Sequential allocation of indivisible goods

Thomas Kalinowski

Institut für Mathematik, Universität Rostock

Newcastle
Tuesday, January 22, 2013

joint work with . . .

Nina Narodytska

Toby Walsh

NICTA and UNSW in Sydney

Outline

Sequential allocation

A simple example

Suppose you are coaching a football team and you want to divide your players into two teams for a practice match.

Sequential allocation

A simple example

Suppose you are coaching a football team and you want to divide your players into two teams for a practice match.

- Nominate two captains and let them take turns in picking team members

Sequential allocation

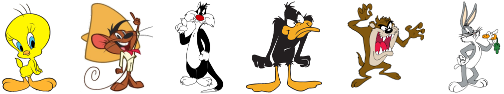
A simple example

Suppose you are coaching a football team and you want to divide your players into two teams for a practice match.

- Nominate two captains and let them take turns in picking team members
- What is the best picking order?
  - alternating: $1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2$
  - alternating and reversing: $1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1$
  - ???

Example: Alternating policy

- Captain 1



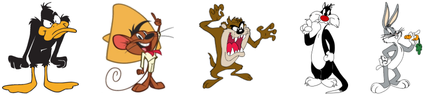
- Captain 2

121212

Example: Alternating policy

- Captain 1

- Captain 2



1**2**1212

Example: Alternating policy

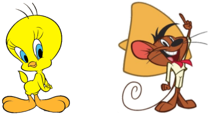
- Captain 1

- Captain 2

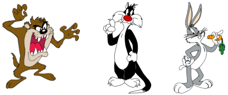
12**1**212

Example: Alternating policy

- Captain 1



- Captain 2



121**2**12

Example: Alternating policy


- Captain 1



- Captain 2


1212**1**2

Example: Alternating policy

- Captain 1

- Captain 2

12121**2**

Example: Alternating policy

- Captain 1

- Captain 2

121212

Example: Alternating and reversing policy

- Captain 1

- Captain 2

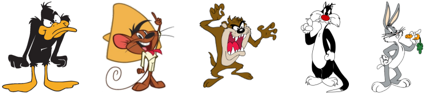
     

122112

Example: Alternating and reversing policy

- Captain 1

- Captain 2



1**2**2112

Example: Alternating and reversing policy

- Captain 1

- Captain 2

12**2**112

Example: Alternating and reversing policy

- Captain 1

- Captain 2

122**1**12

Example: Alternating and reversing policy


- Captain 1



- Captain 2


1222**1**2

Example: Alternating and reversing policy

- Captain 1

- Captain 2

12211**2**

Example: Alternating and reversing policy

- Captain 1

- Captain 2

122112

The order makes a difference

Preference orders

- Captain 1 
- Captain 2 

Alternating

- Captain 1 
- Captain 2 

Alternating and reversing

- Captain 1 
- Captain 2 

Background

- How do we **best** share resources between competing agents?
- **Best** can mean different things (fair, efficient, . . .)
- Resources can be
  - divisible (mineral rights, viewing times, etc.) or
  - indivisible (machines, holiday slots, time slots for landing and take-off, etc.)

Background

- How do we **best** share resources between competing agents?
- **Best** can mean different things (fair, efficient, ...)
- Resources can be
  - divisible (mineral rights, viewing times, etc.) or
  - indivisible (machines, holiday slots, time slots for landing and take-off, etc.)

- The allocation of scarce resources is an abundant problem in many economic and social contexts, in engineering, algorithm design, etc.
- Therefore, it is of great interest to
  - theoretically understand the related phenomena, and
  - develop good allocation mechanisms.

