Complete subgraphs in multipartite graphs

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Abstract

Turán's Theorem states that every graph G of edge density $\|G\|/\binom{|G|}{2} > \frac{k-2}{k-1}$ contains a complete graph K^k and describes the unique extremal graphs. We give a similar Theorem for ℓ-partite graphs. For large ℓ, we find the minimal edge density d_ℓ^k, such that every ℓ-partite graph whose parts have pairwise edge density greater than d_ℓ^k contains a K^k. It turns out that $d_\ell^k = \frac{k-2}{k-1}$ for large enough ℓ. We also describe the structure of the extremal graphs.

1 Introduction and Notation

All graphs in this note are simple and undirected, and we follow the notation of [3]. In particular, K^k is the complete graph on k vertices, $|G|$ stands for the number of vertices and $\|G\|$ denotes the number of edges in G with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, let $N(x)$ be the set of vertices adjacent to x, and let $d(x) := |N(x)|$ be the degree of the vertex. For sets $X, Y \subseteq V(G)$, let $G[X]$ be the graph on X induced by G, $E(X)$ be the edge set of $G[X]$ and $E(X,Y)$ be the set of edges from X to Y.

Let G be an ℓ-partite graph on finite non-empty independent sets $V_1, V_2, \dots V_\ell$. For $X \subseteq V(G)$, we write $X_i := X \cap V_i$. For $i \neq j$, the density between V_i and V_j is defined as

$$d_{ij} := d(V_i, V_j) := \frac{\|G[V_i \cup V_j]\|}{|V_i| \cdot |V_j|}.$$

For a graph H with $|H| \geq \ell$, let $d_\ell(H)$ be the minimum number such that every ℓ-partite graph with $\min d_{ij} > d_\ell(H)$ contains a copy of H. Clearly, $d_\ell(H)$ is monotone decreasing in ℓ. In [2], Bondy et al. study the quantity $d_\ell(H)$, and in particular $d_\ell^3 := d_\ell(K^3)$, i.e. the values for the complete graph on three vertices, the triangle. Their main results about triangles can be written as follows.

Theorem 1. *[2]*

1. $d_3^3 = \tau \approx 0.618$, *the golden ratio, and*
2. d_ω^3 *exists and* $d_\omega^3 = \frac{1}{2}$.

Here, d_ω^3 stands for the corresponding value for graphs with a (countably) infinite number of finite parts. They go on and show that $d_4^3 \geq 0.51$ and speculate that $d_\ell^3 > \frac{1}{2}$ for all finite ℓ. We will show that this speculation is false. In fact, $d_\ell^3 = \frac{1}{2}$ for $\ell \geq 12$ as we will prove in Section 3. In Section 4, we will extend the main proof ideas to show that $d_\ell^k := d_\ell(K^k) = \frac{k-2}{k-1}$ for large enough ℓ.

In order to state our results, we need to define classes $\mathcal{G}_\ell^k$ of extremal graphs. We will do this properly in Section 2. Our main result is the following theorem.

Theorem 2. *Let $k \geq 2$, let ℓ be large enough and let $G = (V_1 \cup V_2 \cup \ldots \cup V_\ell, E)$ be an ℓ-partite graph, such that the pairwise edge densities*

$$d(V_i, V_j) := \frac{\|G[V_i \cup V_j]\|}{|V_i| \cdot |V_j|} \geq \frac{k-2}{k-1} \text{ for } i \neq j.$$

Then G contains a K^k or G is isomorphic to a graph in $\mathcal{G}_\ell^k$.

Corollary 3. *For ℓ large enough, $d_\ell^k = \frac{k-2}{k-1}$.*

The bound on ℓ one may get out of the proof is fairly large, and we think that the true bound is much smaller. For triangles ($k = 3$), we can give a reasonable bound on ℓ. We think that this bound is not sharp, either. We conjecture that $\ell \geq 5$ turns out to be sufficient.

Theorem 4. *Let $\ell \geq 12$ and let $G = (V_1 \cup V_2 \cup \ldots \cup V_\ell, E)$ be an ℓ-partite graph, such that the pairwise edge densities*

$$d(V_i, V_j) := \frac{\|G[V_i \cup V_j]\|}{|V_i| \cdot |V_j|} \geq \frac{1}{2} \text{ for } i \neq j.$$

Then G contains a triangle or G is isomorphic to a graph in $\mathcal{G}_\ell^3$.

Corollary 5. $d_{12}^3 = \frac{1}{2}$.

