}<q_{2}<\cdots<q_{t-1}<p_{t},
$$

and put $q_{0}=0$ and $q_{t}=n+1$. If $a_{n}=0$ denote the elements of $P$ and $Q$ by $p_{1}, \ldots, p_{t}$ and $q_{1}, \ldots, q_{t}$ such that

$$
p_{1}<q_{1}<p_{2}<q_{2}<\cdots<q_{t-1}<p_{t}<q_{t} .
$$

From the proof of Theorem 1 in [8] it follows that in a DT-optimal decomposition $\boldsymbol{a}=\sum_{j=1}^{k} c_{j} \boldsymbol{s}^{(j)}$ every shape matrix is of the form

$$
s_{i}^{(j)}= \begin{cases}1 & \text { for } l_{j} \leq i \leq r_{j}, \\ 0 & \text { otherwise },\end{cases}
$$

with $q_{\tau-1}<l_{j} \leq p_{\tau}$ and $p_{\tau^{\prime}} \leq r_{j}<q_{\tau^{\prime}}$ for some $\tau, \tau^{\prime} \in[t]$. Since the order of the shape matrices is not relevant, we may order them in such a way that $r_{1} \leq \cdots \leq r_{k}$. For $\tau \in[t-1]$, let $k_{0}(\tau)$ be the unique index with $r_{j}<q_{\tau}$ for
$j \leq k_{0}(\tau)$ and $r_{j} \geq q_{\tau}$ for $j>k_{0}(\tau)$, and let $\boldsymbol{a}^{(\tau)}$ denote the remainder after extracting the first $k_{0}(\tau)$ shape matrices, i.e.

$$
\boldsymbol{a}^{(\tau)}=\boldsymbol{a}-\sum_{j=1}^{k_{0}(\tau)} c_{j} \boldsymbol{s}^{(j)} .
$$

In addition, put $k_{0}(0)=0, k_{0}(t)=k, \boldsymbol{a}^{(0)}=\boldsymbol{a}$ and $\boldsymbol{a}^{(t)}=\mathbf{0}$. The interpretation is that the shapes with indices between $k_{0}(\tau-1)$ and $k_{0}(\tau)$ are used to eliminate the $\tau$-th local maximum of the fluence profile. This is illustrated in the following example.

Example 1. Consider the vector $\boldsymbol{a}=(22377985512)$ where $t=2, p_{1}=6$, $q_{1}=8, p_{2}=10, q_{2}=11$. For the decomposition

$$
\begin{aligned}
& \boldsymbol{a}=(1111110000)+3(0001111000)+\left(\begin{array}{l}
11111111111) \\
+(0011111111)+(00011111111)+2(0000111111) \\
+7(0000000001)
\end{array}\right.
\end{aligned}
$$

we have $k=7=k_{0}(2)$ and $k_{0}(1)=2$ and $\boldsymbol{a}^{(1)}=(11233555512)$.
After extracting the first $k_{0}(\tau)$ shapes we obtain a vector $\boldsymbol{a}^{(\tau)}$ which is increasing up to entry $p_{\tau+1}$. This is made precise in the following lemma.

Lemma 1. For $\tau \in[t-1]$ we have

$$
\begin{equation*}
a_{1}^{(\tau)} \leq a_{2}^{(\tau)} \leq \cdots \leq a_{q_{\tau}}^{(\tau)} \tag{6}
\end{equation*}
$$

and the multisets

$$
\begin{align*}
U_{\tau} & =\left\{a_{i}^{(\tau)}-a_{i-1}^{(\tau)}: 1 \leq i \leq q_{\tau}, a_{i}^{(\tau)} \neq a_{i-1}^{(\tau)}\right\},  \tag{7}\\
V_{\tau} & =\left\{a_{i}^{(\tau)}-a_{i-1}^{(\tau)}: q_{\tau}<i \leq p_{\tau+1}, a_{i}^{(\tau)} \neq a_{i-1}^{(\tau)}\right\},  \tag{8}\\
W_{\tau} & =\left\{a_{i}^{(\tau)}-a_{i+1}^{(\tau)}: p_{\tau+1} \leq i<q_{\tau+1}, a_{i}^{(\tau)} \neq a_{i+1}^{(\tau)}\right\} \tag{9}
\end{align*}
$$

are partitions of $a_{q_{\tau}}, a_{p_{\tau+1}}-a_{q_{\tau}}$ and $a_{p_{\tau+1}}-a_{q_{\tau+1}}$, respectively.
Proof. For $j>k_{0}(\tau)$, from $r_{j} \geq q_{\tau}$ it follows that for $i \leq q_{\tau}$,

$$
s_{i}^{(j)}=1 \quad \Longleftrightarrow \quad l_{j} \leq i
$$

In particular, for $i=1, \ldots, q_{\tau}-1$ and $j=k_{0}(\tau)+1, \ldots, k$,

$$
\begin{equation*}
s_{i}^{(j)}=1 \quad \Longrightarrow \quad s_{i+1}^{(j)}=1 . \tag{10}
\end{equation*}
$$

For $0 \leq \tau \leq t-1$, we have

$$
\boldsymbol{a}^{(\tau)}=\sum_{j=k_{0}(\tau)+1}^{k} c_{j} \boldsymbol{s}^{(j)},
$$

hence (10) implies (6). The second statement follows easily, since $a_{i}^{(\tau)}$ is increasing for $1 \leq i \leq p_{\tau+1}$ and decreasing for $p_{\tau+1}<i \leq q_{\tau+1}$.

Observe that $a_{i}^{(\tau)}=a_{i}$ for $i \geq q_{\tau}$, hence $V_{\tau}$ and $W_{\tau}$ depend only on $\boldsymbol{a}$, while $U_{\tau}$ depends also on the pairs

$$
\left(s^{(1)}, c_{1}\right), \ldots,\left(s^{\left(k_{0}(\tau)\right)}, c_{k_{0}(\tau)}\right)
$$

Considering the sequence $\left(U_{\tau}, V_{\tau}, W_{\tau}\right)(\tau=0, \ldots, t)$, where we add $U_{t}=V_{t}=$ $W_{t}=\varnothing$, we will present a method to construct the desired decomposition.

Definition 2. For integers $u, v$ and $w$ with $0 \leq u \leq v \leq L$ and $0 \leq w<v$, a $(u, v, w)$-peak is a triple $(U, V, W)$ of unordered partitions of $u, v-u$ and $v-w$, i.e. a triple of multisets of positive integers with

$$
\sum_{x \in U} x=u, \quad \sum_{x \in V} x=v-u, \quad \sum_{x \in W} x=v-w .
$$

In addition, the triple $(\varnothing, \varnothing, \varnothing)$ is called $(0,0,0)$-peak.
Such peaks carry the essential information on $\boldsymbol{a}^{(\tau)}$ that can be used to determine the next shapes and their coefficients. According to Lemma 1, for $\tau=0, \ldots, t,\left(U_{\tau}, V_{\tau}, W_{\tau}\right)$ is an $\left(a_{q_{\tau}}, a_{p_{\tau+1}}, a_{q_{\tau+1}}\right)$-peak (where $a_{p_{t+1}}=a_{q_{t+1}}=$ 0 ), and for $\tau \leq t-1$, the choice of the pairs

$$
\left(s^{\left(k_{0}(\tau)+1\right)}, c_{k_{0}(\tau)+1}\right), \ldots,\left(s^{\left(k_{0}(\tau+1)\right)}, c_{k_{0}(\tau+1)}\right)
$$

can be considered as the choice of a way to go from the peak $\left(U_{\tau}, V_{\tau}, W_{\tau}\right)$ to the peak $\left(U_{\tau+1}, V_{\tau+1}, W_{\tau+1}\right)$. We claim that the number of shape matrices needed for this step does not depend on the particular $\boldsymbol{a}^{(\tau)}$, but only on the multisets $U_{\tau} \cup V_{\tau}, W_{\tau}$ and $U_{\tau+1}$. To prove this we associate with a $(u, v, w)$-peak $(U, V, W)$ a vector $\boldsymbol{b}=\left(b_{1} \ldots b_{\beta}\right)$ as follows. Put $\alpha=|U|+|V|$, $\beta=\alpha+|W|$, denote the elements of $U \cup V$ by $d_{1}, \ldots, d_{\alpha}$ and the elements of $W$ by $d_{\alpha+1}, \ldots, d_{\beta}$, such that

$$
d_{1} \geq d_{2} \geq \cdots \geq d_{\alpha} \quad \text { and } \quad d_{\alpha+1} \geq d_{\alpha+2} \geq \cdots \geq d_{\beta} .
$$

So, for $U=U_{\tau}, V=V_{\tau}$ and $W=W_{\tau}$ the $d_{i}(i=1, \ldots, \beta)$ are the absolute values of the nonzero differences of consecutive entries of the initial part $\left(a_{1}^{(\tau)} \ldots a_{q_{\tau}+1}^{(\tau)}\right)$ of $\boldsymbol{a}^{(\tau)}$. Now $\boldsymbol{b}$ is defined by

$$
b_{i}= \begin{cases}\sum_{j=1}^{i} d_{j} & \text { for } 1 \leq i \leq \alpha \\ v-\sum_{j=\alpha+1}^{i} d_{j} & \text { for } \alpha+1 \leq i \leq \beta\end{cases}
$$

In addition, let $b_{0}=0$.
Example 2. Let us consider vector $\boldsymbol{a}=(2237798512)$ from Example 1. We start with the peak $U_{0}=\varnothing, V_{0}=\{2,1,4,2\}, W_{0}=\{1,3\}$ with associated vector $\boldsymbol{b}=(468965)$. After extracting the first two shapes as in Example 1 we obtain $\boldsymbol{a}^{(1)}=(1123355512)$ with peak $U_{1}=\{1,1,1,2\}, V_{1}=\{7\}$, $W_{1}=\{12\}$ and associated vector $\boldsymbol{b}=\left(\begin{array}{ll}2345120\end{array}\right)$.