Cake-cutting (fair division)

Cut-and-choose

- Dividing a cake between two persons
- The first person cuts the cake into two parts
- The second person chooses which part to take

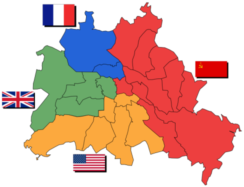
Cake-cutting (fair division)

Cut-and-choose

- Dividing a cake between two persons
- The first person cuts the cake into two parts
- The second person chooses which part to take

More agents

- Different solutions depending on fairness notion
  - Banach, Knaster, Steinhaus 1947
  - Selfridge; Conway 1960
  - Brams, Taylor 1995



From fair division to social welfare maximization

- To compare division mechanisms the agent's shares have to be evaluated using a *utility* function.
- *Fair* division usually tries to balance utilities: Every agent should be satisfied with the outcome.
- Game theory studies the effect of strategic decision making.

A different aspect

A central agency that manages the allocation process might be interested in maximizing a global quality measure, while the opinions of individual agents might be irrelevant.

From fair division to social welfare maximization

- To compare division mechanisms the agent's shares have to be evaluated using a *utility* function.
- *Fair* division usually tries to balance utilities: Every agent should be satisfied with the outcome.
- Game theory studies the effect of strategic decision making.

A different aspect

A central agency that manages the allocation process might be interested in maximizing a global quality measure, while the opinions of individual agents might be irrelevant.

Problem [Bouveret, Lang 2011]

Maximize the *social welfare* over a class of allocation mechanisms.

Formal setup

- n agents compete for k items

Preference order

Permutation π of the set $[k] = \{1, \ldots, k\}$

Formal setup

- n agents compete for k items

Preference order

Permutation π of the set $[k] = \{1, \ldots, k\}$

Preference profile

n-tuple $R = (\pi_1, \ldots, \pi_n)$ of preference orders

Formal setup

- n agents compete for k items

Preference order

Permutation π of the set $[k] = \{1, \ldots, k\}$

Preference profile

n-tuple $R = (\pi_1, \ldots, \pi_n)$ of preference orders

Utilities

Values $k, k-1, k-2, \ldots, 1$

Formal setup

- n agents compete for k items

Preference order

Permutation π of the set $[k] = \{1, \ldots, k\}$

Preference profile

n-tuple $R = (\pi_1, \ldots, \pi_n)$ of preference orders

Utilities

Values $k, k-1, k-2, \ldots, 1$

Additivity assumption

The utility of a subset $A \subseteq [k]$ is the sum of the utilities of the elements of A.

Example for $n = 2$, $k = 6$, alternating

Available items

1–  2 –  3 –  4 –  5 –  6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$$

Example for $n = 2$, $k = 6$, alternating

Available items

1–  2 –  3 –  4 –  5 –  6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$$

Allocation

Agent 1 Agent 2

Utilities

Agent 1:

Agent 2:

Example for $n = 2$, $k = 6$, alternating

Available items

1–  2 –  3 –  4 –  5 –  6 – 

Profile

$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$

Allocation

Agent 1 

Agent 2

Utilities

Agent 1: 6

Agent 2:

Example for $n = 2$, $k = 6$, alternating

Available items

2 –  3 –  4 –  5 –  6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$$

Allocation

Agent 1 

Agent 2 

Utilities

Agent 1: 6

Agent 2: 5

Example for $n = 2$, $k = 6$, alternating

Available items

2 –  3 –  5 –  6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$$

Allocation

Agent 1  

Agent 2 

Utilities

Agent 1: $6 + 5$

Agent 2: 5

Example for $n = 2$, $k = 6$, alternating

Available items

3 –  5 –  6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$$

Allocation

Agent 1  

Agent 2  

Utilities

Agent 1: $6 + 5$

Agent 2: $5 + 3$

Example for $n = 2$, $k = 6$, alternating

Available items

3 –    6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$$

Allocation

Agent 1   

Agent 2  

Utilities

Agent 1: $6 + 5 + 4 = 15$

Agent 2: $5 + 3$

Example for $n = 2$, $k = 6$, alternating

Available items

6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 5, 3, 6)$$

Allocation

Agent 1   

Agent 2   

Utilities

Agent 1: $6 + 5 + 4 = 15$

Agent 2: $5 + 3 + 1 = 9$

$\Longrightarrow$ social welfare $15 + 9 = 24$

Example for $n = 2$, $k = 6$, alternating and reversing

Available items

6 – 

Profile

$$(1, 2, 3, 4, 5, 6), \qquad (1, 4, 2, 3, 5, 6)$$

Allocation

Agent 1   

Agent 2   

Utilities

Agent 1: $6 + 4 + 2 = 12$

Agent 2: $5 + 4 + 1 = 10$

$\Longrightarrow$ social welfare $12 + 10 = 22$

Allocation policies

Policy

$p = p_1 \dots p_k \in [n]^k$ In step i agent p_i picks an item.