2 Extremal graphs

For $\ell \geq (k-1)!$, a graph G is in $\bar{\mathcal{G}}_\ell^k$, if it can be constructed as follows. For a sketch, see the figure below. Let $\{\pi_1, \pi_2, \ldots, \pi_{(k-1)!}\}$ be the set of all permutations of the set $\{1, \ldots, k-1\}$. For $1 \leq i \leq \ell$ and $1 \leq s \leq k-1$, pick integers n_i^s such that

$$\begin{aligned} n_i^{\pi_i(1)} \geq n_i^{\pi_i(2)} \geq \ldots \geq n_i^{\pi_i(k-1)} &\text{ for } 1 \leq i \leq (k-1)!, \\ n_i^1 = n_i^2 = \ldots = n_i^{k-1} &\text{ for } (k-1)! < i \leq \ell, \text{ and} \\ \sum_s n_i^s > 0 &\text{ for } 1 \leq i \leq \ell. \end{aligned}$$

Let

$$\begin{aligned} V(G) &= \{(i, s, t) : 1 \leq i \leq \ell,\ 1 \leq s \leq k-1,\ 1 \leq t \leq n_i^s\}, \text{ and} \\ E(G) &= \{(i, s, t)(i', s', t') : i \neq i',\ s \neq s'\}. \end{aligned}$$

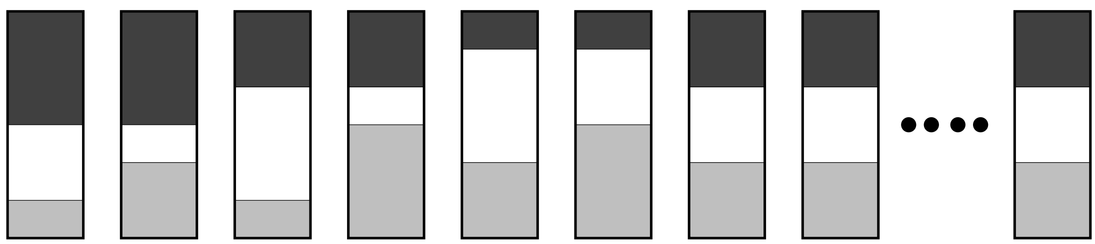

Figure 1: A sketch of a member of $\bar{\mathcal{G}}_\ell^4$, all edges between different colors in different parts exist.

Let $\mathcal{G}_\ell^k$ be the class of graphs which can be obtained from graphs in $\bar{\mathcal{G}}_\ell$ by deletion of some edges in $\{(i, s, k)(i', s', k') : s \neq s' \wedge 1 \leq i < i' \leq (k-1)!\}$.

All graphs in $\mathcal{G}_\ell^k$ are ℓ-partite and $\mathcal{G}_\ell^k$ contains graphs with $\min d_{ij} \geq \frac{k-2}{k-1}$ (e.g., we get $d_{ij} = \frac{k-2}{k-1}$ for all $i \neq j$ if all n_i^s are equal).

For $k = 3$, the density condition is fulfilled for all graphs in $\bar{\mathcal{G}}_\ell^3$, and for all graphs in $\mathcal{G}_\ell^3$ which have $d_{1,2} \geq \frac{1}{2}$. For $k > 3$, this description is not a full characterization of the extremal graphs in the problem, as for some choices of the n_i^s, the resulting graphs will have lower densities than stated in the theorem. We would need some extra conditions on the n_i^s to make sure that the graphs fulfill the density conditions.

3 Theorem 4—triangles

In this section we prove Theorem 4. We will start with a few useful lemmas and this easy fact.

Fact 6. *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph on $2n$ vertices with $\|G\| \geq \frac{1}{2}n^2$, and let X be an independent set. Then $|X_1| \cdot |X_2| \leq \frac{1}{2}n^2$.*

Proof. There are at most n^2 pairs of vertices v_1v_2 with $v_i \in V_i$. If $\frac{1}{2}n^2$ of them are edges, then at most $\frac{1}{2}n^2$ of them can be non-edges. □

An important lemma for the study of d_ω^3 in [2] is the following.

Lemma 7. *[2] Let $G = (V_1 \cup V_2 \cup V_3 \cup V_4, E)$ be a 4-partite graph with $|V_1| = 1$, such that the pairwise edge densities $d(V_i, V_j) > \frac{1}{2}$ for $i \neq j$. Then G contains a triangle.*

With the same proof one gets a slightly stronger result which we will use in our proof. In most cases occurring later, X will be the neighborhood of a vertex, and the Lemma will be used to bound the degree of the vertex. For the sake of exposition, we present a slightly modified version of the proof here.