Now we want to show that under the condition of minimal DT the elimination of the first maximum in $\boldsymbol{a}^{(\tau)}$ can be reduced to the reduction of the associated vector $\boldsymbol{b}$ to an increasing sequence. The following example illustrates the method for our example vector.

Example 3. Start with the following reduction of $\boldsymbol{b}=\left(\begin{array}{lll}468965\end{array}\right)$.

$$
\boldsymbol{b}-3(111100)-(0111110)(468965)(124555) .
$$

The two shapes can be described by the parameters $\left(l_{1}^{\prime}, r_{1}^{\prime}\right)=(1,4)$ and $\left(l_{2}^{\prime}, r_{2}^{\prime}\right)=(2,5)$, indicating the first and the last 1 -entry. For a corresponding reduction of $\boldsymbol{a}$ we require that we extract one shape with coefficient 3 and parameters $\left(l_{1}, r_{1}\right)$ and another shape with coefficient 1 and parameters $\left(l_{2}, r_{2}\right)$ such that

$$
\begin{array}{ll}
a_{l_{1}}-a_{l_{1}-1}=b_{l_{1}^{\prime}}-b_{l_{1}^{\prime}-1}=4, & a_{r_{1}}-a_{r_{1}+1}=b_{r_{1}^{\prime}}-b_{r_{1}^{\prime}+1}=3, \\
a_{l_{2}}-a_{l_{2}-1}=b_{l_{2}^{\prime}}-b_{l_{2}^{\prime}-1}=2, & a_{r_{2}}-a_{r_{2}+1}=b_{r_{2}^{\prime}}-b_{r_{2}^{\prime}+1}=1 .
\end{array}
$$

Clearly, the first two shapes in Example 1 (in opposite order) with $\left(l_{1}, r_{1}\right)=$ $(4,7)$ and $\left(l_{2}, r_{2}\right)=(1,6)$ satisfy these requirements.

The following lemma makes this correspondence between the reductions of $\boldsymbol{a}$ and $\boldsymbol{b}$ rigorous.

Lemma 2. Fix some $\tau, 0 \leq \tau \leq t-1$, and let $\boldsymbol{b}=\left(b_{1} \ldots b_{\beta}\right)$ be the vector associated with the ( $a_{q_{\tau}}, a_{p_{\tau+1}}, a_{q_{\tau+1}}$ )-peak $\left(U_{\tau}, V_{\tau}, W_{\tau}\right)$, defined according to
(7)-(9), where $\alpha=\left|U_{\tau} \cup V_{\tau}\right|$ and $\beta=\alpha+\left|W_{\tau}\right|$. Also let $U^{\prime}$ be a partition of $a_{q_{\tau+1}}$, and let $c_{1}, \ldots, c_{\rho}$ be positive integers with

$$
\begin{equation*}
\sum_{j=1}^{\rho} c_{j}=a_{p_{\tau+1}}-a_{q_{\tau+1}} . \tag{11}
\end{equation*}
$$

Then the following statements are equivalent.

1. There exist integers $l_{j}, r_{j}$ with $1 \leq l_{j} \leq p_{\tau+1} \leq r_{j}<q_{\tau+1}(j=1, \ldots, \rho)$, such that for $\boldsymbol{a}^{\prime}=\boldsymbol{a}^{(\tau)}-\sum_{j=1}^{\rho} c_{j} \boldsymbol{s}^{(j)}$, where

$$
s_{i}^{(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j} \leq i \leq r_{j} \\
0 & \text { otherwise }
\end{array} \quad(j=1, \ldots, \rho ; i=1, \ldots, n)\right.
$$

we have
(a) $0 \leq a_{1}^{\prime} \leq a_{2}^{\prime} \leq \cdots \leq a_{q_{\tau+1}}^{\prime}$
(b) $\left\{a_{i}^{\prime}-a_{i-1}^{\prime}: 1 \leq i \leq q_{\tau+1}, a_{i}^{\prime} \neq a_{i-1}^{\prime}\right\}=U^{\prime} \quad\left(\right.$ where $\left.a_{0}^{\prime}=0\right)$.
2. There exist integers $l_{j}^{\prime}, r_{j}^{\prime}$ with $1 \leq l_{j}^{\prime} \leq r_{j}^{\prime} \leq \beta-1$ for $j=1, \ldots, \rho$, such that for $\boldsymbol{b}^{\prime}=\boldsymbol{b}-\sum_{j=1}^{\rho} c_{j} \boldsymbol{f}^{(j)}$, where

$$
f_{i}^{(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j}^{\prime} \leq i \leq r_{j}^{\prime} \\
0 & \text { otherwise }
\end{array} \quad(j=1, \ldots, \rho ; i=1, \ldots, \beta)\right.
$$

we have
(a) $b_{1}^{\prime} \leq b_{2}^{\prime} \leq \cdots \leq b_{\beta}^{\prime}=b_{\beta}$
(b) $\left\{b_{i}^{\prime}-b_{i-1}^{\prime}: 1 \leq i \leq \beta, b_{i}^{\prime} \neq b_{i-1}^{\prime}\right\}=U^{\prime} \quad\left(\right.$ where $\left.b_{0}^{\prime}=0\right)$.

Observe that the sum over $\tau$ of the right hand side of (11) is the minimal DT of a decomposition of $\boldsymbol{a}$ [8]. Hence (11) together with the first statement (for all $\tau$ ) characterize the decompositions with minimal DT.

Proof. Let

$$
\begin{aligned}
& R_{1}=\left\{i: 1 \leq i \leq p_{\tau+1}, a_{i}^{(\tau)} \neq a_{i-1}^{(\tau)}\right\}, \\
& R_{2}=\left\{i: p_{\tau+1} \leq i<q_{\tau+1}, a_{i}^{(\tau)} \neq a_{i+1}^{(\tau)}\right\} .
\end{aligned}
$$

Clearly,

$$
U_{\tau} \cup V_{\tau}=\left\{a_{i}^{(\tau)}-a_{i-1}^{(\tau)}: i \in R_{1}\right\} \quad \text { and } \quad W_{\tau}=\left\{a_{i}^{(\tau)}-a_{i+1}^{(\tau)}: i \in R_{2}\right\} .
$$

By construction of $\boldsymbol{b}$, we also have
$U_{\tau} \cup V_{\tau}=\left\{b_{i}-b_{i-1}: 1 \leq i \leq \alpha\right\} \quad$ and $\quad W_{\tau}=\left\{b_{i}-b_{i+1}: \alpha \leq i \leq \beta-1\right\}$.
Together this implies that there are bijections

$$
\varphi_{1}: R_{1} \rightarrow\{1, \ldots, \alpha\}, \quad \varphi_{2}: R_{2} \rightarrow\{\alpha, \ldots, \beta-1\}
$$

such that

$$
\begin{aligned}
& a_{i}^{(\tau)}-a_{i-1}^{(\tau)}=b_{\varphi_{1}(i)}-b_{\varphi_{1}(i)-1} \text { for } i \in R_{1} \quad \text { and } \\
& a_{i}^{(\tau)}-a_{i+1}^{(\tau)}=b_{\varphi_{2}(i)}-b_{\varphi_{2}(i)+1} \text { for } i \in R_{2} .
\end{aligned}
$$

The proof of the characterization of the minimal DT (Theorem 1 in [8]) implies that in a decomposition with minimal DT we must have $a_{l_{i}}>a_{l_{i}-1}$ and $a_{r_{i}}>a_{r_{i}}+1$ for all $i$. As observed above, $l_{j}, r_{j}(j=1, \ldots, \rho)$ as in the first statement occur in a DT-optimal decomposition, so we have $l_{j} \in R_{1}$ and $r_{j} \in R_{2}$ for all $j$. Similarly, for $l_{j}^{\prime}, r_{j}^{\prime}(j=1, \ldots, \rho)$ as in the second statement we have $l_{j}^{\prime} \leq \alpha$ and $r_{j}^{\prime} \geq \alpha$ for all $j$. Suppose that $l_{j}, r_{j}(j=1, \ldots, \rho)$ satisfy the conditions of the first statement. The difference of the entries number $i$ and $i-1$ changes only when $l_{j}=i$ or $r_{j}=i-1$ for some $j$. Thus, if $i \notin R_{1}$ and $i-1 \notin R_{2}$ we have

$$
a_{i}^{\prime}-a_{i-1}^{\prime}=a_{i}^{(\tau)}-a_{i-1}^{(\tau)}=0
$$

Hence, for $i=1, \ldots, q_{\tau+1}$,

$$
a_{i}^{\prime}-a_{i-1}^{\prime} \neq 0 \Longrightarrow i \in R_{1} \text { or } i-1 \in R_{2} .
$$

Put

$$
\begin{aligned}
& C_{1}(i)=\left\{j \in[\rho]: \quad l_{j}=i\right\} \quad \text { for } i \in R_{1}, \\
& C_{2}(i)=\left\{j \in[\rho]: r_{j}=i\right\} \quad \text { for } i \in R_{2} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& a_{i}^{\prime}-a_{i-1}^{\prime}=a_{i}^{(\tau)}-a_{i-1}^{(\tau)}-\sum_{j \in C_{1}(i)} c_{j} \quad \text { for } i \in R_{1} \\
& a_{i}^{\prime}-a_{i+1}^{\prime}=a_{i}^{(\tau)}-a_{i+1}^{(\tau)}-\sum_{j \in C_{2}(i)} c_{j} \quad \text { for } i \in R_{2}
\end{aligned}
$$

