Maximal flat antichains of minimum weight

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Outline

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Notation

- B_n denotes the power set of $[n] := \{1, 2, \ldots, n\}$ ordered by inclusion.
- Let $\mathcal{A}$ denote an antichain in B_n.
- The *size* and the *volume* of $\mathcal{A}$ are

$$|\mathcal{A}| \qquad \text{and} \qquad v(\mathcal{A}) := \sum_{A \in \mathcal{A}} |A|.$$

- $\mathcal{A}$ is called *flat*, if $\mathcal{A} \subseteq \binom{[n]}{k} \cup \binom{[n]}{k+1}$ for some k.

The Flat Antichain Theorem

Theorem (Kisvölcsey, Lieby)

For every antichain $\mathcal{A}$ in B_n, there is a flat antichain $\mathcal{A}'$ with

$$|\mathcal{A}'| = |\mathcal{A}| \quad \textit{and} \quad v(\mathcal{A}') = v(\mathcal{A}).$$

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- Define an equivalence relation on the set of all antichains:

$$\mathcal{A} \sim \mathcal{B} \quad \Leftrightarrow \; |\mathcal{A}| = |\mathcal{B}| \text{ and } v(\mathcal{A}) = v(\mathcal{B}).$$

- There is a flat antichain in each equivalence class.

An extremal property of flat antichains

- weight function $w : \{0\} \cup [n] \to \mathbb{R}^+$
- weight of a family $\mathcal{F} \subseteq 2^{[n]}$,

$$w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w(|F|).$$

- (w_i) convex (concave) $\Rightarrow$ The flat AC have minimum (maximum) weight within their equivalence classes. (Griggs, Hartmann, Leck, Roberts)
- In particular the flat antichains have minimum LYM-value $\mathrm{LYM}(\mathcal{F}) = \sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1}$ within their classes.

The problem

- Let $1 < k < n$ and $w_k, w_{k+1} \in \mathbb{R}^+$ be given.
- What is the minimum weight

$$w(\mathcal{A}) = w_k|\mathcal{A}_k| + w_{k+1}|\mathcal{A}_{k+1}|$$

  of a maximal flat antichain $\mathcal{A} = \mathcal{A}_k \cup \mathcal{A}_{k+1}$, where $\mathcal{A}_i \subseteq \binom{[n]}{i}$?

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  of a maximal flat antichain $\mathcal{A} = \mathcal{A}_k \cup \mathcal{A}_{k+1}$, where $\mathcal{A}_i \subseteq \binom{[n]}{i}$?
- More general: Given $1 < k < l \leq n$ and $w_k, w_l \in \mathbb{R}^+$, what is the minimum weight

$$w(\mathcal{A}) = w_k|\mathcal{A}_k| + w_l|\mathcal{A}_l|$$

  of a maximal antichain $\mathcal{A} = \mathcal{A}_k \cup \mathcal{A}_l$?

Special cases

$$w(\mathcal{A}) = w_k|\mathcal{A}_k| + w_l|\mathcal{A}_l| \to \min$$

- size: $w_k = w_l = 1$
- volume: $w_k = k$, $w_l = l$
- LYM: $w_k = \binom{n}{k}^{-1}$, $w_l = \binom{n}{l}^{-1}$

Outline

A graph formulation

- With an antichain $\mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_3$ we associate a graph $G(\mathcal{A}) = (V, E)$:

$$V = [n], \qquad E = \binom{[n]}{2} \setminus \mathcal{A}_2.$$

- $\mathcal{A}$ is a maximal antichain iff every edge of $G(\mathcal{A})$ is contained in a triangle and $\mathcal{A}$ is the set of triangles in $G(\mathcal{A})$.
- Let T denote the set of triangles in $G(\mathcal{A})$.
- $w(\mathcal{A}) = w_3|T| + w_2\left(\binom{n}{2} - |E|\right) \to \min$

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- Let T denote the set of triangles in $G(\mathcal{A})$.
- $w(\mathcal{A}) = w_3|T| + w_2\left(\binom{n}{2} - |E|\right) \to \min$
- We divide by w_2 and put $\lambda := w_3/w_2$
- $|E| - \lambda|T| \to \max$ subject to the condition that every edge is contained in a triangle.
- We call the graphs satisfying this condition T-graphs.