Lemma 8. *Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a 3-partite graph and X an independent set, such that the pairwise edge densities $d(V_i, V_j) \geq \frac{1}{2}$ for $i \neq j$ and $|X_i| \geq \frac{1}{2}|V_i|$ for $1 \leq i \leq 3$, with a strict inequality for at least two of the six inequalities. Then G contains a triangle.*

Proof. In the following, all indices are computed modulo 3. For $i \in \{1, 2, 3\}$, consider the 4-partite graph $G[X_i, Y_i, X_{i+1}, Y_{i+1}]$. For the different choices of i, we get the three inequalities

$$d(X_i, Y_{i+1}) + d(Y_i, X_{i+1}) + d(Y_i, Y_{i+1}) \geq 2.$$

Indeed, if we fix the number of edges between V_i and V_{i+1} and the sizes of $X_i, Y_i, X_{i+1}, Y_{i+1}$, the above sum is minimized if we minimize the number of edges between Y_i and Y_{i+1}. As

$$|X_i| \cdot |Y_{i+1}| + |Y_i| \cdot |X_{i+1}| \leq \frac{1}{2}|V_i| \cdot |V_{i+1}|$$

and $d(V_i, V_{i+1}) \geq \frac{1}{2}$, the sum must be at least 2. As we have strict inequality in at least two of the six inequalities in the statement of the lemma, at least one of the three sums is in fact greater than 2, and so

$$\sum_{i=1}^{3} d(X_i, Y_{i-1}) + d(X_i, Y_{i+1}) + d(Y_{i-1}, Y_{i+1}) = \sum_{i=1}^{3} d(X_i, Y_{i+1}) + d(Y_i, X_{i+1}) + d(Y_i, Y_{i+1}) > 6,$$

and thus for some $i \in \{1, 2, 3\}$,

$$d(X_i, Y_{i-1}) + d(X_i, Y_{i+1}) + d(Y_{i-1}, Y_{i+1}) > 2.$$

Picking independently at random vertices $x \in X_i, y \in Y_{i-1}, z \in Y_{i+1}$, the expected number of edges in $G[\{x, y, z\}]$ is $d(X_i, Y_{i-1}) + d(X_i, Y_{i+1}) + d(Y_{i-1}, Y_{i+1}) > 2$, and therefore $G[X_i \cup Y_{i-1} \cup Y_{i+1}]$ contains a triangle. □

As a corollary from Fact 6 and Lemma 8 we get

Corollary 9. *For $\ell \geq 3$, let $G = (V_1 \cup V_2 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph on $n\ell$ vertices with edge densities $d_{ij} \geq \frac{1}{2}$, which does not contain a triangle. Then for every independent set $X \subseteq V(G)$, $|X| \leq \frac{(\ell+1)n}{2}$.*

Proof. We may assume that $|X_1| \geq |X_2| \geq \ldots \geq |X_\ell|$. By Lemma 8, $|X_3| \leq \frac{1}{2}n$ and by Fact 6, $|X_1| + |X_2| \leq \frac{3}{2}n$. $\square$

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Suppose that G contains no triangle. Without loss of generality we may assume that each of the $\ell \geq 12$ parts of G contains exactly n vertices, where n is a sufficiently large even integer. Otherwise, multiply each vertex in each part V_i by a factor of $\frac{n}{|V_i|}$, which has no effect on the densities or the membership in $\mathcal{G}^3_\ell$, and creates no triangles.

For a vertex x, let $d_i(x) := |N(x) \cap V_i|$. For each edge $xy \in E(G)$, choose i and j such that $x \in V_i$ and $y \in V_j$, and let

$$s(xy) := d(x) - d_j(x) + d(y) - d_i(y).$$

We have

$$\sum_{xy \in E(G)} s(xy) = \frac{1}{2} \sum_{\substack{x \in V(G) \\ y \in N(x)}} s(xy) = \sum_{x \in V(G)} \left(d(x)^2 - \sum_{j=1}^{\ell} d_j(x)^2 \right).$$

The set $N(x)$ is independent, so by Lemma 8, for fixed x at most two of the $d_j(x)$ may be larger than $\frac{n}{2}$, and by Fact 6, $d_j(x) d_k(x) \leq \frac{1}{2}n^2$ for every vertex $x \in V_i$ and $j \neq k$.