By condition (a) of the first statement we have $a_{i}^{\prime}-a_{i+1}^{\prime} \leq 0$ for $i=$ $0, \ldots, q_{\tau+1}-1$. For $i \in R_{2}$ this yields

$$
\sum_{j \in C_{2}(i)} c_{j} \geq a_{i}^{(\tau)}-a_{i+1}^{(\tau)},
$$

and together with

$$
\sum_{i \in R_{2}} \sum_{j \in C_{2}(i)} c_{j}=\sum_{j=1}^{\rho} c_{j}=a_{p_{\tau+1}}-a_{q_{\tau+1}}=\sum_{i \in R_{2}}\left(a_{i}^{(\tau)}-a_{i+1}^{(\tau)}\right)
$$

we obtain for $i \in R_{2}$,

$$
\sum_{j \in C_{2}(i)} c_{j}=a_{i}^{(\tau)}-a_{i+1}^{(\tau)},
$$

and thus $a_{i}^{\prime}-a_{i+1}^{\prime}=0$ for $i \in R_{2}$. So the only nonzero differences $a_{i}^{\prime}-a_{i-1}^{\prime}$ come from indices $i \in R_{1}$. Now put $l_{j}^{\prime}=\varphi_{1}\left(l_{j}\right)$ and $r_{j}^{\prime}=\varphi_{2}\left(r_{j}\right)(j=1, \ldots, \rho)$ and let $\boldsymbol{b}^{\prime}$ be defined as in the second statement. Then $l_{j}^{\prime}=\varphi_{1}(i)$ iff $j \in C_{1}(i)$ and $r_{j}^{\prime}=\varphi_{2}(i)$ iff $j \in C_{2}(i)$, hence for $i \in R_{1}$ we have

$$
\begin{aligned}
b_{\varphi_{1}(i)}^{\prime}-b_{\varphi_{1}(i)-1}^{\prime} & =b_{\varphi_{1}(i)}-b_{\varphi_{1}(i)-1}-\sum_{j: l_{j}^{\prime}=\varphi_{1}(i)} c_{j} \\
& =b_{\varphi_{1}(i)}-b_{\varphi_{1}(i)-1}-\sum_{j \in C_{1}(i)} c_{j} \\
& =a_{i}-a_{i-1}-\sum_{j \in C_{1}(i)} c_{j} \\
& =a_{i}^{\prime}-a_{i-1}^{\prime}
\end{aligned}
$$

and for $i \in R_{2}$,

$$
\begin{aligned}
b_{\varphi_{2}(i)}^{\prime}-b_{\varphi_{2}(i)+1}^{\prime} & =b_{\varphi_{2}(i)}-b_{\varphi_{2}(i)+1}-\sum_{j: r_{j}^{\prime}=\varphi_{2}(i)} c_{j} \\
& =b_{\varphi_{2}(i)}-b_{\varphi_{2}(i)+1}-\sum_{j \in C_{2}(i)} c_{j} \\
& =a_{i}-a_{i+1}-\sum_{j \in C_{2}(i)} c_{j} \\
& =a_{i}^{\prime}-a_{i+1}^{\prime}=0 .
\end{aligned}
$$

So the second statement holds, and since all the arguments are reversible, we have proved that $l_{j}, r_{j}(j=1, \ldots, \rho)$ satisfy the conditions of the first statement iff $l_{j}^{\prime}=\varphi_{1}\left(l_{j}\right), r_{j}^{\prime}=\varphi_{2}\left(r_{j}\right)(j=1, \ldots, \rho)$ satisfy the conditions of the second statement, and this proves the lemma.

In fact the proof shows even more than just the equivalence of the two statements: knowing $l_{j}^{\prime}$ and $r_{j}^{\prime}(j=1, \ldots, \rho)$ and $R_{1}$ and $R_{2}$, we can determine the $l_{j}, r_{j}(j=1, \ldots, \rho)$ and $R^{\prime}=\left\{i: 1 \leq i \leq q_{\tau+1}, a_{i}^{(\tau+1)} \neq a_{i-1}^{(\tau+1)}\right\}$
in a number of steps that is bounded by a constant. Lemma 2 motivates the following definitions. For a given vector $\boldsymbol{b}$ (associated with the peak $\left.\left(U_{\tau}, V_{\tau}, W_{\tau}\right)\right)$ and a partition $U^{\prime}$ of $a_{q_{\tau+1}}$ we want to know how many shapes are needed to reduce $\boldsymbol{a}^{(\tau)}$ such that the peak for $\boldsymbol{a}^{(\tau+1)}$ starts with $U^{\prime}$.

Definition 3. Let $\boldsymbol{b}=\left(b_{1} \ldots b_{\beta}\right)$ be the vector associated with some $(u, v, w)-$ peak $(U, V, W)$ where $\alpha=|U \cup V|$ and $\beta=\alpha+|W|$, and let $U^{\prime}$ be a partition of $w$. Let $T$ be the set of positive integers $\rho$ such that there are integers $l_{1}, \ldots, l_{\rho}, r_{1}, \ldots, r_{\rho}$ and coefficients $c_{1}, \ldots, c_{\rho} \in \mathbb{N} \backslash\{0\}$ such that

1. $\sum_{j=1}^{\rho} c_{j}=v-w$,
2. $1 \leq l_{j} \leq r_{j} \leq \beta-1$ for $j=1,2, \ldots, \rho$.
and for $\boldsymbol{b}^{\prime}=\boldsymbol{b}-\sum_{j=1}^{\rho} c_{j} \boldsymbol{f}^{(j)}$, where

$$
f_{i}^{(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j} \leq i \leq r_{j}, \\
0 & \text { otherwise },
\end{array} \quad(j=1, \ldots, \rho ; i=1, \ldots, \beta)\right.
$$

we have
3. $b_{1}^{\prime} \leq b_{2}^{\prime} \leq \cdots \leq b_{\beta}^{\prime}=b_{\beta}=w$ and
4. $\left\{b_{i}^{\prime}-b_{i-1}^{\prime}: 1 \leq i \leq \beta, b_{i}^{\prime} \neq b_{i-1}^{\prime}\right\}=U^{\prime}\left(\right.$ with $\left.b_{0}^{\prime}=0\right)$.

Then we define

$$
\rho\left(\boldsymbol{b}, U^{\prime}\right)= \begin{cases}\min T & \text { if } T \neq \varnothing \\ \infty & \text { if } T=\varnothing\end{cases}
$$

Definition 4. Let $(U, V, W)$ and $\left(U^{\prime}, V^{\prime}, W^{\prime}\right)$ be a $(u, v, w)$-peak and a $\left(u^{\prime}, v^{\prime}, w^{\prime}\right)$-peak, respectively, where $u^{\prime}=w$. Then we put

$$
\delta\left((U, V, W),\left(U^{\prime}, V^{\prime}, W^{\prime}\right)\right)=\rho\left(\boldsymbol{b}, U^{\prime}\right),
$$

where $\boldsymbol{b}$ is the vector associated with $(U, V, W)$.
In order to model the decomposition process we define a digraph $G=$ $(\mathcal{V}, \mathcal{E})$. The vertex set is

$$
\mathcal{V}=\left\{\left(\tau, U, V_{\tau}, W_{\tau}\right): \quad 0 \leq \tau \leq t, U \text { is a partition of } a_{q_{\tau}}\right\},
$$



Figure 2: The digraph for the vector $\boldsymbol{a}$.
where

$$
\begin{aligned}
V_{\tau} & =\left\{a_{i}-a_{i-1}: q_{\tau}<i \leq p_{\tau+1}, a_{i} \neq a_{i-1}\right\}, \\
W_{\tau} & =\left\{a_{i}-a_{i+1}: p_{\tau+1} \leq i<q_{\tau+1}, \quad a_{i} \neq a_{i+1}\right\}
\end{aligned}
$$

for $0 \leq \tau \leq t$. Observe that there is only one vertex with first component 0 , namely $\left(0, \varnothing, V_{0}, W_{0}\right)$ corresponding to $\boldsymbol{a}^{(0)}=\boldsymbol{a}$ and there is only one vertex with first component $t$, namely $(t, \varnothing, \varnothing, \varnothing)$ corresponding to the zero vector. In general, the vertices with first component $\tau$ represent the possibilities for $\left(U_{\tau}, V_{\tau}, W_{\tau}\right)$, and for each $\tau$, the vertices with first component $\tau$ differ only in the second component $U_{\tau}$, because $V_{\tau}$ and $W_{\tau}$ depend only on $\boldsymbol{a}$. In the arc set $\mathcal{E}$ we include all arcs of the form

$$
\left(\left(\tau, U, V_{\tau}, W_{\tau}\right),\left(\tau+1, U^{\prime}, V_{\tau+1}, W_{\tau+1}\right)\right)
$$

for $\tau=0, \ldots, t-1$.
Example 4. Figure 2 shows $G$ for $\boldsymbol{a}=\left(\begin{array}{ll}13 & 2434\end{array}\right)$, where the vertices are labeled as follows.

$$
\begin{array}{lll}
a=(0, \varnothing,\{1,2\},\{1\}), & b=(1,\{2\},\{2\},\{1\}), & c=(1,\{1,1\},\{2\},\{1\}), \\
d=(2,\{3\},\{1\},\{4\}), & e=(2,\{2,1\},\{1\},\{4\}), & f=(2,\{1,1,1\},\{1\},\{4\}) \\
g=(3, \varnothing, \varnothing, \varnothing) . &
\end{array}
$$

We define the arc weights in $G$ to be the distances of the corresponding peaks, i.e.

$$
\delta\left(\left(\tau, U, V_{\tau}, W_{\tau}\right),\left(\tau+1, U^{\prime}, V_{\tau+1}, W_{\tau+1}\right)\right)=\delta\left(\left(U, V_{\tau}, W_{\tau}\right),\left(U^{\prime}, V_{\tau+1}, W_{\tau+1}\right)\right)
$$

for $0 \leq \tau \leq t-1$ and all partitions $U$ and $U^{\prime}$ of $a_{q_{\tau}}$ and $a_{q_{\tau+1}}$, respectively. Observe that in this definition we used the fact that $\left(U, V_{\tau}, W_{\tau}\right)$ and $\left(U^{\prime}, V_{\tau+1}, W_{\tau+1}\right)$ are an $\left(a_{q_{\tau}}, a_{p_{\tau+1}}, a_{q_{\tau+1}}\right)$-peak and an $\left(a_{q_{\tau+1}}, a_{p_{\tau+2}}, a_{q_{\tau+2}}\right)-$ peak, respectively. This assures that the condition $u^{\prime}=w$ in the definition of $\delta$ is satisfied.