An upper bound

Theorem

For any T-graph on n vertices and any $\lambda > 0$ we have

$$|E| - \lambda |T| \leq \frac{(n+\lambda)^2}{8\lambda}.$$

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Corollary

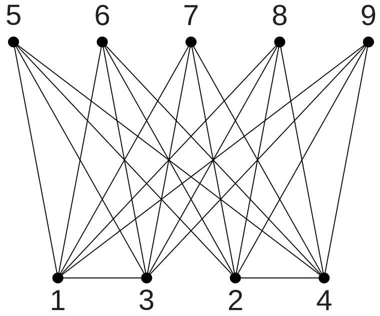
If $\mathcal{A} \subseteq \binom{[n]}{2} \cup \binom{[n]}{3}$ is a maximal antichain, then

$$w(\mathcal{A}) \geq \binom{n}{2} - \frac{(n+\lambda)^2}{8\lambda}.$$

A construction

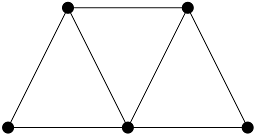
- The graph $K^+_{2s,n-2s}$:

$$E = ([2s] \times ([n] \setminus [2s])) \cup \{\{i, i+s\} : i = 1, 2, \ldots, s\},$$

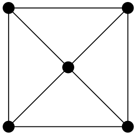


$K^+_{4,5}$

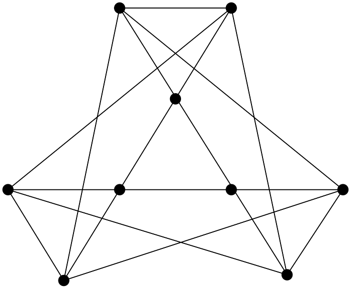
Exceptional cases for small n



G_{5a}



G_{5b}



G_9

Theorem

Let $\mathcal{A} \subseteq \binom{[n]}{2} \cup \binom{[n]}{3}$ be a maximal antichain. Then

$$|\mathcal{A}| \geq \binom{n}{2} - \left\lfloor \frac{(n+1)^2}{8} \right\rfloor,$$

and equality holds if and only if

$$\begin{array}{rlrl}
\text{(i)} & n \in \{5,9\} & \textit{and} & G(\mathcal{A}) \in \{G_{5a}, G_{5b}, G_9\}, \ \textit{or} \\
\text{(ii)} & n \equiv 0 \pmod 4 & \textit{and} & G(\mathcal{A}) \cong K^+_{n/2,n/2}, \ \textit{or} \\
\text{(iii)} & n \equiv 1 \pmod 4 & \textit{and} & G(\mathcal{A}) \cong K^+_{(n-1)/2,(n+1)/2} \\
 & & \textit{or} & G(\mathcal{A}) \cong K^+_{(n+3)/2,(n-3)/2}, \ \textit{or} \\
\text{(iv)} & n \equiv 2 \pmod 4 & \textit{and} & G(\mathcal{A}) \cong K^+_{(n+2)/2,(n-2)/2}, \ \textit{or} \\
\text{(v)} & n \equiv 3 \pmod 4 & \textit{and} & G(\mathcal{A}) \cong K^+_{(n+1)/2,(n-1)/2}.
\end{array}$$

Outline

The hypergraph formulation

- $\mathcal{A} \subseteq \binom{[n]}{k} \cup \binom{[n]}{l}$
- Similar to the $(2,3)$-case we are looking for a k-uniform hypergraph (k-graph) $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with the property that every edge is contained in some complete k-graph on l vertices. (Call these hypergraphs $(k,l)-$graphs.)
- Subject to this condition we have to maximize $e_k - \lambda e_l$, where e_k is the number of edges and e_l is the number of complete k-graphs on l vertices.
- $w(\mathcal{A}) = \binom{n}{k} - (e_k - \lambda e_l)$

The bound

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  - By deleting $o(n^k)$ edges we can obtain a hypergraph $\mathcal{H}' = (\mathcal{V}, \mathcal{E}')$ without complete k-graphs on l vertices.
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- $|\mathcal{E}'| \leq t_k(n, l)$
- For $e \in \mathcal{E}$ let $t(e)$ denote the number of complete k-graphs on l vertices containing e (in $\mathcal{H}$).
- $|\mathcal{E}'| \leq \sum_{e \in \mathcal{E}'} t(e) \leq \left(\binom{l}{k} - 1\right) e_l$