Thus, for fixed $d(x) \geq n$, the sum $\sum d_j(x)^2$ is maximized if

$$d_j(x) = \begin{cases} n, & \text{if } j = 1 \text{ and } d(x) \geq n, \\ \frac{n}{2}, & \text{if } 2 \leq j \leq \left\lfloor \frac{2d(x)}{n} \right\rfloor - 1, \\ d(x) - j\frac{n}{2}, & \text{if } j = \left\lfloor \frac{2d(x)}{n} \right\rfloor, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

in which case

$$\sum_{j=1}^{\ell} d_j(x)^2 = n^2 + (d(x) - n)\tfrac{n}{2}.$$

For fixed $d(x) < n$, we have

$$\sum_{j=1}^{\ell} d_j(x)^2 \leq d(x)^2 < (n + d(x))\tfrac{n}{2} = n^2 + (d(x) - n)\tfrac{n}{2}.$$

Therefore, using that $\sum d(x) = 2\|G\| \geq \binom{\ell}{2} n^2$,

$$\begin{aligned} \frac{1}{\|G\|} \sum_{xy \in E(G)} s(xy) &\geq \frac{2}{\sum d(x)} \sum_{x \in V(G)} \left(d(x)^2 - n^2 - (d(x) - n)\tfrac{n}{2}\right) \\ &= \frac{2 \sum d(x)^2}{\sum d(x)} - n - \frac{\ell n^3}{\sum d(x)} \\ &\geq \frac{2}{\ell n} \sum d(x) - n - \frac{\ell n^3}{\sum d(x)} \\ &\geq (\ell - 2)n - \frac{2n}{\ell - 1}. \end{aligned}$$

We conclude that there is an edge $xy \in E(G)$ with $s(xy) \geq (\ell-2)n - \frac{2n}{\ell-1}$. By symmetry, we may assume that $x \in V_{11}$ and $y \in V_{12}$. Note that $N(x)$ and $N(y)$ are disjoint as otherwise there would be a triangle. Let $G' := G[\bigcup_{i=1}^{10} V_i]$. Let

$$X := N(x) \cap V(G'),\ Y := N(y) \cap V(G'),\ \text{and } Z := V(G') \setminus (X \cup Y).$$

Note that $|Z| \leq \frac{2}{11}n$. By Lemma 8, at most two of the sets X_i and at most two of the sets Y_i are greater than $\frac{n}{2}$, so we assume in the following that $|X_i| \leq \frac{n}{2}$ for $1 \leq i \leq 8$ and $|Y_i| \leq \frac{n}{2}$ for $1 \leq i \leq 6$. Further, we may assume that $|X_9| \leq \min\{|X_{10}|, |Y_7|, |Y_8|\}$. Let $X \subseteq X' \subseteq X \cup Z$ and $Y \subseteq Y' \subseteq Y \cup Z$ such that

1. $X' \cap Y' = \emptyset$,
2. $X' \cup Y' = V(G')$,
3. $|Y_i'| = \max\{|Y_i|, \frac{n}{2}\}$ for $1 \leq i \leq 8$, and
4. $|X_i'| = \max\{|X_i|, \frac{n}{2}\}$ for $9 \leq i \leq 10$.

Let $H := G' - E(V_{10}, V_7 \cup V_8)$. Let $H' \supseteq H[X \cup Y]$ be the complete bipartite graph on X' and Y', minus the edges inside the V_i and the edges between V_{10} and $V_7 \cup V_8$.

We want to bound $\|H\|$ from above. We have

$$d_H(z) \leq \begin{cases} \frac{8}{2}n, & \text{for } z \in Z_{10}, \\ \frac{10}{2}n, & \text{for } z \in Z_9, \text{ and} \\ \frac{9}{2}n, & \text{for } z \in Z \setminus (Z_9 \cup Z_{10}), \end{cases}$$

by Corollary 9. On the other hand, we have, using that $\frac{n}{2} \leq |X_7'| + |X_8'| \leq n$,

$$d_{H'}(z) \geq \begin{cases} \frac{8}{2}n, & \text{for } z \in Z_9, \text{ and} \\ \frac{7}{2}n, & \text{for } z \in Z \setminus Z_9. \end{cases}$$

To see that $d_{H'}(z) \geq \frac{7}{2}n$ for $z \in Z_{10}$, note that $Z_{10} \subseteq Y_{10}'$ if $|Y_{10}'| < \frac{1}{2}$, and thus $d_{H'}(z) \geq \frac{6}{2}n + |X_9'|$.