Truthful behaviour

Among the available items, the agent always picks the best according to her ranking.

Individual utilities

$u_i(R, p)$ – Utility of agent i for profile R and policy p

Social welfare

$$\mathrm{sw}(R, p) = \sum_{i=1}^{n} u_i(R, p)$$

Problem formulation

For a given probability P on the set $\mathcal{R}$ of all profiles we consider

Expected utilities and social welfare

$$\overline{u}_i(p) = \sum_{R \in \mathcal{R}} P(R) u_i(R, p) \quad \text{and} \quad \overline{\mathrm{sw}}(p) = \sum_{R \in \mathcal{R}} P(R)\, \mathrm{sw}(R, p)$$

- Linearity of expectation: $\overline{\mathrm{sw}}(p) = \sum_{i=1}^{n} \overline{u}_i(p)$.
- Here P is always the uniform distribution on $\mathcal{R}$.

Conjecture [Bouveret & Lang 2011]

The expected social welfare is maximized by the alternating policy

$$p = 12 \ldots (n-1)n\; 12 \ldots (n-1)n \;\ldots\ldots 12 \ldots (n-1)n \;\ldots$$

Main results

Theorem (K,Narodytska,Walsh 2013+)

The expected utilities $\overline{u}_i(p)$ can be computed in linear time.

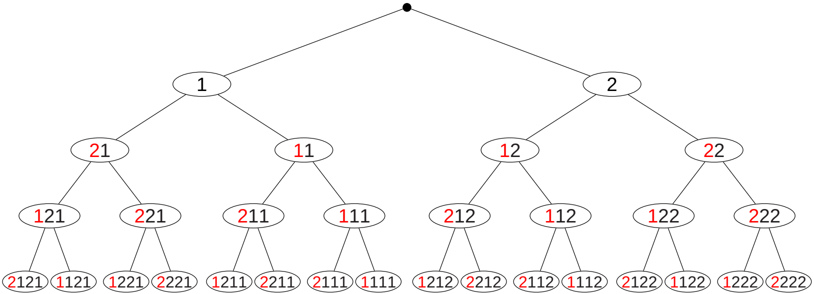
Theorem (K,Narodytska,Walsh 2013+)

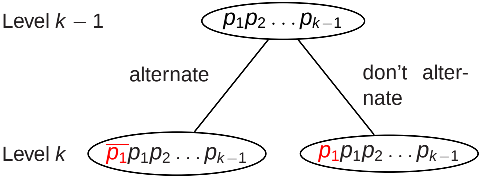
For a linear utility function and $n = 2$ agents the expected social welfare is maximized by the alternating policy $p = 121212\ldots$

Theorem (K,Narodytska,Walsh 2013+)

For Borda utility and n agents the expected social welfare is $\dfrac{nk^2}{n+1} + O(k)$ and this is asymptotically optimal.

