ROOTED INDUCED TREES IN TRIANGLE-FREE GRAPHS

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ABSTRACT. For a graph G, let $t(G)$ denote the maximum number of vertices in an induced subgraph of G that is a tree. Further, for a vertex $v \in V(G)$, let $t(G,v)$ denote the maximum number of vertices in an induced subgraph of G that is a tree, with the extra condition that the tree must contain v. The minimum of $t(G)$ ($t(G,v)$, respectively) over all connected triangle-free graphs G (and vertices $v \in V(G)$) on n vertices is denoted by $t_3(n)$ ($t_3^*(n)$). Clearly, $t(G,v) \leq t(G)$ for all $v \in V(G)$. In this note, we solve the extremal problem of maximizing $|G|$ for given $t(G,v)$, given that G is connected and triangle-free. We show that $|G| \leq 1 + \frac{(t(G,v)-1)t(G,v)}{2}$ and determine the unique extremal graphs. Thus, we get as corollary that $t_3(n) \geq t_3^*(n) = \lceil \frac{1}{2}(1+\sqrt{8n-7}) \rceil$, improving a recent result by Fox, Loh and Sudakov.

All graphs in this note are simple and finite. For notation not defined here we refer the reader to Diestel's book [1].

For a graph G, let $t(G)$ denote the maximum number of vertices in an induced subgraph of G that is a tree. The problem of bounding $t(G)$ was first studied by Erdős, Saks and Sós [2] for certain classes of graphs, one of them being triangle-free graphs. Let $t_3(n)$ be the minimum of $t(G)$ over all connected triangle-free graphs G on n vertices. Erdős, Saks and Sós showed that

$$\Omega\left(\frac{\log n}{\log\log n}\right) \leq t_3(n) \leq O(\sqrt{n}\log n).$$

This was recently improved by Matoušek and Šámal [4] to

$$e^{c\sqrt{\log n}} \leq t_3(n) \leq 2\sqrt{n}+1,$$

for some constant c. For the upper bound, they construct graphs as follows. For $k \geq 1$, let B_k be the bipartite graph obtained from the path $P^k = v_1 \dots v_k$ if we replace v_i by $\frac{k+1}{2} - |\frac{k+1}{2} - i|$ independent vertices for $1 \leq i \leq k$. This graph has $|B_k| = \left\lfloor \frac{(k+1)^2}{4} \right\rfloor$ vertices, yielding the bound.

For a vertex $v \in V(G)$, let $t(G,v)$ denote the maximum number of vertices in an induced subgraph of G that is a tree, with the extra condition that the tree must contain v. Similarly as above, we define $t_3^*(n)$ as the minimum of $t(G,v)$ over all connected graphs G with $|G| = n$ and vertices $v \in V(G)$. As $t(G,v) \leq t(G)$ for every graph, this can be used to bound $t_3(n)$. In a very recent paper, Fox, Loh and Sudakov do exactly that to show that

$$\sqrt{n} \leq t_3^*(n) \leq t_3(n) \text{ and } t_3^*(n) \leq \lceil \tfrac{1}{2}(1+\sqrt{8n-7}) \rceil.$$

For the upper bound, they construct graphs similarly as above. For $k \geq 1$, let G_k be the bipartite graph obtained from the path $P^k = v_0 v_1 \dots v_{k-1}$ if we replace v_i by $k-i$ independent vertices $V_i := \{v_i^1, \dots, v_i^{k-i}\}$ for $1 \leq i \leq k-1$. No induced tree containing v_0 and a vertex in V_j contains more than one vertex in any of the V_i, for $1 \leq i < j$. Thus, G_k contains no induced tree containing v_0 with more than k vertices. This graph has $|G_k| = 1 + \frac{(k-1)k}{2}$ vertices, yielding the bound.

In this note, we show that this upper bound is tight, and that the graphs G_k are, in a way, the unique extremal graphs. This improves the best lower bound on $t_3(n)$ by a factor of roughly $\sqrt{2}$. In [3], the authors relax the problem to a continuous setting to achieve their lower bound on $t_3^*(n)$. While most of our ideas are inspired by this proof, we will skip this initial step and get a much shorter and purely combinatorial proof of our tight result.

Theorem A. *Let G be a connected triangle-free graph on n vertices, and let $v \in V(G)$. If G contains no tree through v on $k+1$ vertices as an induced subgraph, then $n \leq 1 + \frac{(k-1)k}{2}$. Further, equality holds only if G is isomorphic to G_k with $v = v_0$.*

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Let $N(v)$ denote the neighborhood of a vertex v, and let $N[v] := N(v) \cup \{v\}$ be the closed neighborhood of v. In the proof of Theorem A, we will use the following related statement.

