Multileaf Collimator Field Segmentation without Tongue-and-groove Effect

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INFORMS International Hong Kong / June 26, 2006

Outline

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A treatment couch

Intensity modulated Radiation therapy (IMRT)

- Goal: effective destruction of the tumor while maintaining the functionality of the healthy tissue.

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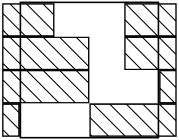
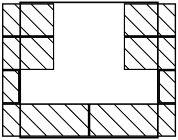
- Goal: effective destruction of the tumor while maintaining the functionality of the healthy tissue.
- A homogeneous field is emmitted from the linear accelerator.
- For a higher resolution in the irradiated area a modulation of the intensity is helpful.
- The modulation should be obtained by a relative simple but flexible technology.

A multileaf collimator

Modulation by superposition of homogeneous fields



2 MU



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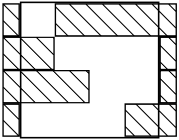

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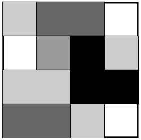
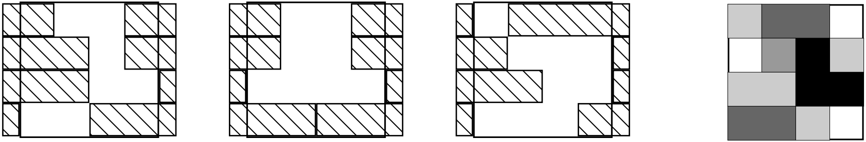


Modulation by superposition of homogeneous fields



$$2 \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 0 \\ 0 & 2 & 4 & 1 \\ 1 & 1 & 4 & 4 \\ 3 & 3 & 1 & 0 \end{pmatrix}$$

Outline

A fluence matrix

- Characterization of the required fluence distribution by an integer matrix.
- Example:

$$A = \begin{pmatrix} 4 & 5 & 0 & 1 & 4 & 5 \\ 2 & 4 & 1 & 3 & 1 & 4 \\ 2 & 3 & 2 & 1 & 2 & 4 \\ 5 & 3 & 3 & 2 & 5 & 3 \end{pmatrix}$$

The homogeneous fields

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A Segmentation

$$\begin{pmatrix} 4&5&0&1&4&5\\ 2&4&1&3&1&4\\ 2&3&2&1&2&4\\ 5&3&3&2&5&3 \end{pmatrix} = 3\begin{pmatrix} 0&0&0&0&1&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&1&1 \end{pmatrix} + 3\begin{pmatrix} 1&1&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$$

$$+1\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&1&1&1\\ 0&0&0&0&1&0\\ 0&1&1&1&1&0 \end{pmatrix} + 1\begin{pmatrix} 0&0&0&1&1&1\\ 0&0&0&1&0&0\\ 0&1&1&0&0&0\\ 1&1&1&0&0&0 \end{pmatrix} + 1\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&1&1&1&0 \end{pmatrix}$$

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The interleaf collision constraint (ICC)

- no overlapping of opposite leaves in consecutive rows

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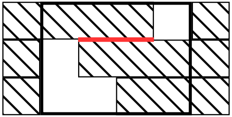
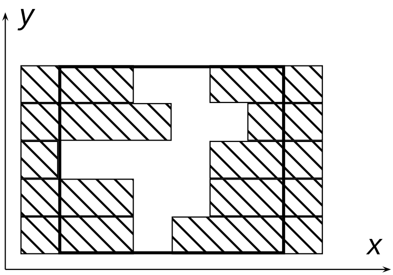


$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ is not a segment.}$$

The Tongue-and-Groove design



TG-underdosage

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

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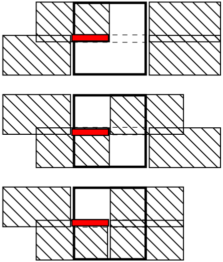

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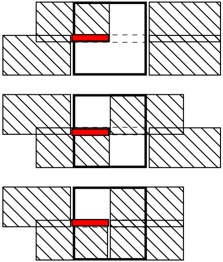

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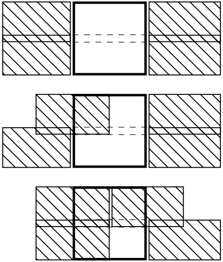

The Tongue-and-groove constraint (TGC)

$$a_{i,j} \leq a_{i+1,j} \wedge s_{i,j} = 1 \Rightarrow \; s_{i+1,j} = 1 \; (i \in [m-1], \; j \in [n]),$$
$$a_{i,j} \leq a_{i-1,j} \wedge s_{i,j} = 1 \Rightarrow \; s_{i-1,j} = 1 \; (i \in [2,m], \; j \in [n]).$$

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- The overlap between (i,j) and $(i+1,j)$ receives a fluence of $\min\{a_{i,j}, a_{i+1,j}\}$.