Example 5. The first two shape matrices in Example 1 correspond to the arc $(a, b)$ of weight 2: the residual vector $\boldsymbol{a}^{(1)}$ corresponds to a peak starting with $U^{\prime}=\{2,1,1,1\}$, the second component of vertex $b$. The whole decomposition from this example corresponds to the path $a, b, f$ of weight 7 . Following the path $a, c, f$ instead yields better decompositions, for instance

$$
\begin{aligned}
& \boldsymbol{a}=(0011110000)+3(0001111000)+2(1111111111) \\
& +(0001111111)+2(0000011111)+7(0000000001)
\end{aligned}
$$

with $\boldsymbol{a}^{(1)}=(2223355512)$. Similarly, for the path $a, d, f$ we obtain the decomposition

$$
\begin{aligned}
& \boldsymbol{a}=\left(\begin{array}{ll}
00 & 01110000
\end{array}\right)+2(1111111000)+\left(\begin{array}{l}
0011111000
\end{array}\right) \\
& +3(0001111111)+2(0000011111)+7(0000000001)
\end{aligned}
$$

with $\boldsymbol{a}^{(1)}=(0003555512)$. The weight $\infty$ for the arc $(a, f)$ comes from the fact that it is not possible to find a leaf sequence and coefficients summing up to 4 , such that the reduced vector has associated vector ( 1234512 ).

With any decomposition we can associate a path

$$
\left(0, \varnothing, V_{0}, W_{0}\right),\left(1, U_{1}, V_{1}, W_{1}\right), \ldots,(t, \varnothing, \varnothing, \varnothing)
$$

in $G$. The minimal number of shape matrices needed to realize a decomposition corresponding to a given path equals the weight of this path.

Lemma 3. In time $O(1)$ we can determine the values $\rho\left(\boldsymbol{b}, U^{\prime}\right)$ for all vectors $\boldsymbol{b}$ that are associated with some $(u, v, w)-$ peak and for all partitions $U^{\prime}$ of $w$. In addition, we obtain values $c_{j}, l_{j}^{\prime}, r_{j}^{\prime}\left(j=1, \ldots, \rho\left(\boldsymbol{b}, U^{\prime}\right)\right)$ satisfying the conditions of Definition 3.

Proof. The total number of vectors $\boldsymbol{b}$ associated with some $(u, v, w)$-peak when $u, v$ and $w$ run through all the possible values is

$$
\sum_{v=1}^{L} \sum_{w=0}^{v-1} \mathcal{P}_{v} \mathcal{P}_{v-w}
$$

where $\mathcal{P}_{i}$ is the number of partitions of $i \in \mathbb{N}$. Fix one of these vectors $\boldsymbol{b}$. We consider all the sets $S=\left\{\left(l_{j}^{\prime}, r_{j}^{\prime}, c_{j}\right): j=1, \ldots, \rho\right\}(\rho \in \mathbb{N})$, such that the vectors $\boldsymbol{f}^{(1)}, \ldots, \boldsymbol{f}^{(\rho)}$, defined as in Definition 3 and the coefficients $c_{1}, \ldots, c_{\rho}$ satisfy the conditions in Definition 3. We claim that there are at most

$$
v^{v-w} \leq L^{L}
$$

possibilities for $S$. Writing $\sum_{k=1}^{c_{j}} \boldsymbol{f}^{(j)}$ for $c_{j} \boldsymbol{f}^{(j)}$ we can express $\sum_{j=1}^{\rho} c_{j} \boldsymbol{f}^{(j)}$ as a sum of $\sum_{j=1}^{\rho} c_{j}=v-w(0,1)$-vectors. In order to satisfy conditions 1 and 3 of Definition 3, for $i=\alpha, \ldots, \beta-1$, in exactly $b_{i}-b_{i+1}$ of these $(0,1)$-vectors must be 0 at position $i+1$ and a 1 at position $i$. So we may assume that the $v-w$ right leaf positions are fixed. Since for each right leaf position there are at most $v$ left leaf positions the claim follows. For each $S$ the resulting partition $U^{\prime}$ of $w$ can be computed in $O(1)$ steps, since $\rho$ is bounded by $v-w \leq L$, and $\beta$ is bounded by $2 L$. Thus the number of peaks is bounded by a constant, the number of sets $S$ to be checked for each peak is bounded by a constant, for each of these sets the number of steps for the checking is bounded by a constant, and this completes the proof.

Lemma 4. In time $O(n)$ we can determine the arc weights $\delta(e)$ for all $e \in \mathcal{E}$ and for each arc e a sequence

$$
\left(s^{(1)}, c_{1}\right), \ldots,\left(s^{(\delta(e))}, c_{\delta(e)}\right)
$$

realizing its weight.
Proof. By Lemma 3 we may assume that we know all the $\rho\left(\boldsymbol{b}, U^{\prime}\right)$. First we determine in time $O(n)$ the sets

$$
\begin{aligned}
P & =\left\{p_{1}, \ldots, p_{t}\right\}, & Q=\left\{q_{0}, \ldots, q_{t}\right\}, & \\
R_{1, \tau} & =\left\{i: q_{\tau}<i \leq p_{\tau+1}, a_{i} \neq a_{i-1}\right\} & & (\tau=0, \ldots, t-1), \\
R_{2, \tau} & =\left\{i: p_{\tau+1} \leq i<q_{\tau+1}, a_{i} \neq a_{i+1}\right\} & & (\tau=0, \ldots, t-1),
\end{aligned}
$$

and the partitions $V_{\tau}$ and $W_{\tau}(\tau=0, \ldots, t)$. By induction, we assume that we have already determined the weights of the arcs up to layer $\tau$ for some $\tau, 0 \leq \tau \leq t-1$. The number of vertices in layers $\tau$ and $\tau+1$ are bounded by $\mathcal{P}_{a_{q \tau}}$ and $\mathcal{P}_{a_{\tau+1}}$, respectively. So the number of arcs is bounded by $\mathcal{P}_{L}^{2}$. Fix some $\left(\tau, U_{\tau}, V_{\tau}, W_{\tau}\right)$ and $\left(\tau+1, U_{\tau+1}, V_{\tau+1}, W_{\tau+1}\right)$. Also by induction, we assume that we know the set

$$
R_{1}=\left\{i: 1 \leq i \leq p_{\tau+1}, a_{i}^{(\tau)} \neq a_{i-1}^{(\tau)}\right\}
$$

for some possible $\boldsymbol{a}^{(\tau)}$ corresponding to $\left(\tau, U_{\tau}, V_{\tau}, W_{\tau}\right)$. Now by Lemma 2 (and its proof) we obtain

$$
\delta\left(\left(\tau, U_{\tau}, V_{\tau}, W_{\tau}\right),\left(\tau+1, U_{\tau+1}, V_{\tau+1}, W_{\tau+1}\right)\right)
$$

and a sequence realizing this value in constant time from the corresponding data for $\boldsymbol{b}$ and $U^{\prime}$ where $\boldsymbol{b}$ is the vector associated with $\left(U_{\tau}, V_{\tau}, W_{\tau}\right)$ and $U^{\prime}=U_{\tau+1}$. If $\tau \leq t-2$ this also yields

$$
R_{1}^{\prime}=\left\{i: 1 \leq i \leq p_{\tau+2}, a_{i}^{(\tau+1)} \neq a_{i-1}^{(\tau+1)}\right\}
$$

for some possible $\boldsymbol{a}^{(\tau+1)}$ corresponding to $\left(\tau+1, U_{\tau+1}, V_{\tau+1}, W_{\tau+1}\right)$. So the weights for all arcs between adjacent layers can be determined in time $O(1)$. Since the number of layers $t+1$ is bounded by $n$, the lemma is proved.

Now the search for a decomposition with minimal DC amounts to the search for a path of minimal weight in a layered digraph with at most $n$ layers where the number of vertices per layer is bounded by the constant $\mathcal{P}_{L}$. This can be done in time $O(n)$ [11]. Thus we have proved

Theorem 1. L-One Row-Min DT-Min DC can be solved in time $O(n)$.

## 3 Multiple row intensity maps

In this section we generalize the basic idea of Section 2 to prove that for bounded $L$ the DC-minimization is polynomially solvable also for multiple row matrices. The problem $L-$ Min DT-Min DC is:

Instance: An integer matrix $A=\left(a_{i, j}\right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ with $0 \leq a_{i, j} \leq L(i \in[m], j \in$ $[n]$ ).

Problem: Find a shape matrix decomposition of $A$ with in first instance minimal DT and in second instance minimal DC.

Assume we have already determined the minimal DT $c$. For any decomposition of $A=\sum_{i=1}^{k} c_{i} S_{i}$, the coefficients form a partition $c=c_{1}+c_{2}+\cdots+c_{k}$. First we consider the problem to check for a given partition if there is a decomposition of $A$ with coefficients $c_{1}, \ldots, c_{k}$. This problem can be solved by checking the rows of $A$ independently. For the moment we omit the row index and denote by $\boldsymbol{a}=\left(a_{1} \ldots a_{n}\right)$ a fixed row of $A$ and as before we put $a_{0}=a_{n+1}=0$. Compared to the single row case an additional difficulty in the multiple row case arises from the fact that the minimal DT that would be
sufficient for a decomposition of $\boldsymbol{a}$ might be smaller than $c$. As a consequence we cannot use Lemma 2, where condition (11) is essential. Here the order of the elements of the considered partition must be taken into consideration. For instance, for $\boldsymbol{b}=(250)$ there is a decomposition with coefficients 4,1 and 1 , namely

$$
\boldsymbol{b}=4\left(\begin{array}{lll}
0 & 1 & 0
\end{array}\right)+\left(\begin{array}{lll}
1 & 1 & 0
\end{array}\right)+\left(\begin{array}{lll}
1 & 0 & 0
\end{array}\right),
$$

while there is no decomposition with these coefficients for $\boldsymbol{b}^{\prime}=(350)$. So instead of peaks we have to consider ordered peaks to be defined below. Also, in order to describe the decomposition, we attach to a peak a multiset $X$ of coefficients, and call the result an extended ordered peak. This is made precise in the following definition.