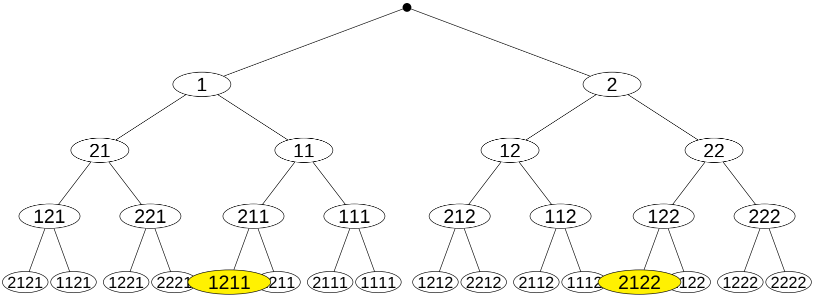
The policy tree





$$\overline{1} = 2, \overline{2} = 1$$

The policy tree

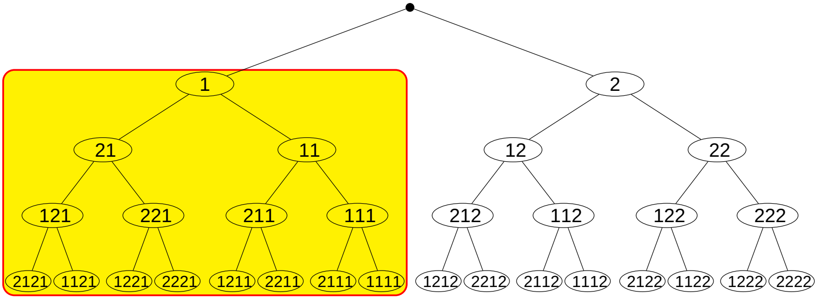


Reducing symmetry

$$\overline{u}_1(1211) = \overline{u}_2(2122), \qquad \overline{u}_2(1211) = \overline{u}_1(2122).$$

$$\Longrightarrow \overline{\mathrm{sw}}(1211) = \overline{\mathrm{sw}}(2122)$$

The policy tree



Reducing symmetry

$$\overline{u}_1(1211) = \overline{u}_2(2122), \qquad \overline{u}_2(1211) = \overline{u}_1(2122).$$

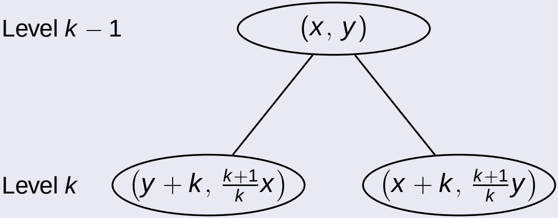
$$\Longrightarrow \overline{\mathrm{sw}}(1211) = \overline{\mathrm{sw}}(2122)$$

$\Longrightarrow$ It is sufficient to consider the left subtree.

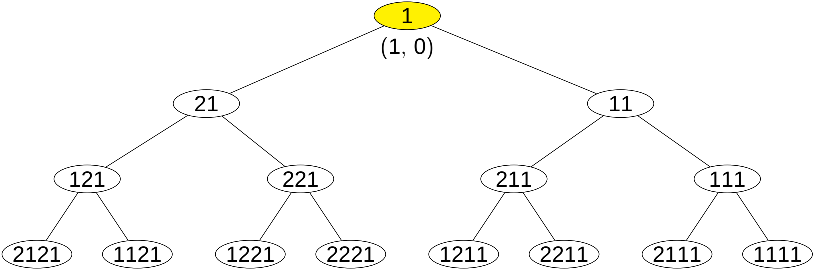
Recursive computation of the expected utilities

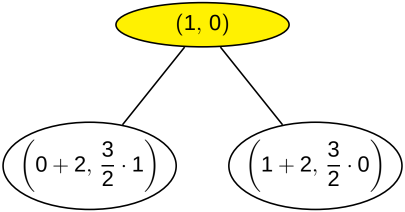
- With a policy we associate a pair (x, y) where
  - x is the expected utility for the starting agent,
  - y is the expected utility for the other agent.
- The root node ($k = 1$): $(1, 0)$