Definition. A-segment

An *A–segment* is an $m \times n$-matrix $S = (s_{i,j})$ with entries from $\{0, 1\}$, such that there exist integers $l_i,\ r_i$ $(i \in [m])$ with:

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The segmentation problem

- Input: a nonnegative integer $m \times n$–matrix A.
- Output: A segmentation $A = \sum_{i=1}^{k} \lambda_i S_i$ with
  - small total irradiation time, $TNMU := \sum_{i=1}^{k} \lambda_i \;\longrightarrow\; \min$
  - a small number of segments, $k \;\longrightarrow\; \min$

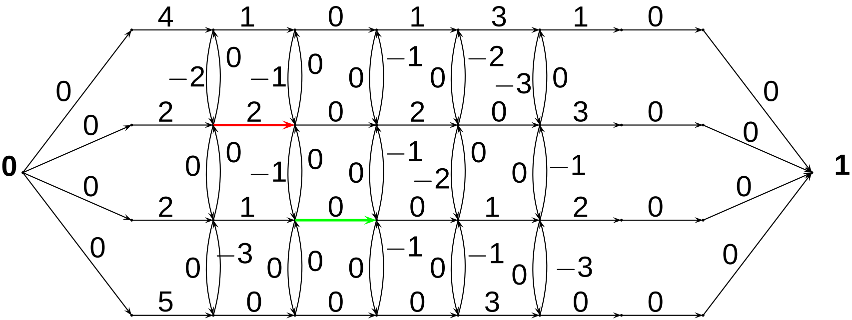
The segmentation problem

$$
(P)\left\{
\begin{aligned}
\sum_{S} \lambda_S \quad &\to \min \quad && \text{subject to} \\
\lambda_S \quad &\geq 0 && \forall \text{ segments } S, \\
\sum_{S:s_{i,j}=1} \lambda_S \quad &= a_{i,j} && \forall (i,j) \in [m] \times [n].
\end{aligned}
\right.
$$

The dual problem

$$(D)\begin{cases} \displaystyle\sum_{(i,j)\in[m]\times[n]} a_{i,j}g(i,j) & \to \max \quad \text{subject to} \\ \displaystyle\sum_{(i,j):s_{i,j}=1} g(i,j) & \leq 1 \qquad \forall \text{ segments } S. \end{cases}$$

The segmentation graph



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$$w((i,j-1),(i,j)) = \max\{0, a_{i,j} - a_{i,j-1}\}$$

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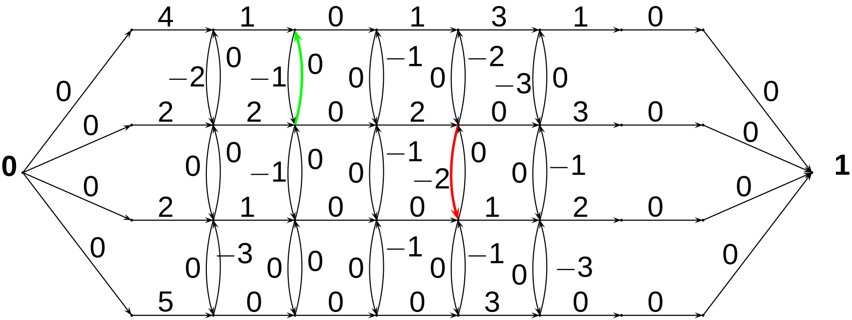

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$$w((i,j),(i \pm 1, j)) = \min\{0, a_{i\pm 1,j} - a_{i,j}\}$$