Definition 5. For integers $v$ and $w$ with $0 \leq w<v \leq L$ an extended ordered $(v, w)$-peak is a pair $(\boldsymbol{b}, X)$ of an integer vector $\boldsymbol{b}=\left(b_{1} b_{2} \ldots b_{\beta}\right)$, such that there is an integer $\alpha$ with $1 \leq \alpha<\beta$ and

$$
\begin{array}{r}
0<b_{1}<b_{2}<\cdots<b_{\alpha}=v, \\
v=b_{\alpha}>b_{\alpha+1}>\cdots>b_{\beta}=w,
\end{array}
$$

and a multiset $X$ of positive integers. In addition, a pair $(\boldsymbol{b}, X)$, where $\boldsymbol{b}=()$ is the empty tuple and $X$ is a multiset of positive integers, is called extended ordered ( 0,0 )-peak.

Example 6. (( 25743$),\{1,2,2,3,3\})$ is an extended ordered (7,3)-peak (with $\alpha=3, \beta=5$ ).

Let $p_{1}, \ldots, p_{t}$ and $q_{0}, \ldots, q_{t}$ be defined as in the preceding section. Then for a decomposition

$$
\boldsymbol{a}=\sum_{j=1}^{k} c_{j} \boldsymbol{s}^{(j)}
$$

we can define $k_{0}(\tau)$ and $\boldsymbol{a}^{(\tau)}(\tau=0, \ldots, t)$ as before. Now for $\tau=0, \ldots, t$, we associate with $\boldsymbol{a}^{(\tau)}$ an extended ordered $\left(a_{p_{\tau+1}}, a_{q_{\tau+1}}\right)$-peak $\left(\boldsymbol{b}^{(\tau)}, X_{\tau}\right)$ as follows. For $\tau<t$, let

$$
\begin{aligned}
& I_{\tau}=\left\{i: 1 \leq i \leq p_{\tau+1}, a_{i}^{(\tau)} \neq a_{i-1}^{(\tau)}\right\}, \\
& J_{\tau}=\left\{i: p_{\tau+1}<i \leq q_{\tau+1}, a_{i}^{(\tau)} \neq a_{i-1}^{(\tau)}\right\},
\end{aligned}
$$

denote the elements of $I_{\tau}$ by $i_{1}, \ldots, i_{\alpha}$ and the elements of $J_{\tau}$ by $i_{\alpha+1}, \ldots, i_{\beta}$ such that $i_{1}<i_{2}<\cdots<i_{\beta}$, and put

$$
b_{0}=0, \quad b_{l}=a_{i_{l}} \quad(l=1, \ldots, \beta) .
$$

Let $X_{0}=\left\{c_{1}, \ldots, c_{k}\right\}$ and

$$
X_{\tau+1}=X_{\tau} \backslash\left\{c_{k_{0}(\tau)+1}, c_{k_{0}(\tau)+2}, \ldots, c_{k_{0}(\tau+1)}\right\} \quad(\tau=0, \ldots, t-1)
$$

Now for $\tau<t,\left(\boldsymbol{b}^{(\tau)}, X_{\tau}\right)$ describes the initial part of $\boldsymbol{a}^{(\tau)}$ (up to column $q_{\tau+1}$ ) together with the coefficients available for the remaining shape matrices. In the final state ( $\tau=t$ ) we have the zero row $\boldsymbol{a}^{(t)}=0$ and a multiset $X_{t}$ of coefficients, that are not needed for the considered row. With the zero row we associate the empty tuple $\boldsymbol{b}^{(t)}=()$, and thus we obtain from any decomposition a sequence $\left(\boldsymbol{b}^{(0)}, X_{0}\right),\left(\boldsymbol{b}^{(1)}, X_{1}\right), \ldots,\left(\boldsymbol{b}^{(t)}, X_{t}\right)$ of extended ordered peaks. This is illustrated by the following example.

Example 7. Suppose $\boldsymbol{a}=(243163061)$ is a row in an intensity matrix with minimal DT $c=18$, and we are checking the partition $c=5+3+2+$ $2+2+1+1+1+1$. Then from the decomposition

$$
\begin{aligned}
& \quad(243163061)=2(111000000)+(010000000)+3(000010000) \\
& +2(000011000)+(011111000)+5(000000010)+(000000011)
\end{aligned}
$$

we obtain

| $\tau$ | $\boldsymbol{a}^{(\tau)}$ | $b^{(\tau)}$ | $X_{\tau}$ |
| :---: | :---: | :---: | :---: |
| 0 | (243163061) | (2431) | \{5,3,2,2,2,1,1,1,1\} |
| 1 | (011163061) | (16.30) | \{5,3,2,2,1,1,1\} |
| 2 | (000000061) | $\left(\begin{array}{llll}6 & 1 & 0\end{array}\right)$ | \{5,2,1,1\} |
| 3 | (000000000) | () | \{2,1\} |

The vectors $\boldsymbol{b}^{(\tau)}$ provide enough information to construct the decomposition. This follows from the simple observation, that w.l.o.g. a plateau, i.e. a sequence of consecutive entries of equal value

$$
a_{i_{1}}=a_{i_{1}+1}=\cdots=a_{i_{2}}
$$

can be considered as one single entry. That means we can always choose the shapes in such a way that the entries corresponding to a plateau are either all 0 or all 1 . This is intuitively clear and proved formally in the next lemma. The idea is to modify any decomposition without changing the coefficients such that the new decomposition has the required property. This is illustrated by the following example.

Example 8. We take the decomposition

$$
(24551)=4\left(\begin{array}{llll}
0 & 0 & 0 & 1
\end{array}\right)+2\left(\begin{array}{lllll}
0 & 1 & 1 & 0 & 0
\end{array}\right)+2\left(\begin{array}{llll}
1 & 1 & 1 & 0
\end{array}\right)+\left(\begin{array}{llll}
0 & 0 & 1 & 1
\end{array}\right),
$$

where the first three shapes are not constant on the plateau of 5 -entries. By replacing $s_{3}$ by $s_{4}$ in every shape we get the new decomposition

$$
\left(\begin{array}{ll}
2 & 4
\end{array}\right)
$$

Lemma 5. Let $\boldsymbol{a}=\sum_{j=1}^{k} c_{j} \boldsymbol{s}^{(j)}$ be a decomposition with

$$
s_{i}^{(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j} \leq i \leq r_{j} \\
0 & \text { otherwise }
\end{array} \quad(j=1, \ldots, k) .\right.
$$

There are integers $l_{j}^{\prime}$ and $r_{j}^{\prime}(j=1, \ldots, k)$ with the following properties.

1. We have $\boldsymbol{a}=\sum_{j=1}^{k} c_{j} \boldsymbol{s}^{(j)}$ where

$$
s_{i}^{\prime(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j}^{\prime} \leq i \leq r_{j}^{\prime} \\
0 & \text { otherwise }
\end{array} \quad(j=1, \ldots, k) .\right.
$$

2. 

$$
\begin{equation*}
a_{i}=a_{i-1} \Longrightarrow s_{i}^{(j)}=s_{i-1}^{\prime(j)} \quad(i=2, \ldots, n ; j=1, \ldots, k) . \tag{12}
\end{equation*}
$$

Proof. In order to satisfy the last condition, we have to replace the shape matrices with $s_{i}^{(j)} \neq s_{i-1}^{(j)}$ but $a_{i}=a_{i-1}$ for some $i$. Our strategy is to modify the given shape matrices as follows. For each plateau we choose one representative, for instance the rightmost entry, and adapt the entries for each shape matrix to the chosen column. This corresponds to the following shifting of the leaves: If the left leaf covers a part of the plateau it is shifted to the left until the whole plateau is open, and if the right leaf covers a part of the plateau it is shifted to the left until the whole plateau is covered.

First observe that $s_{i}^{(j)}$ can differ from $s_{i-1}^{(j)}$ only if $i=l_{j}$ or $i-1=r_{j}$. So for (12) it is sufficient that, for all $j$, we have

$$
\begin{equation*}
a_{l_{j}^{\prime}} \neq a_{l_{j}^{\prime}-1} \quad \text { and } \quad a_{r_{j}^{\prime}} \neq a_{r_{j}^{\prime}+1} . \tag{13}
\end{equation*}
$$

Suppose $a_{l_{j}}=a_{l_{j}-1}$ for some $j$. Then $i_{1}<l_{j} \leq i_{2}$ for some $i_{1}, i_{2}$ with

$$
\begin{equation*}
a_{i_{1}}=a_{i_{1}+1}=\cdots=a_{i_{2}}=a \quad \text { and } \quad a_{i_{1}-1}, a_{i_{2}+1} \neq a . \tag{14}
\end{equation*}
$$

Since we want to adapt the entries of the shape matrix to the rightmost column $i_{2}$ we have to shift the left leaf to the left and put $l_{j}^{\prime}=i_{1}$. Similarly, if $a_{r_{j}}=a_{r_{j}+1}$, then $i_{1} \leq r_{j}<i_{2}$ for some $i_{1}$, $i_{2}$ with (14), and in order to
adapt the entries of the shape matrix to column $i_{2}$, we have to shift the right leaf to the left and put $r_{j}^{\prime}=i_{1}-1$. In summary, for $j \in[k]$ we put

$$
\begin{aligned}
& l_{j}^{\prime}= \begin{cases}l_{j} & \text { if } a_{l_{j}} \neq a_{l_{j}-1}^{\prime}, \\
\max \left\{i<l_{j}: a_{i} \neq a_{l_{j}}\right\}+1 & \text { if } a_{l_{j}}=a_{l_{j}-1}^{\prime},\end{cases} \\
& r_{j}^{\prime}= \begin{cases}r_{j} & \text { if } a_{r_{j}} \neq a_{r_{j}+1}^{\prime}, \\
\max \left\{i<r_{j}: a_{i} \neq a_{r_{j}}\right\} & \text { if } a_{r_{j}}=a_{r_{j}+1}^{\prime} .\end{cases}
\end{aligned}
$$