Therefore, taking into account a possible double count of edges in the bipartite graph $H'[Z]$, we have

$$\|H\| \leq \|H'\| + n|Z| + \frac{1}{4}|Z|^2.$$

Now,

$$\begin{aligned} \|H'\| = 39\frac{n^2}{2} & \\ & + |X_9'| \cdot |Y_{10}'| + |X_{10}'| \cdot |Y_9'| && (1) \\ & + |X_7'| \cdot |Y_8'| + |X_8'| \cdot |Y_7'| && (2) \\ & + |X_9'|(|Y_7'| + |Y_8'|) + (|X_7'| + |X_8'|)|Y_9'|. && (3) \end{aligned}$$

For fixed $|X_9'| \geq \frac{n}{2}$, (1) is maximized for minimal $|X_{10}'| \geq |X_9'|$, and (3) is maximized for maximal $|Y_7'| + |Y_8'|$. For fixed $|Y_7'| + |Y_8'|$, (2) is maximized for maximal $|Y_8'| - |Y_7'|$. Thus, (1)+(2)+(3) is maximized for

$$|X_{10}'| = |Y_7'| = |X_9'|,$$

in which case $(1) + (2) + (3) = 2n^2$. This shows that $\|H'\| \leq 43\frac{n^2}{2}$, and thus

$$\|H\| \leq 43\frac{n^2}{2} + |Z|n + \tfrac{1}{4}|Z|^2 \leq 43\frac{n^2}{2} + \frac{23}{121}n^2.$$

On the other hand, by the density condition,

$$\|H\| \geq 43\frac{n^2}{2},$$

so $|E(H') \setminus E(H)| \leq \frac{23}{121}n^2$. In particular, no vertex z can have large neighborhoods in both X and Y, as $N(z)$ is an independent set and this would force $|E(H') \setminus E(H)|$ to be large. To be more precise, let $\bar{X} := X \setminus X_{10}$ and $\bar{Y} := Y \setminus Y_{10}$, then we have

$$(|N(z) \cap \bar{Y}| - n)|N(z) \cap \bar{X}| < \frac{23}{121}n^2, \tag{4}$$

as every vertex in $N(z) \cap \bar{X}_i$ forces $|(N(z) \cap \bar{Y}) \setminus V_i| > |N(z) \cap \bar{Y}| - n$ missing edges. Note that $|\bar{X}|, |\bar{Y}| \leq 5n$ by Corollary 9. Let $G'' := G' - V_{10}$, and let

$$\begin{aligned} X'' &:= \{v \in V(G'') : |N(v) \cap \bar{X}| > \tfrac{1}{2}|\bar{X}|\}, \\ Y'' &:= \{v \in V(G'') : |N(v) \cap \bar{Y}| > \tfrac{1}{2}|\bar{Y}|\}, \text{ and} \\ Z'' &:= V(G'') \setminus (X'' \cup Y''). \end{aligned}$$

The sets X'' and Y'' are disjoint by (4). As any two vertices in X'' (or Y'') have a common neighbor, X'' and Y'' are independent sets.

If $z \in Z''$ and $|N(z) \cap \bar{Y}| \geq \frac{6}{5}n$, then

$$\begin{aligned} d_H(z) &\leq |N(z) \cap \bar{Y}| + |N(z) \cap \bar{X}| + |N(z) \cap Z| \\ &\leq_{(4)} |N(z) \cap \bar{Y}| + \frac{23n^2}{121(|N(z) \cap \bar{Y}| - n)} + |Z|. \end{aligned} \tag{5}$$

The last expression is a convex function in $|N(z) \cap \bar{Y}|$ and thus maximized on the boundary of the interval $\left[\frac{6}{5}n, \frac{5}{2}n\right]$. In the case $|N(z) \cap \bar{Y}| = \frac{5}{2}n$, (5) gives

$$d_H(z) \leq \frac{5}{2}n + \frac{46}{363}n + \frac{2}{11}n < 2.81n.$$

For $|N(z) \cap \bar{Y}| = \frac{6}{5}n$, (5) gives

$$d_H(z) \leq \frac{6}{5}n + \frac{115}{121}n + \frac{2}{11}n < 2.4n.$$

We get the same upper bound with a symmetric argument for $|N(z) \cap X''| \geq \frac{6}{5}n$ (the symmetric statement of (4) also holds). Finally, if $|N(z) \cap X''| + |N(z) \cap Y''| \leq \frac{12}{5}n$, then

$$d_H(z) \leq \frac{12}{5}n + \frac{2}{11}n < 2.6n.$$

Every vertex $z \in Y'_i \cap Z''$ is incident to at least $\frac{1}{2}|\bar{X}| - |X_i| \geq \frac{1}{2}\left(9n - |\bar{Y}| - |Z|\right) - n \geq \frac{10}{11}n$ edges in $E(H') \setminus E(H)$. So we have