The recursion for $k \geqslant 2$

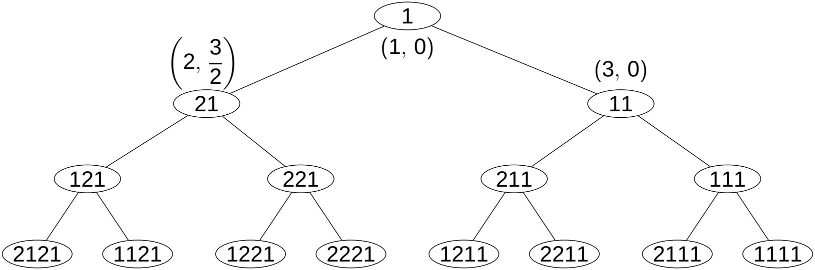


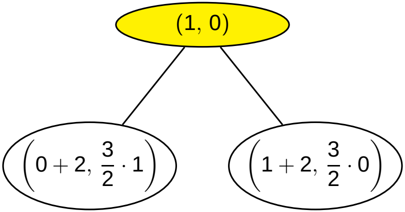
Utilities for the first four levels



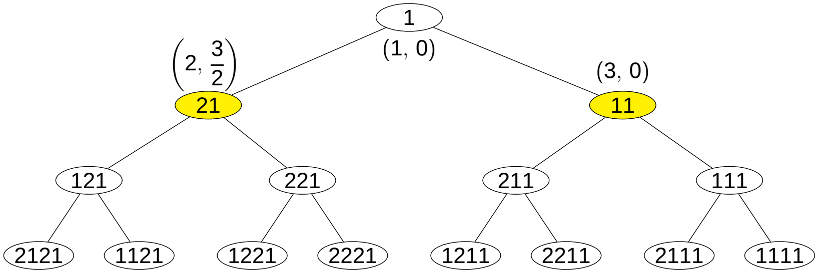


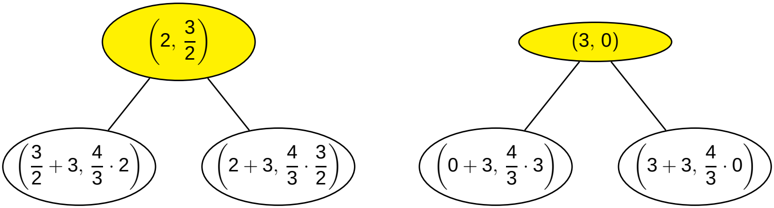
Utilities for the first four levels



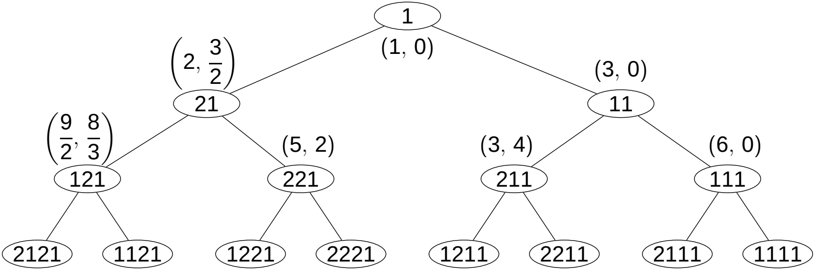


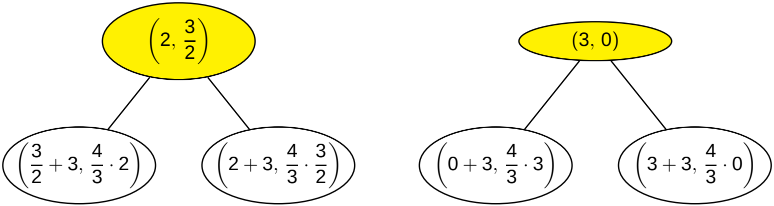
Utilities for the first four levels



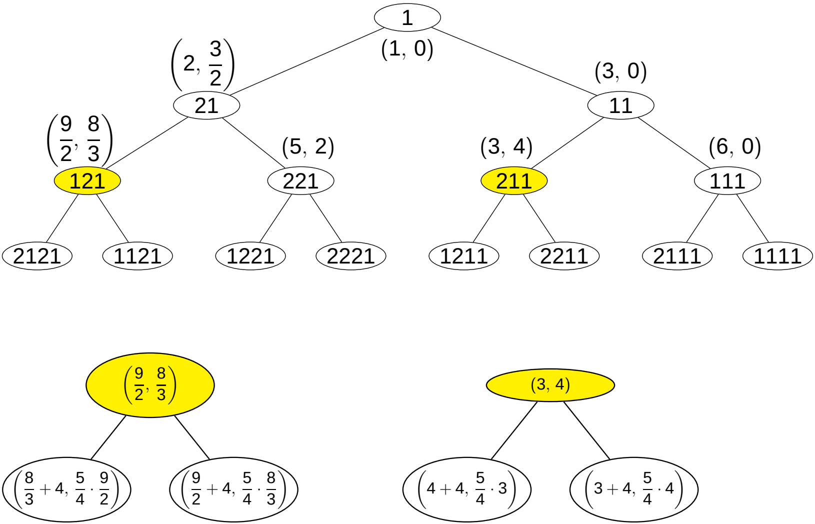


Utilities for the first four levels

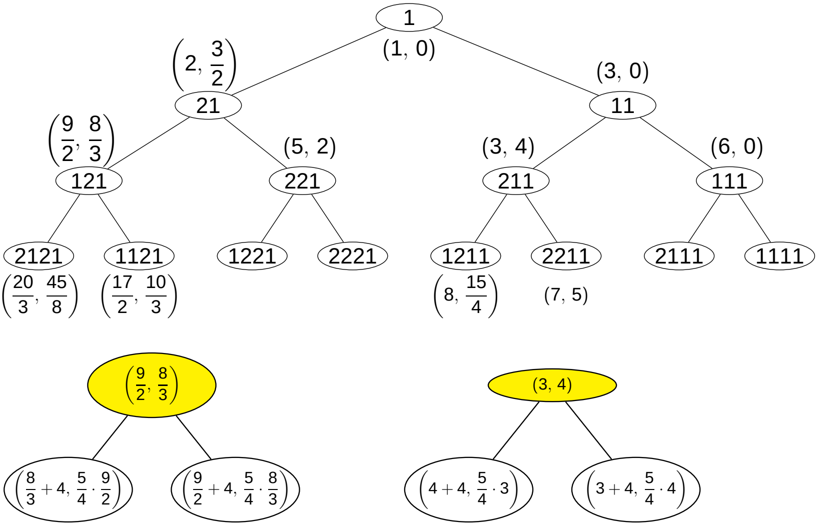