Theorem B. *Let G be a connected triangle-free graph, and let $v \in V(G)$. If G contains no tree through v on $k+1$ vertices as an induced subgraph, then $|V(G) \setminus N[v]| \leq \frac{(k-2)(k-1)}{2}$.*

Proof of Theorems A and B. Let $A(k)$ be the statement that Theorem A is true for the fixed value k, and let $B(k)$ be the statement that Theorem B is true for k. We will use induction on k to show $A(k)$ and $B(k)$ simultaneously.

To start, note that $A(k)$ and $B(k)$ are trivially true for $k \leq 2$. Now assume that $A(\ell)$ and $B(\ell)$ hold for all $\ell < k$ for some $k \geq 3$, and we will show $B(k)$. We may assume that every vertex in $N(v)$ is a cut vertex in G (otherwise delete it and proceed with the smaller graph, which is connected and triangle-free). Further, $N(v)$ is an independent set as G is triangle-free. Let $N(v) = \{x_1, x_2, \dots, x_r\}$, and let X_i be a component of $G \setminus N[v]$ adjacent only to x_i for $1 \leq i \leq r$. Note that $G \setminus N[v]$ may contain other components but we do not need to worry about them.

Let $k_i + 1$ be the size of a largest induced tree in $x_i \cup X_i$ containing x_i. As $N(v)$ is independent, we can glue these r trees together in v to create an induced tree through v on $1 + r + \sum k_i$ vertices, so $1 + r + \sum k_i \leq k$ (and in particular $k_i + 1 < k$). By $A(k_i + 1)$ we have $|X_i| \leq \frac{k_i(k_i+1)}{2}$.

Now replace each $G[x_i \cup X_i]$ by a graph isomorphic to G_{k_i} with $v_0 = x_i$ (all other components of $G \setminus N[v]$ remain untouched), reducing the total number of vertices by at most $\sum k_i$. Note that this new graph G' is triangle-free and connected. Since every maximal induced tree in G through v must contain a vertex x_i for some $1 \leq i \leq r$, and therefore exactly k_i vertices of X_i, every induced tree through v in G' has fewer than k vertices. Therefore, by $B(k-1)$,

$$|V(G) \setminus N[v]| \leq |V(G') \setminus N[v]| + \sum k_i \leq \frac{(k-3)(k-2)}{2} + k - r - 1 \leq \frac{(k-2)(k-1)}{2},$$

establishing $B(k)$. Equality can hold only for $r = 1$, if $G[x_1 \cup X_1]$ is isomorphic to G_{k-1} by $A(k-1)$, and if $G \setminus N[v]$ contains no vertices outside X_1.

To show $A(k)$ we can no longer assume that all vertices in $N(v)$ are cut vertices, we now have to consider all the vertices we may have deleted in the beginning of the proof of $B(k)$. We need to show that $|N(v)| = k - 1$ and that $N(x) = N(x_1)$ for all $x \in N(v)$.

The first statement follows as $G[N[v]]$ is a star which implies $|N(v)| \leq k - 1$, and equality must hold if $n = 1 + \frac{(k-1)k}{2}$.

Now let $x \in N(v)$. If $N(x) \cap X_1 = \emptyset$, then $G[x \cup v \cup T]$ is a tree for any induced tree T through x_1 in $G[x_1 \cup X_1]$. In particular, if $|T| = k - 1$, this tree contains $k + 1$ vertices, a contradiction. If $N(x) \cap X_1 \neq \emptyset$, then $G[x \cup X_1]$ is isomorphic to G_{k-1} by $A(k-1)$ as above. By the structure of G_{k-1}, this implies that $N(x) = N(x_1)$, showing $A(k)$. □

As a corollary we get the exact value for $t_3^*(n)$, which is an improved lower bound for $t_3(n)$.

Corollary 1. $\lceil \frac{1}{2}(1 + \sqrt{8n-7}) \rceil = t_3^*(n) \leq t_3(n) \leq 2\sqrt{n} + 1.$

Concluding remarks

One may speculate that, similarly to the role of the G_k for $t_3^*(n)$, the graphs B_k are extremal graphs for $t_3(n)$. This is not true for $k = 5$, though, as $K_{5,5}$ minus a perfect matching has no induced tree with more than 5 vertices, and B_5 has only 9 vertices, as was pointed out to me by Christian Reiher. We currently know of no other examples beating the bound from B_k. In fact, with a similar but somewhat more involved proof as above one can show that B_k is extremal under the added condition that G has diameter $k - 1$.

References

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