$$|Y' \cap Z''| \leq \frac{23 \cdot 11}{121 \cdot 10}n < 0.21n,$$

and similarly,

$$|X' \cap Z''| < 0.21n.$$

Thus,

$$|Z''| < 0.42n.$$

Like above, we may assume (after possibly renumbering the sets) that $|X_i''| \leq \frac{n}{2}$ for $1 \leq i \leq 7$ and $|Y_i''| \leq \frac{n}{2}$ for $1 \leq i \leq 5$. Further, we may assume that $|X_8''| \leq \min\{|X_9''|, |Y_6''|, |Y_7''|\}$ (switch Ys and Xs if necessary). Let $H'' := G'' - E(V_9, V_6 \cup V_7)$. By the density condition,

$$\|H''\| \geq 34\frac{n^2}{2}.$$

On the other hand, we can repeat the above arguments for H for H'', and create a bipartite graph H''' on $X''' \supseteq X''$ and $Y''' \supseteq Y''$ with $d_{H'''}(z) \geq \frac{6}{2}n$ for all $z \in Z''$, and conclude that

$$\|H''\| \leq 34\frac{n^2}{2} - \left(\tfrac{6}{2} - 2.81\right) n|Z''| + \frac{1}{4}|Z''|^2 \leq 34\frac{n^2}{2} - 0.08n|Z''|.$$

Therefore, $\|H''\| = 34\frac{n^2}{2}$ and $Z'' = \emptyset$. This shows that $d_\ell^3 = d_{12}^3 = \frac{1}{2}$.

But more is true, $G[\bigcup_{i \leq 8} V_i] = H'' \setminus V_9$ is a complete bipartite graph minus the edges inside the V_i, and we may assume that $|X_1''| = |X_2''| = |X_3''| = \frac{1}{2}n$, as at most one of the $|X_i''|$ and at most one of the $|Y_i''|$ $(1 \leq i \leq 8)$ may be greater than $\frac{1}{2}n$ by the density condition. For $9 \leq k \leq \ell$, $1 \leq i \leq 8$, $1 \leq j \leq 8$ with $i \neq j$, for every $v \in V_k$, we have $|N(v) \cap X_i''| \cdot |N(v) \cap Y_j''| = 0$ as otherwise there is a triangle. Thus, $|N(v) \cap (V_1 \cup V_2 \cup V_3)| \leq \frac{3}{2}n$ with equality only for $N(v) \cap X'' = \emptyset$ or $N(v) \cap Y'' = \emptyset$. Since $d_{ik} \geq \frac{1}{2}$, equality must hold for every $v \in V_k$, showing that G is isomorphic to a graph in $\mathcal{G}_\ell^3$. $\square$

4 Theorem 2—complete subgraphs

Graphs which have almost enough edges to force a K^k either contain a K^k or have a structure very similar to the Turán graph. This is described by the following theorem from [1], where a more general version is credited to Erdös and Simonovits.

Theorem 10. *[1, Theorem VI.4.2] Let $k \geq 3$. Suppose a graph G contains no K^k and*

$$\|G\| = \left(1 - \frac{1}{k-1} + o(1)\right)\binom{|G|}{2}.$$

Then G contains a $(k-1)$-partite graph of minimal degree $\left(1 - \frac{1}{k-1} + o(1)\right)|G|$ as an induced subgraph.

Proof of Theorem 2. For the ease of reading and since we are not trying to minimize the needed ℓ, we will use a number of variables ℓ_i and $c_i > 0$ depending on ℓ. As ℓ is chosen larger, the ℓ_i grow without bound and the c_i approach 0.

Let G be an ℓ-partite graph with $V(G) = V_1 \cup V_2 \cup \ldots \cup V_\ell$ with densities $d_{ij} \geq \frac{k-2}{k-1}$, and suppose that G contains no K^k. Without loss of generality we may assume that each of the V_i contains exactly n vertices, where n is an integer divisible by $k-1$.