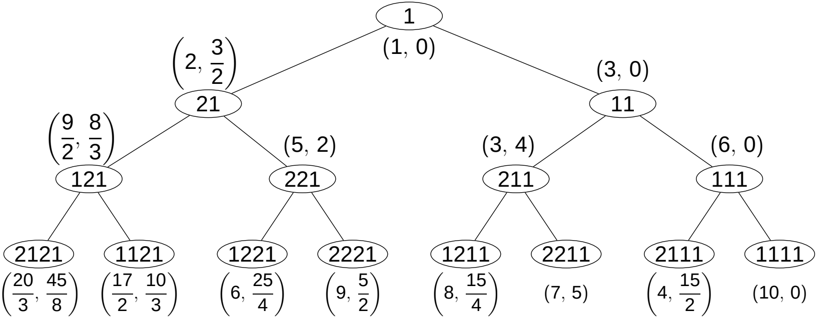
Utilities for the first four levels

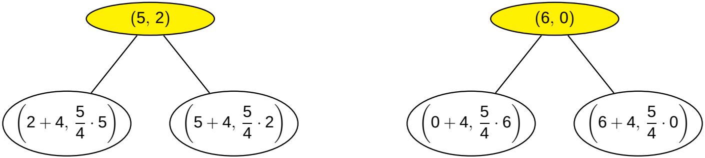


Utilities for the first four levels



Utilities for the first four levels





Solution for the alternating policy

- Let $\widehat{p}^k$ denote the alternating policy of length k, $\widehat{p}^k = 1212\ldots$

Theorem (K,Narodytska,Walsh 2013+)

The expected social welfare for the alternating policy is

$$\overline{\mathrm{sw}}\left(\widehat{p}^k\right) = \frac{k(2k+1)}{3} + O(\sqrt{k}).$$