Then (13) is valid for all $j$, hence (12) is satisfied. In order to check the first condition of the lemma, fix some $i \in[n]$. If $s_{i}^{(j)}=s_{i}^{(j)}$ for all $j$, then

$$
\sum_{j=1}^{k} c_{j} s_{i}^{(j)}=\sum_{j=1}^{k} c_{j} s_{i}^{(j)}=a_{i}
$$

So assume $s_{i}^{(j)} \neq s_{i}^{(j)}$ for some $j$. By construction this can be the case only if $a_{i}=a_{i-1}$ or $a_{i}=a_{i+1}$. Now let $i_{1}$ and $i_{2}$ be the indices with $i_{1} \leq i \leq i_{2}$,

$$
a_{i_{1}}=a_{i_{1}+1}=\cdots=a_{i}=\cdots=a_{i_{2}} \quad \text { and } \quad a_{i_{1}-1}, a_{i_{2}+1} \neq a_{i} .
$$

We claim that $s_{i}^{(j)}=s_{i_{2}}^{(j)}(j=1, \ldots, k)$. If $s_{i_{2}}^{(j)}=0, l_{j}>i_{2}$ or $r_{j}<i_{2}$. By construction, in the first case $l_{j}^{\prime}>i_{2}$ and in the second case $r_{j}^{\prime}<i_{1}$, so in both cases $s_{i}^{(j)}=0$. If $s_{i_{2}}^{(j)}=1, l_{j} \leq i_{2}$ and $r_{j} \geq i_{2}$. By construction, $l_{j}^{\prime} \leq i_{1}$ and $r_{j}^{\prime} \geq i_{2}$, hence $s_{i}^{(j)}=1$ and the claim is proved. From this it follows that

$$
\sum_{j=1}^{k} c_{j} s_{i}^{(j)}=\sum_{j=1}^{k} c_{j} s_{i_{2}}^{(j)}=a_{i_{2}}=a_{i}
$$

and since this argument works for any $i \in[n]$ the first condition of the lemma is satisfied.

By Lemma 5 applied to $\boldsymbol{a}^{(\tau)}$, w.l.o.g. we may assume that $a_{l_{j}}^{(\tau)} \neq a_{l_{j}-1}^{(\tau)}$ and $a_{r_{j}}^{(\tau)} \neq a_{r_{j}+1}^{(\tau)}$ for all $j>k_{0}(\tau)$. With this assumption the next lemma, whose proof is obvious, justifies that we use the $\boldsymbol{b}^{(\tau)}$ instead of the $\boldsymbol{a}^{(\tau)}$.

Lemma 6. For fixed $\tau, 0 \leq \tau \leq t-1$, let $\boldsymbol{b}^{(\tau)}$ and $X_{\tau}$ be defined as described above and let $\left\{c_{1}, \ldots, c_{\rho}\right\} \subseteq X_{\tau}$ be fixed. If $a_{q_{\tau+1}} \neq 0$ let $\boldsymbol{g}=\left(g_{1} \ldots g_{\gamma}\right)$ be some vector with

$$
0<g_{1}<\cdots<g_{\gamma}=a_{q_{\tau+1}} .
$$

Then the following statements are equivalent.

1. There exist integers $l_{j}, r_{j}$ with $1 \leq l_{j} \leq r_{j}<q_{\tau+1}, a_{l_{j}}^{(\tau)} \neq a_{l_{j}-1}^{(\tau)}$ and $a_{r_{j}}^{(\tau)} \neq a_{r_{j}+1}^{(\tau)}(j=1, \ldots, \rho)$ such that for $\boldsymbol{a}^{\prime}=\boldsymbol{a}^{(\tau)}-\sum_{j=1}^{\rho} c_{j} \boldsymbol{s}^{(j)}$, where

$$
s_{i}^{(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j} \leq i \leq r_{j} \\
0 & \text { otherwise }
\end{array} \quad(j=1, \ldots, \rho ; i=1, \ldots, n)\right.
$$

we have
(a) $0 \leq a_{1}^{\prime} \leq a_{2}^{\prime} \leq \cdots \leq a_{q_{\tau+1}}^{\prime}=a_{q_{\tau+1}}$
(b) If $a_{q_{\tau+1}} \neq 0$ there are exactly $\gamma$ indices $1 \leq i_{1}<\cdots<i_{\gamma} \leq q_{\tau+1}$ with $a_{i_{j}}^{\prime} \neq a_{i_{j}-1}^{\prime}$ for $j \in[\gamma]$ (where $a_{0}^{\prime}=0$ ) and we have

$$
\left(\begin{array}{llll}
a_{i_{1}} & a_{i_{2}} & \ldots & a_{i_{\gamma}}
\end{array}\right)=\boldsymbol{g} .
$$

2. There exist integers $l_{j}^{\prime}$, $r_{j}^{\prime}$ with $1 \leq l_{j}^{\prime} \leq r_{j}^{\prime} \leq \beta-1$ for $j=1, \ldots, \rho$, such that for $\boldsymbol{b}^{\prime}=\boldsymbol{b}-\sum_{j=1}^{\rho} c_{j} \boldsymbol{f}^{(j)}$, where

$$
f_{i}^{(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j}^{\prime} \leq i \leq r_{j}^{\prime} \\
0 & \text { otherwise },
\end{array} \quad(j=1, \ldots, \rho ; i=1, \ldots, \beta),\right.
$$

we have
(a) $b_{1}^{\prime} \leq b_{2}^{\prime} \leq \cdots \leq b_{\beta}^{\prime}=b_{\beta}=a_{q_{\tau+1}}$
(b) If $a_{q_{\tau+1}} \neq 0$ there are exactly $\gamma$ indices $1 \leq i_{1}<\cdots<i_{\gamma} \leq \beta$ with $b_{i_{j}}^{\prime} \neq b_{i_{j}-1}^{\prime}$ for $j \in[\gamma]$ (where $b_{0}^{\prime}=0$ ) and we have

$$
\left(\begin{array}{llll}
b_{i_{1}} & b_{i_{2}} & \ldots & b_{i_{\gamma}}
\end{array}\right)=\boldsymbol{g}
$$

Now for $\tau=0,1, \ldots, t-1$ the choice of the pairs

$$
\left(s^{k_{0}(\tau)+1}, c_{k_{0}(\tau)+1}\right), \ldots,\left(s^{k_{0}(\tau+1)}, c_{k_{0}(\tau+1)}\right)
$$

can be viewed as a way to go from the extended ordered $\left(a_{p_{\tau+1}}, a_{q_{\tau+1}}\right)$-peak $\left(\boldsymbol{b}^{(\tau)}, X_{\tau}\right)$ to the extended ordered $\left(a_{p_{\tau+2}}, a_{q_{\tau+2}}\right)$-peak $\left(\boldsymbol{b}^{(\tau+1)}, X_{\tau+1}\right)$ (with $\left.a_{p_{t+1}}=a_{q_{t+1}}=0\right)$.
Definition 6. Let $0 \leq w<v$ and let $(\boldsymbol{b}, X)$ be an extended ordered $(v, w)-$ peak, and let $v^{\prime}, w^{\prime}$ be integers with $w \leq v^{\prime} \leq L$ and $0 \leq w^{\prime}<v^{\prime}$ or $v^{\prime}=w^{\prime}=0$. In addition let $X^{\prime}$ be a submultiset of $X$ and denote the elements of $X^{\prime}$ by $x_{1}, \ldots, x_{\left|X^{\prime}\right|}$. We call an extended ordered ( $\left.v^{\prime}, w^{\prime}\right)$-peak $\left(\boldsymbol{b}^{\prime}, X \backslash X^{\prime}\right)$ accessible from $(\boldsymbol{b}, X)$ if there are integers $l_{1}^{\prime}, \ldots, l_{\left|X^{\prime}\right|}^{\prime}, r_{1}^{\prime}, \ldots, r_{\left|X^{\prime}\right|}^{\prime}$ such that

1. $1 \leq l_{j}^{\prime} \leq r_{j}^{\prime} \leq \beta-1$ for $j=1, \ldots,\left|X^{\prime}\right|\left(\right.$ where $\left.\boldsymbol{b}=\left(b_{1} \ldots b_{\beta}\right)\right)$.
and for $\boldsymbol{b}^{\prime \prime}=\boldsymbol{b}-\sum_{j=1}^{\left|X^{\prime}\right|} x_{j} \boldsymbol{f}^{(j)}$, where

$$
f_{i}^{(j)}=\left\{\begin{array}{ll}
1 & \text { if } l_{j}^{\prime} \leq i \leq r_{j}^{\prime} \\
0 & \text { otherwise }
\end{array} \quad\left(j=1, \ldots,\left|X^{\prime}\right| ; i=1, \ldots, \beta\right),\right.
$$

we have $\boldsymbol{b}^{\prime \prime}=\mathbf{0}$ if $v^{\prime}=w^{\prime}=0$ and otherwise
2. $b_{1}^{\prime \prime} \leq b_{2}^{\prime \prime} \leq \cdots \leq b_{\beta}^{\prime \prime}=b_{\beta}=w$ and
3. If $i_{1}<i_{2}<\cdots<i_{\gamma^{\prime}}$ are the indices with $b_{i_{j}}^{\prime \prime} \neq b_{i_{j}-1}^{\prime \prime}$ for $j \in[\gamma]$ (where $b_{0}^{\prime \prime}=0$ ), then

$$
\left(\begin{array}{llll}
b_{i_{1}}^{\prime \prime} & b_{i_{2}}^{\prime \prime} & \ldots & b_{i_{\gamma^{\prime}}}^{\prime \prime}
\end{array}\right)=\left(\begin{array}{llll}
b_{1}^{\prime} & b_{2}^{\prime} & \ldots & b_{\gamma^{\prime}}^{\prime}
\end{array}\right) .
$$