We have

$$\|G\| \geq \left(1 - \frac{1}{k-1} - \frac{1}{\ell}\right)\binom{|G|}{2}.$$

Let H be the $(k-1)$-partite subgraph of G guaranteed by Theorem 10, with parts

$$V(H) = X^1 \cup X^2 \cup \ldots \cup X^{k-1}$$

and $Z := V(G) \setminus V(H)$. Further by Theorem 10, there is a $c_1 > 0$ depending on ℓ, so that $|Z| \leq c_1|G|$, and this c_1 becomes arbitrarily small if ℓ is chosen large enough. In particular, $|Z_i| \leq 2c_1 n$ for at least half the indices $1 \leq i \leq \ell$. By the pigeon hole principle, we can renumber the V_i and the X^j, such that $|Z_i| \leq 2c_1 n$ and $|X_i^1| \geq |X_i^2| \geq \ldots \geq |X_i^{k-1}|$ for $1 \leq i \leq \ell_1$, where $\ell_1 := \frac{\ell}{2(k-1)!}$.

For $c_2 = 2(k-1)c_1$, there is at most one index $i \leq \ell_1$ with $|X_i^1| > \left(\frac{1}{k-1} + c_2\right) n$, as otherwise there is a pair $(V_i, V_{i'})$ with

$$\begin{aligned} d_{ii'} &\leq 1 - \frac{1}{n^2} \sum_{j=1}^{k} |X_i^j| \cdot |X_{i'}^j| \\ &< 1 - \left(\frac{1}{k-1} + c_2\right)^2 - (k-2)\left(\frac{1 - \left(\frac{1}{k-1} + c_2\right) - 2c_1}{k-2}\right)^2 + 4c_1 \\ &\leq \frac{k-2}{k-1} - \frac{2c_2}{k-1} + 4c_1 \\ &= \frac{k-2}{k-1}. \end{aligned}$$

So we may assume that

$$\left(\frac{1}{k-1} - kc_2\right) n \leq |X_i^j| \leq \left(\frac{1}{k-1} + c_2\right) n$$

for $1 \leq i \leq \ell_1 - 1$ and $1 \leq j \leq k-1$. This implies that

$$\|G[X_i^j, X_{i'}^{j'}]\| > |X_i^j| \cdot |X_{i'}^{j'}| - c_3 n^2$$

for $i \neq i'$, $j \neq j'$, $1 \leq i, i' \leq \ell_1 - 1$, $1 \leq j, j' \leq k-1$ and some $c_3 > 0$ with $c_3 \to 0$.

For every $v \in \bigcup_{i \leq \ell_1 - 1} V_i$, find a maximum set $\mathcal{P}_v$ of pairs (i_s, j_s) with

$$\begin{aligned} &(1,1) \leq (i_s, j_s) \leq (\ell - 1, k-1), \\ &i_s \neq i_{s'},\ j_s \neq j_{s'}, \text{ and} \\ &|N(v) \cap X_{i_s}^{j_s})| > c_4 n, \end{aligned}$$

where $c_4 := k\sqrt{c_3}$. If there is a vertex v with $|\mathcal{P}_v| = k-1$, then we have a K^k as follows. If we pick a vertex v_s independently at random in each $|N(v) \cap X_{i_s}^{j_s})|$, then the probability that $v_s v_{s'}$ is an edge is larger than $\frac{c_4^2 - c_3}{c_4^2} = \frac{k^2-1}{k^2}$, and therefore the expected total number of such edges is greater than $\frac{k^2-1}{k^2}\binom{k-1}{2} > \binom{k-1}{2} - 1$. Thus, there is a choice for the v_s inducing a K^{k-1} in $N(v)$.

So we may assume that $|\mathcal{P}_v| \leq k-2$ for all v. For $1 \leq i \leq \ell_1 - 1$, assign $v \in Z_i$ to one set $Y_i^j \supseteq X_i^j$, if there is no pair (i,j) in $\mathcal{P}_v$. If there is more than one available set, arbitrarily pick one of them.

Now we reorder the V_i and Y^j again to guarantee that $|Y_i^1| \geq \ldots \geq |Y_i^{k-1}|$ for $1 \leq i \leq \ell_2$, with $\ell_2 := \frac{\ell_1 - 1}{(k-1)!}$. In the following, only consider indices $i \leq \ell_2$. Note that for $v \in Y_i^j$, $|N(v) \cap Y_i^{j'}| < (c_4 + 2c_1)n$ for all but at most $k-2$ different j', as $Y_i^{j'} \setminus X_i^{j'} \subseteq Z_{j'}$.