The expected utility difference between the agents is

$$d_k := \overline{u}_1\left(\widehat{p}^k\right) - \overline{u}_2\left(\widehat{p}^k\right) = \frac{k}{3} + O(\sqrt{k}).$$

Reformulation

Defect pairs

For a policy p we measure the deviation from $\widehat{p}^k$ by the pair

$$(x_p, y_p) = \left(\overline{u}_1(p) - \overline{u}_1\left(\widehat{p}^k\right), \overline{u}_2(p) - \overline{u}_2\left(\widehat{p}^k\right)\right)$$

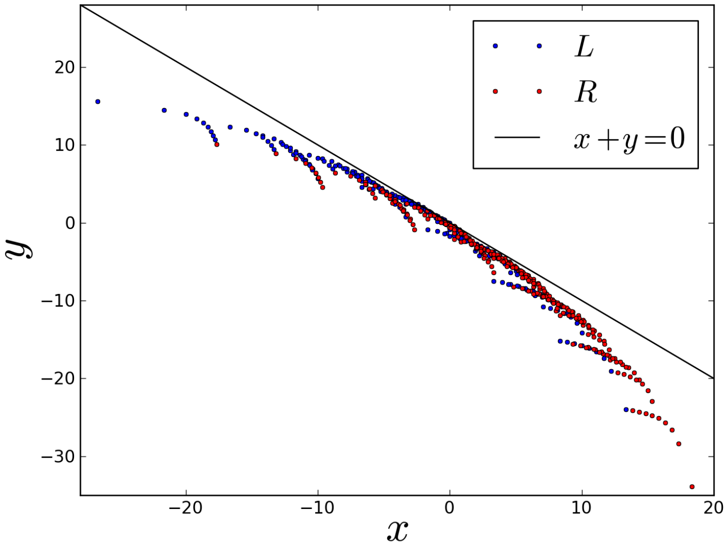
- $\overline{\mathrm{sw}}(p) - \overline{\mathrm{sw}}(\widehat{p}^k) = x_p + y_p$
- The optimality of $\widehat{p}^k$ for all k is equivalent to

Theorem

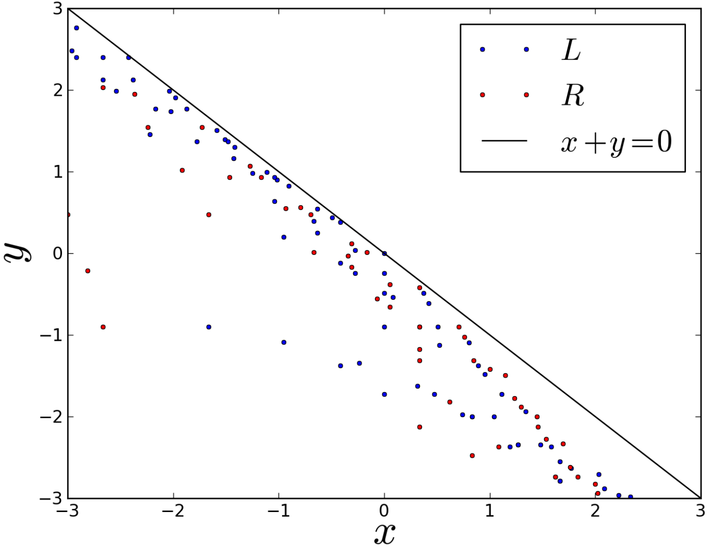
For all $k \geqslant 1$, if (x, y) is the defect pair for a policy of length k then

$$x + y \leqslant 0.$$

Defect pairs for $k = 10$



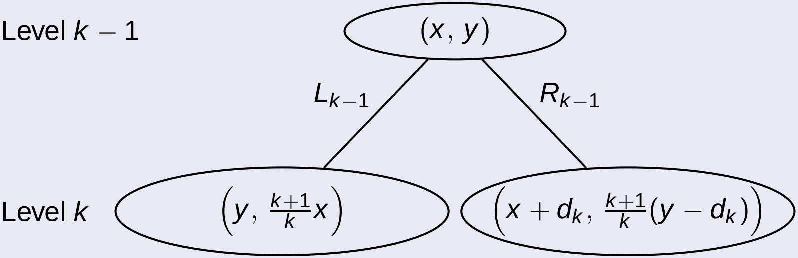
“Small” defect pairs for $k = 10$



Recursion for defect pairs

- $(0, 0)$ is the only defect pair for $k = 1$.