The definition can be interpreted as follows. Assume $a_{p_{1}}=v, a_{q_{1}}=w$, $a_{p_{2}}=v^{\prime}, a_{q_{2}}=w^{\prime}$, let $\boldsymbol{b}^{(0)}$ be associated with $\boldsymbol{a}^{(0)}$ as above, and let $\boldsymbol{b}^{\prime}=$ $\left(b_{1}^{\prime} \ldots b_{\beta^{\prime}}^{\prime}\right)$ be a vector with

$$
0<b_{1}^{\prime}<\cdots<b_{\alpha^{\prime}}^{\prime}=v^{\prime}, v^{\prime}=b_{\alpha^{\prime}}^{\prime}>\cdots>b_{\beta^{\prime}}^{\prime}=w^{\prime}
$$

Then $\left(\boldsymbol{b}^{\prime}, X \backslash X^{\prime}\right)$ is accessible from $\left(\boldsymbol{b}^{(0)}, X\right)$ iff we can assign shape matrices $s^{(j)}$ to the elements of $X^{\prime}$, described by $l_{j}, r_{j}\left(j=1, \ldots,\left|X^{\prime}\right|\right)$ with $r_{j}<q_{1}$ for all $j$, such that for

$$
\boldsymbol{a}^{(1)}=\boldsymbol{a}^{(0)}-\sum_{j=1}^{\left|X^{\prime}\right|} x_{j} \boldsymbol{s}^{(j)}
$$

we have $a_{1}^{(1)} \leq a_{2}^{(1)} \leq \cdots \leq a_{p_{2}}^{(1)}$ and the extended ordered $\left(v^{\prime}, w^{\prime}\right)$-peak associated with $\boldsymbol{a}^{(1)}$ is $\left(\boldsymbol{b}^{\prime}, X \backslash X^{\prime}\right)$.

Example 9. Let $\boldsymbol{a}=(025574335682), X=\{5,3,2,2,2,1,1,1\}$ and $X^{\prime}=\{3,1\}$. The associated extended ordered (7,3)-peak is $(\boldsymbol{b}, X)$ where $\boldsymbol{b}=(25743)$. We want to determine the extended ordered ( 8,0 )-peaks $\left(\boldsymbol{b}^{\prime}, X \backslash X^{\prime}\right)$ that are accessible from $(\boldsymbol{b}, X)$, where

$$
\boldsymbol{b}^{\prime}=\left(\begin{array}{llllllll}
b_{1}^{\prime} & \ldots & b_{\gamma-1}^{\prime} & b_{\gamma}^{\prime} & 5 & 6 & 8 & 2
\end{array}\right)
$$

with $b_{\gamma}^{\prime}=3$. We obtain that $\left(\boldsymbol{b}^{(1)}, X \backslash X^{\prime}\right)$ and $\left(\boldsymbol{b}^{(2)}, X \backslash X^{\prime}\right)$ are accessible from $(\boldsymbol{b}, X)$, where $\boldsymbol{b}^{(1)}=(235682)$ and $\boldsymbol{b}^{(2)}=(135682)$ :

$$
\left.\begin{array}{l}
\left(\begin{array}{l}
2
\end{array} 23 l l l\right.
\end{array}\right)=\boldsymbol{b}-\left(\begin{array}{l}
0
\end{array} 3\right.
$$

This corresponds to the following possible beginnings of a decomposition.

$$
\begin{aligned}
& \text { ( } 025574335682 \text { ) } \\
& -\left(\begin{array}{ll}
003330000000)
\end{array}\right. \\
& -\left(\begin{array}{llllllll}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array} 000\right. \text { ) } \\
& =\left(\begin{array}{ll}
022233335682)
\end{array}\right. \\
& \text { (025574335682) } \\
& \text { - ( } 003330000000 \text { ) } \\
& \text { - ( } 0111111000000) \\
& =\left(\begin{array}{lll}
0 & 1 & 1 \\
1 & 3 & 3
\end{array} 335682\right) \text {. }
\end{aligned}
$$

On the other hand one can check that ( $\left(\begin{array}{ll}5 & 6 \\ \hline\end{array}\right.$ 2) , $\left.X \backslash X^{\prime}\right)$ is not accessible from ( $\boldsymbol{b}, X$ ) because it is not possible to find $\left(l_{1}, r_{1}\right)$ and $\left(l_{2}, r_{2}\right)$ with $r_{1}, r_{2}<7$ such that after subtracting the corresponding shape matrices with coefficients 3 and 1 from $\boldsymbol{a}$ we obtain a row vector $\boldsymbol{a}^{\prime}$ with $a_{1}^{\prime}=\cdots=a_{i}^{\prime}=0, a_{i+1}^{\prime}=$ $\cdots=a_{7}^{\prime}=3$ for some $i, 1 \leq i \leq 6$. A similar remark applies to (1235682).

Lemma 7. Let $(\boldsymbol{b}, X)$ be an extended ordered $(v, w)-p e a k$. Then the set of all $\left(\boldsymbol{b}^{\prime}, X \backslash X^{\prime}\right)$ that are accessible from $(\boldsymbol{b}, X)$ can be determined in time $O(1)$.

Proof. Observe that the accessibility does not depend on the whole vector $\boldsymbol{b}^{\prime}$ but only on the initial part ( $b_{1}^{\prime} \ldots b_{\gamma^{\prime}}^{\prime}=w$ ). So in order to determine the accessible extended ordered peaks it is sufficient to determine the pairs $\left(\left(b_{1}^{\prime} \ldots b_{\gamma^{\prime}}^{\prime}\right), X \backslash X^{\prime}\right)$ of initial parts and multisets of coefficients. Let $\boldsymbol{b}=$ $\left(b_{1} \ldots b_{\beta}\right)$ and let $\alpha$ be the unique index with $b_{\alpha}=v$. We have $b_{1}<\cdots<b_{\alpha}$ and $b_{\alpha}>\cdots>b_{\beta}$. So for $1 \leq k \leq v-1$ there are at most two indices $i$ and $i^{\prime}$ with $1 \leq i, i^{\prime} \leq \beta-1$ and $b_{i}=k, b_{i^{\prime}}=k$ (namely the first one with $1 \leq i \leq \alpha-1$ and the second one with $\alpha+1 \leq i^{\prime} \leq \beta-1$ ). The only index $i$ with $b_{i}=v$ is $i=\alpha$, and so we have

$$
\sum_{i=1}^{\beta-1} b_{i} \leq v+2 \sum_{k=1}^{v-1} k \leq L^{2} .
$$

Hence it is sufficient to consider at most $\mathcal{P}_{L^{2}}$ candidates for $X^{\prime}$, where each of these has at most $L^{2}$ elements. Fix one of these $X^{\prime}$. Indexing the elements of $X^{\prime}$ as in Definition 6, for each $x_{j} \in X^{\prime}$ there are at most $\binom{2 L-1}{2}$ choices for $\boldsymbol{f}^{(j)}$. So the total number of choices for the pairs $\left(\boldsymbol{f}^{(j)}, x_{j}\right)$ that have to be considered is bounded by

$$
\left[\binom{2 L-1}{2}\right]^{\left|X^{\prime}\right|} \leq\left[\binom{2 L-1}{2}\right]^{L^{2}} .
$$

For each of these choices the time needed to determine the resulting $\boldsymbol{b}^{\prime \prime}$ is bounded by a constant. Precisely, in order to subtract one of the $x_{j} \boldsymbol{f}^{(j)}$ we have to do at most $2 L$ subtractions. So after at most $L^{2} \cdot 2 L$ subtractions
we have determined $\boldsymbol{b}^{\prime \prime}$. Finally, in order to determine the corresponding ( $b_{1}^{\prime} \ldots b_{\gamma^{\prime}}^{\prime}$ ) according to condition 3 of Definition 6, we have to run through the at most $2 L$ entries of $\boldsymbol{b}^{\prime}$. This proves the lemma, since the number of steps to determine the required data is bounded by

$$
\mathcal{P}_{L^{2}}\left[\binom{2 L-1}{2}\right]^{L^{2}}\left(L^{2}+1\right) 2 L .
$$

In order to model the decomposition we construct sets $\mathcal{V}_{0}, \ldots, \mathcal{V}_{t}$ of extended ordered peaks. Put $\mathcal{V}_{0}=\left\{\left(\boldsymbol{b}^{(0)}, X_{0}\right)\right\}$ and suppose we have already constructed $\mathcal{V}_{0}, \ldots, \mathcal{V}_{\tau}$ for some $\tau$ with $0 \leq \tau<t$. Now we put

$$
\begin{aligned}
& \mathcal{V}_{\tau+1}=\left\{\left(\boldsymbol{b}^{\prime}, X^{\prime}\right):\left(\boldsymbol{b}^{\prime}, X^{\prime}\right) \text { is an }\left(a_{p_{\tau+2}}, a_{q_{\tau+2}}\right)-\right.\text { peak that } \\
& \text { is accessible from some } \left.(\boldsymbol{b}, X) \in \mathcal{V}_{\tau}\right\} .
\end{aligned}
$$

Here, in order to avoid case distinctions for $\tau=t$, we put $a_{p_{t+1}}=1$ and $a_{q_{t+1}}=0$. The elements of $\mathcal{V}_{\tau}$ represent the possibilities for $\left(\boldsymbol{b}^{(\tau)}, X_{\tau}\right)$. There is a decomposition of the row with coefficients $c_{1}, \ldots, c_{k}$ iff $\mathcal{V}_{t} \neq \varnothing$. Note that a natural interpretation of this construction is a breadth first search (BFS) in the tree with vertex set $\mathcal{V}_{0} \cup \ldots \cup \mathcal{V}_{t}$ starting at $\left(\boldsymbol{b}^{(0)}, X_{0}\right)$, where two vertices $(\boldsymbol{b}, X)$ and $\left(\boldsymbol{b}^{\prime}, X^{\prime}\right)$ are connected by an edge iff $(\boldsymbol{b}, X) \in \mathcal{V}_{\tau}$, $\left(\boldsymbol{b}^{\prime}, X^{\prime}\right) \in \mathcal{V}_{\tau+1}$ for some $\tau$ and $\left(\boldsymbol{b}^{\prime}, X^{\prime}\right)$ is accessible from $(\boldsymbol{b}, X)$.