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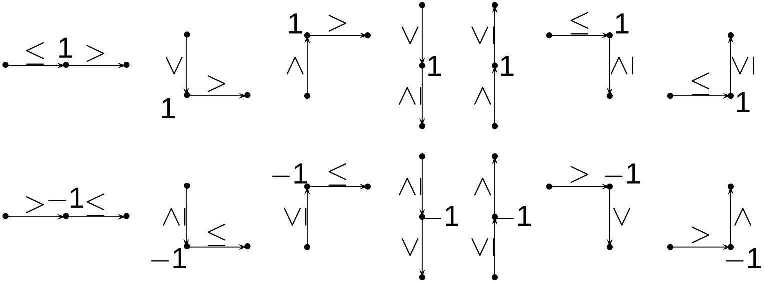
Theorem

The minimal TNMU of a segmentation of a nonnegative matrix A equals $c(A)$, the maximal weight of a $(\mathbf{0},\mathbf{1})-$path in G.

Outline

A dual feasible solution

- With each path P we associate a dual feasible solution.



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$$\sum_{(i,j)\in[m]\times[n]} g(i,j)a_{i,j} = w(P)$$

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Outline of the Algorithm

$A^{(0)} := A$, $k := 0$.
while $A^{(k)} \neq 0$ **do**
  $k := k + 1$
  Determine an $A^{(k-1)}-$segment $S^{(k)}$ such that
  $A^{(k)} := A^{(k-1)} - S^{(k)}$ is nonnegative and
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- Let's assume for the moment that it is always possible to determine $S^{(k)}$ with the required properties.

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Every $A^{(k)}-$segment ($k = 0, 1, 2, \ldots$) is also an A-segment.

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- Let S be any $A^{(k)}-$segment ($k \geq 1$).

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- $s_{i,j} = 1 \ \Rightarrow \ s_{i\pm1,j} = 1$

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Theorem

The algorithm yields only $A-$segments.

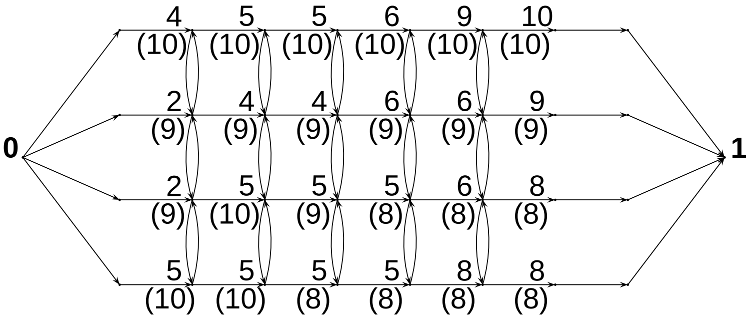
The construction of segment $S^{(k)}$

- Let w_{k-1} denote the weight function on the arcs of G with respect to $A^{(k-1)}$.
- We put

$$\begin{aligned}
\alpha_1^{(k-1)}(i,j) &= \max\{w_{k-1}(P) \,:\, P \text{ is a } (0,(i,j))-\text{path in } G\},\\
\alpha_2^{(k-1)}(i,j) &= \max\{w_{k-1}(P) \,:\, P \text{ is a } ((i,j),1)-\text{path in } G\},\\
\alpha^{(k-1)}(i,j) &= \alpha_1^{(k-1)}(i,j) + \alpha_2^{(k-1)}(i,j).
\end{aligned}$$

- Define $S^{(k)}$ by

$$s_{i,j}^{(k)} = \begin{cases} 1 & \text{if } a_{i,j}^{(k-1)} > 0,\ \alpha^{(k-1)}(i,j) = c\left(A^{(k-1)}\right) \text{ and} \\ & \alpha_1^{(k-1)}(i,j) = a_{i,j}^{(k-1)}, \\ 0 & \text{otherwise.} \end{cases}$$



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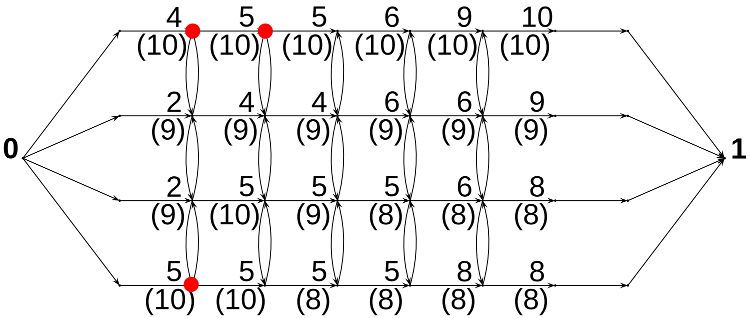