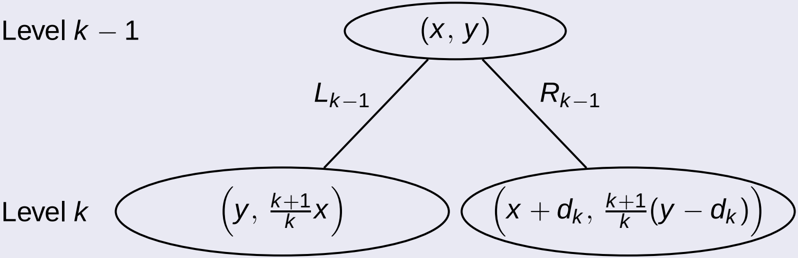
Recursion for $k \geqslant 2$



Recursion for defect pairs

- $(0, 0)$ is the only defect pair for $k = 1$.

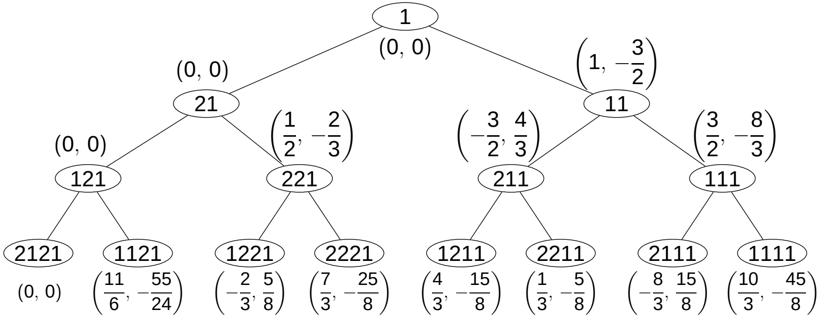
Recursion for $k \geqslant 2$



where d_k is the utility difference for the alternating policy:

$$d_k = \overline{u}_1\left(\widehat{p}^k\right) - \overline{u}_2\left(\widehat{p}^k\right).$$

Defect pairs in the policy tree



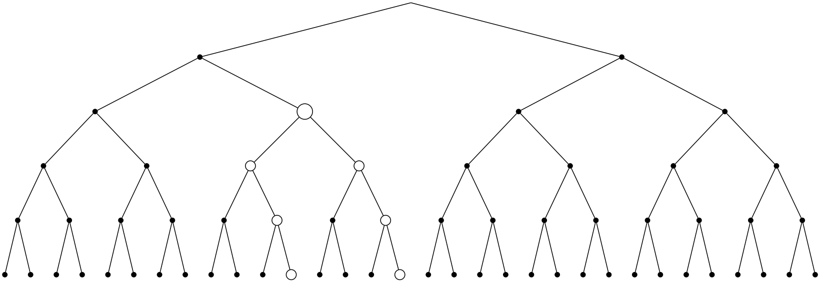
Proof strategy

Prove by induction on k the following statement:

Proposition

If (x, y) is a defect pair for a policy of length k then

1. $x + y \leqslant 0$,
2. $\forall m \geqslant 1 : \ (x', y') = R_{k+m-1} \circ \cdots \circ R_k(x, y) \implies x' + y' \leqslant 0$,
3. $\forall m \geqslant 1 : \ (x', y') = R_{k+m-1} \circ \cdots \circ R_{k+1} \circ L_k(x, y) \implies x' + y' \leqslant 0$.



Outlook

- Generalize the optimality result for the alternating policy to
  - more than two agents,
  - convex utility functions, i.e. the utility difference between consecutive items decreases with the rank,
  - different probability distributions on the set of profiles.
- Study different social welfare measures.
- What happens if agents behave strategically?