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- $|\mathcal{E}'| \leq \sum_{e \in \mathcal{E}'} t(e) \leq \left(\binom{l}{k} - 1\right) e_l$
- $e_k - e_l = |\mathcal{E}'| - e_l + o(n^k) \leq \frac{\binom{l}{k}-2}{\binom{l}{k}-1} t_k(n, l) + o(n^k)$

Numerical Examples

- $(k, l) = (3, 4)$:

$$e_3 - e_4 \leq \frac{2}{3} t_3(n, 4) + o(n^3) \leq \frac{2}{3} \frac{3 + \sqrt{17}}{12} \frac{1}{6} n^3 + o(n^3)$$

(Chung, Lu)

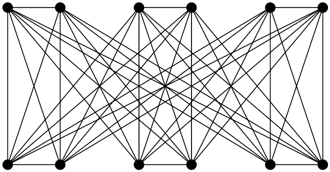
- $(k, l) = (2, 4)$:

$$e_2 - e_4 \leq \frac{4}{5} t_2(n, 4) + o(n^2) = \frac{4}{15} n^2 + o(n^2).$$

Outline

A construction for $(k, l) = (2, 4)$

- Assume $n = 4t$.
- $E = [1, 2t] \times [2t + 1, 4t] \cup \{(2i - 1, 2i) \ : \ i = 1, 2, \dots, 2t\}$



- $e_2 - e_4 = \frac{3}{16} n^2 + \frac{n}{2}$

Conjecture

In any $(2, 4)$-graph we have $e_2 - e_4 \leq \frac{3}{16} n^2 + o(n^2)$.

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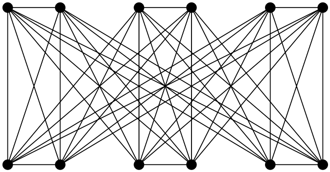

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Conjecture

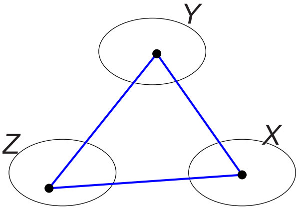
In any $(2, 4)$-graph we have $e_2 - e_4 \leq \frac{3}{16}n^2 + o(n^2)$.

- Under the additional assumption that the number of triangles is $o(n^3)$ the conjecture follows from the removal lemma.

A construction for $(k, l) = (3, 4)$

- Assume $n = 3t$ with $t \equiv 1$ or $3 \pmod 6$.
- Let S be a Steiner triple system on $\{1, \dots, t\}$.
- Vertex set $X \cup Y \cup Y$ with $X = \{x_1, \dots, x_t\}$, $Y = \{y_1, \dots, y_t\}$, $Z = \{z_1, \dots, z_t\}$.
- Triple system $T = T_1 \cup T_2 \cup T_3$ by

$$T_2 = \{x_i x_j y_k,\ y_i y_j z_k,\ z_i z_j x_k \ :\ i, j, k \in [t], i \neq j\},$$
$$T_3 = \{x_i x_j z_k,\ y_i y_j x_k,\ z_i z_j y_k \ :\ ijk \in S\}.$$



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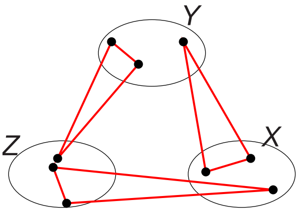

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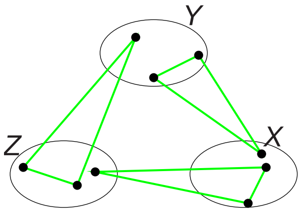

Lemma

T is a $(3,4)$-graph and $e_3 - e_4 = \frac{n^3}{27} + \frac{n^2}{18} - \frac{n}{2}$.

Conjecture

In any $(3,4)$-graph we have $e_3 - e_4 \leq \frac{n^3}{27} + o(n^3)$.

Outline

Problems

- upper bounds without the regularity lemma
- proof of the optimality of the construction in the $(2, 4)$-case
- constructions for the cases $(2, l)$ and $(k, k + 1)$