$$A = \begin{pmatrix} 4 & 5 & 0 & 1 & 4 & 5 \\ 2 & 4 & 1 & 3 & 1 & 4 \\ 2 & 3 & 2 & 1 & 2 & 4 \\ 5 & 3 & 3 & 2 & 5 & 3 \end{pmatrix} \quad \Rightarrow \quad S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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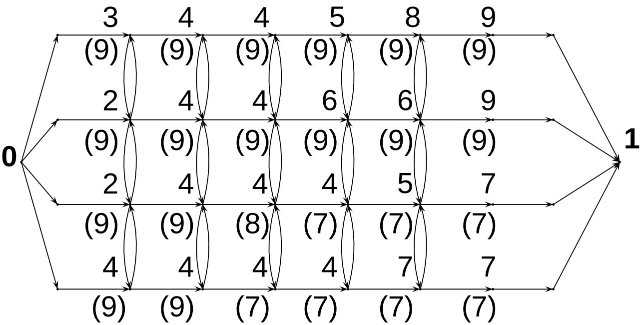

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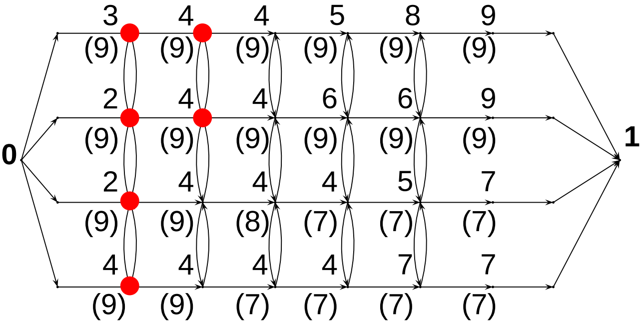

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Example

- Continuing this way we obtain the segmentation

$$\begin{pmatrix} 4&5&0&1&4&5\\ 2&4&1&3&1&4\\ 2&3&2&1&2&4\\ 5&3&3&2&5&3 \end{pmatrix} = \begin{pmatrix} 1&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 1&1&0&0&0&0\\ 1&1&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 1&1&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&0&0&0&0 \end{pmatrix}$$

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The algorithm works

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In every step the described method yields an $A^{(k-1)}$-segment $S^{(k)}$ with $c(A^{(k-1)} - S^{(k)}) = c(A^{(k-1)}) - 1$.

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- *Remark.* The resulting segmentation is unidirectional, i.e. the leaves move only from left to right.

Heuristic for the number of segments

- while ($A \neq 0$)
  - Determine a pair (u, S) such that
    - $A' := A - uS$ is nonnegative,
    - $s_{i,j} = 1 \wedge s_{i+1,j} = 0 \quad \Rightarrow \quad a_{i,j} \geq a_{i+1,j} + u$,
    - $s_{i,j} = 1 \wedge s_{i-1,j} = 0 \quad \Rightarrow \quad a_{i,j} \geq a_{i-1,j} + u$.

Outline

- $15 \times 15-$matrices with random entries from $\{0, 1, \ldots, L\}$, ($L = 3, \ldots, 16$)
- computation time
  - pure TNMU-minimization: few seconds for 1000 segmentations

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  - with heuristic for the number of segments:
    - 15 minutes for 1000 matrices
    - maximal time for a single matrix 13 seconds

Test results on random matrices

L	TNMU	segments
4	21.2	18.0
6	30.3	22.6
8	39.2	25.7
10	48.2	28.3
12	57.2	30.5
14	66.0	32.2
16	74.8	33.9

Table: Results with elimination of Tongue-and-groove underdosage.

L	TNMU	segments
4	19.5	14.5
6	27.6	17.2
8	35.7	19.1
10	43.8	20.7
12	51.8	21.9
14	59.8	23.0
16	67.7	24.0

Table: Results without elimination of Tongue-and-groove underdosage.

Outline

- The TNMU–problem is solved in the cases
  - without ICC, without TGC
  - with ICC, without TGC
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- The case without ICC, with TGC is open and of increasing importance from a practical point of view.
- Better methods for the minimization of the number of segments are needed.

Thank you for your attention!