Let $\bar{Y}^i \subseteq Y^i$ be the set of all vertices $v \in Y^i$ with $|N(v) \cap Y^j)| < \frac{1}{2}\left(\frac{1}{k-1} + c_5\right) \ell_2 n$ for some $j \neq i$, $c_5 := c_2 + c_4$. Note that the sets $Y^i \setminus \bar{Y}^i$ are independent, as the intersection of the neighborhoods of every two vertices in this set contain a K^{k-2}. Every vertex in $v \in \bar{Y}_j^i$ may have up to

$$\Big((c_4 + 2c_1)(\ell_2 - k + 1) + k - 2\Big) n$$

neighbors in Y^i. But, at the same time, v has at least

$$|Y^{i'}| - \frac{1}{2}\left(\frac{1}{k-1} + c_5\right) \ell_2 n - n > \frac{1}{3k} \ell_2 n$$

non-neighbors in some $Y^{i'} \setminus V_j$, $i' \neq i$. Then

$$\begin{aligned}
\|G[V_1 \cup \dots V_{\ell_2}]\| &\leq \sum_{\substack{i \neq i' \\ j < j'}}^{\ell_2} |Y_i^j| \cdot |Y_{i'}^{j'}| + \sum_{i \leq \ell_2} |\bar{Y}^i| \left(((c_4 + 2c_1)(\ell_2 - k + 1) + k - 2)n - \tfrac{1}{3k}\ell_2 n \right) \\
&\leq \sum_{\substack{i \neq i' \\ j < j'}} |Y_i^j| \cdot |Y_{i'}^{j'}| + \sum_{i \leq \ell_2} |\bar{Y}^i| \underbrace{\left(c_4 + 2c_1 + \tfrac{k}{\ell_2} - \tfrac{1}{3k} \right)}_{<0 \text{ for large enough } \ell} \ell_2 n \\
&\leq \binom{\ell_2}{2} n^2 - \sum_{\substack{i \leq \ell_2 \\ j < j'}} |Y_i^j| \cdot |Y_i^{j'}| \\
&\leq \binom{\ell_2}{2} \tfrac{k-2}{k-1}\, n^2,
\end{aligned}$$

where equality only holds if $|\bar{Y}_i| = 0$ for all i, and $|Y_i^j| = \frac{n}{k-1}$ for $1 \leq j \leq k-1$ and all but at most one index i.

This completes the proof of $d_\ell^k = \frac{k-2}{k-1}$ for large enough ℓ. We are left to analyze the extremal graphs. After reordering, we have $|Y_i^j| = \frac{n}{k-1}$ and $d(Y_i^j, Y_{i'}^{j'}) = 1$ for $1 \leq j, j' \leq k-1$ and $1 \leq i, i' \leq k$, if $i \neq i'$ and $j \neq j'$.

Let $v \in V_{i'}$ for some $i' > k$. Then $|N(v) \cap \bigcup_{i \leq k} V_i| \leq \frac{k(k-2)}{k-1} n$, as otherwise there is a K^{k-1} in $N(v)$. On the other hand, equality must hold for all vertices $v \in V_{i'}$ due to the density condition. Therefore, $N(v) \cap \bigcup_{i \leq k} V_i = V_i \setminus Y_j$ for some $1 \leq j \leq k-1$. Define $Y_{i'}^j$ accordingly for all $i' > k$, and let $Y^j = \bigcup_i Y_i^j$. Then $V = \bigcup Y^j$. For every permutation π of the set $\{1, \dots, k-1\}$, there can be at most one set V_i with $|Y_i^{\pi(1)}| \geq |Y_i^{\pi(2)}| \geq \dots \geq |Y_i^{\pi(k-1)}|$ and $|Y_i^{\pi(1)}| > |Y_i^{\pi(k-1)}|$. Otherwise, this pair of sets would have density smaller than $\frac{k-2}{k-1}$. Thus, all but at most $(k-1)!$ of the V_i have $|Y_i^j| = \frac{n}{k-1}$ for $1 \leq j \leq k-1$. Therefore, all extremal graphs are in $\mathcal{G}_\ell^k$. □

5 Open problems

As mentioned above, the characterization of the extremal graphs is not complete for $k > 3$. We need to determine all parameters n_i^s so that the resulting graphs in $\bar{\mathcal{G}}_\ell^k$ fulfill the density conditions.

The other obvious question left open is a good bound on ℓ depending on k in Theorem 2, and the determination of the exact values of d_ℓ^k for smaller ℓ. In particular, is it true that $d_5^3 = \frac{1}{2}$?

Another interesting open topic is the behavior of $d_\ell(H)$ for non-complete H. Bondy et al. [2] show that

$$\lim_{\ell \to \infty} d_\ell(H) = \frac{\chi(H) - 2}{\chi(H) - 1},$$

but it should be possible to show with similar methods as in this note that $d_\ell(H) = \frac{\chi(H)-2}{\chi(H)-1}$ for large enough ℓ depending on H.

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