Lemma 8. For given $\mathcal{V}_{\tau}$, the set $\mathcal{V}_{\tau+1}$ can be determined in time $O\left(n^{L+1}\right)$.
Proof. The sum of the elements of $X_{0}$ (the minimal DT) equals

$$
c=\max _{1 \leq i \leq m} \sum_{j=1}^{n} \max \left\{0, a_{i, j}-a_{i, j-1}\right\} \leq n L .
$$

In any partition $c=c_{1}+\cdots+c_{k}$ where the $c_{i}(i \in[k])$ are the coefficients of a decomposition of $A$, we have $c_{i} \leq L$ for $i \in[k]$. Hence such a partition can be described by an $L$-tuple $\left(\lambda_{1}, \ldots, \lambda_{L}\right)$ of integers, where $\lambda_{r}$ is the number of summands equal to $r$ for $r \in[L]$. Then $\lambda_{r} \leq n L / r(r \in[L])$, and so there are $O\left(n^{L}\right)$ choices for $X_{0}$. The multiset $X$ in $(\boldsymbol{b}, X) \in \mathcal{V}_{\tau}$ is a partition of some $c^{\prime}$ with $0 \leq c^{\prime} \leq c \leq n L$ with all summands less than or equal to $L$. So there are $n L$ possibilities for $c^{\prime}$, and for each of these there are $O\left(n^{L}\right)$ possible partitions. Thus the number of choices for $X$ is bounded by $O\left(n^{L+1}\right)$. The vectors $\boldsymbol{b}$ in the elements of $\mathcal{V}_{\tau}$ differ only in the initial part ( $b_{1} \ldots b_{\gamma}$ ), where
$b_{\gamma}=a_{q_{\tau}}$. But these initial parts are in bijection to the ordered partitions of $a_{q_{\tau}}$, and of these there are (see for instance [1])

$$
\sum_{i=1}^{a_{q \tau}}\binom{a_{q \tau}-1}{i-1} \leq L\binom{L}{\left\lfloor\frac{L}{2}\right\rfloor} .
$$

Since $L$ is bounded by a constant we obtain that $\left|\mathcal{V}_{\tau}\right|$ is bounded by $O\left(n^{L+1}\right)$. By Lemma 7, for each $(\boldsymbol{b}, X) \in \mathcal{V}_{\tau}$ the set of accessible ( $\boldsymbol{b}^{\prime}, X \backslash X^{\prime}$ ) can be determined in time bounded by a constant, and this yields the claim.

Lemma 9. For a fixed partition $c=c_{1}+\cdots+c_{k}$, it can be checked in time $O\left(n^{L+2}\right)$ if there is a decomposition of $\boldsymbol{a}$ with coefficients $c_{1}, \ldots, c_{k}$.

Proof. We only have to check if $\mathcal{V}_{t} \neq \varnothing$. Since $t \leq n$ the claim is an immediate consequence of Lemma 8.

Now we can prove
Theorem 2. The problem L-Min DT-Min DC can be solved in time $O\left(m n^{2 L+2}\right)$.

Proof. Obviously,

$$
c=\max _{1 \leq i \leq m} \sum_{j=1}^{n} \max \left\{0, a_{i, j}-a_{i, j-1}\right\}
$$

can be determined in time $O(m n)$. As in the proof of Lemma 8 the number of partitions of $c=c_{1}+\cdots+c_{k}$ that have to be considered is bounded by $O\left(n^{L}\right)$. By Lemma 9, for a fixed partition $c=c_{1}+\cdots+c_{k}$ it can be checked in time $O\left(m n^{L+2}\right)$ if there is a decomposition of $A$ with coefficients $c_{1}, \ldots, c_{k}$, and this concludes the proof.

Observe that for the algorithm in this section it is not necessary that $c$ is equal to the minimal value of the DT: For any value $c$ we can determine a decomposition with $D T=c$ and minimal DC. So one could try to increase $c$ step by step (starting with the minimal DT) in order to reduce the necessary number of shape matrices. This approach was considered in [15].

We finish this section with a remark concerning practical aspects of this result. Though the time complexity of the DC -minimization is polynomial in $m$ and $n$ the exponent grows linearly with $L$ and also the $L$-dependent constants that were used to estimate the time-complexities of the different steps of the algorithm, grow rapidly with $L$. So we expect an efficient algorithm only for very small $L$. In the proof of the polynomiality we constructed
the whole sets $\mathcal{V}_{\tau}(\tau=1, \ldots, t)$, i.e. we performed a BFS as described before Lemma 8. But in order to decide if there is a decomposition with the considered coefficients we need to know only if $\mathcal{V}_{t}$ is nonempty, and in order to reconstruct a decomposition basically one path from the unique element of $\mathcal{V}_{0}$ to some element of $\mathcal{V}_{t}$ is sufficient. So for practical purposes it is natural to use depth first search (DFS) instead of BFS.

## 4 Test results

We implemented the algorithm described above and Tables 1 and 2 show test results for random $10 \times 10-$ and $15 \times 15$-matrices, respectively. The computations where done on a 2 GHz workstation and we determined the minimal DC for optimal DT for 1000 randomly generated matrices with maximal entry $L$. The entry in column 'max. time' is the maximal time needed for one single matrix, and the entry in column 'total time' is the time needed for all the 1000 matrices. For comparison the tables also contain heuristic results that were obtained with a slightly improved version of the algorithm described in [8].

|  | exact |  |  | heuristic |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| L | DC | max. time | total time | DC | total time |
| 4 | 7.6 | 1 s | 13 s | 7.8 | 1.0 s |
| 7 | 8.8 | 50 s | 5.6 min | 9.3 | 1.2 s |
| 10 | 9.5 | 5.6 min | 41.3 min | 10.3 | 1.4 s |

Table 1: Average number of segments for random $10 \times 10$-matrices with maximal entry $L$. Each entry is averaged over 1000 matrices.

|  | exact |  |  | heuristic |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| L | DC | max. time | total time | DC | total time |
| 4 | 10.7 | 1 s | 31 s | 10.9 | 5.4 s |
| 7 | 12.3 | 6.5 min | 1.6 h | 13.0 | 6.8 s |
| 10 | 13.2 | 10.0 h | 44.7 h | 14.5 | 7.6 s |

Table 2: Average number of segments for random $15 \times 15$-matrices with maximal entry $L$. Each entry is averaged over 1000 matrices.

In order to evaluate the performance of the heuristic we determined the differences between the heuristic values and the exact minima. Tables 3 and 4 show the frequencies of the values of the differences when 1000 matrices

| L | 0 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- |
| 4 | 876 | 123 | 1 | 0 |
| 7 | 525 | 456 | 19 | 0 |
| 10 | 306 | 584 | 104 | 6 |

Table 3: Frequencies of the differences between the heuristic DC and the exact minimum for $10 \times 10-$ matrices.

| L | 0 | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 4 | 809 | 189 | 2 | 0 | 0 |
| 7 | 327 | 585 | 86 | 2 | 0 |
| 10 | 85 | 551 | 335 | 28 | 1 |

Table 4: Frequencies of the differences between the heuristic DC and the exact minimum for $15 \times 15$ matrices.
where treated for each value of $L$. We conclude that for the considered range of parameters the exact algorithm yields only small improvements in terms of the DC, while the computational effort is extremely high already for small values of $L$. So for practical purposes the heuristic seems to be a good compromise between computation time and accuracy of the optimization. Finally, we also tested our algorithm with 13 clinical matrices of size about $10 \times 10$ with 10 fluence levels. The results are shown in Table 5. In

|  |  | exact |  | heuristic |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| no. | size | DT | DC | CPU-time | DC | CPU-time |
| 1 | $10 \times 11$ | 16 | 7 | 0.04 s | 8 | 0.01 s |
| 2 | $10 \times 9$ | 16 | 7 | 0.19 s | 7 | 0.00 s |
| 3 | $9 \times 9$ | 20 | 8 | 0.39 s | 8 | 0.01 s |
| 4 | $9 \times 9$ | 19 | 7 | 0.04 s | 8 | 0.00 s |
| 5 | $10 \times 8$ | 15 | 7 | 0.01 s | 7 | 0.00 s |
| 6 | $9 \times 9$ | 17 | 8 | 0.70 s | 9 | 0.00 s |
| 7 | $10 \times 8$ | 18 | 7 | 0.03 s | 7 | 0.00 s |
| 8 | $14 \times 12$ | 22 | 9 | 1.30 s | 9 | 0.01 s |
| 9 | $14 \times 10$ | 26 | 9 | 25.77 s | 10 | 0.00 s |
| 10 | $14 \times 10$ | 22 | 8 | 0.62 s | 9 | 0.00 s |
| 11 | $15 \times 10$ | 22 | 10 | 7.88 s | 10 | 0.00 s |
| 12 | $15 \times 11$ | 23 | 9 | 1.96 s | 10 | 0.01 s |
| 13 | $14 \times 10$ | 23 | 9 | 2.36 s | 9 | 0.01 s |

Table 5: Test results for clinical matrices.
order to indicate how these clinical matrices look like, we include the matrix
corresponding to the first row of Table 5:

$$
A=\left(\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 2 & 9 & 8 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 4 & 0 & 2 & 4 & 0 & 0 \\
0 & 0 & 0 & 2 & 7 & 5 & 6 & 5 & 5 & 8 & 0 \\
0 & 0 & 0 & 2 & 4 & 6 & 4 & 4 & 2 & 4 & 1 \\
0 & 2 & 3 & 3 & 8 & 7 & 4 & 4 & 2 & 5 & 10 \\
0 & 3 & 3 & 4 & 4 & 7 & 5 & 6 & 3 & 7 & 9 \\
0 & 0 & 1 & 6 & 10 & 8 & 8 & 8 & 8 & 7 & 10 \\
0 & 0 & 4 & 6 & 4 & 4 & 3 & 2 & 6 & 10 & 9 \\
0 & 0 & 6 & 3 & 1 & 2 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

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