On k-Ordered Bipartite Graphs

Jill R. Faudree
University of Alaska Fairbanks
Fairbanks, AK 99775
ffjrf@aurora.uaf.edu

Ronald J. Gould
Emory University
Atlanta, GA 30322
rg@mathcs.emory.edu

Florian Pfender
Emory University
Atlanta, GA 30322
fpfende@mathcs.emory.edu

Allison Wolf
College of Computing
Georgia Institute of Technology
Atlanta, GA 30332
awolf@cc.gatech.edu

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Abstract

In 1997, Ng and Schultz introduced the idea of cycle orderability. For a positive integer k, a graph G is *k-ordered* if for every ordered sequence of k vertices, there is a cycle that encounters the vertices of the sequence in the given order. If the cycle is also a hamiltonian cycle, then G is said to be *k-ordered hamiltonian.* We give minimum degree conditions and sum of degree conditions for nonadjacent vertices that imply a balanced bipartite graph to be k-ordered hamiltonian. For example, let G be a balanced bipartite graph on $2n$ vertices, n sufficiently large. We show that for any positive integer k, if the minimum degree of G is at least $(2n+k-1)/4$, then G is k-ordered hamiltonian.

1 Introduction

Over the years, hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example hamiltonian connectedness. Recently a new strong hamiltonian property was introduced in [3].

We say a graph G on n vertices, $n \geq 3$, is *k-ordered* for an integer k, $1 \leq k \leq n$, if for every sequence $S = (x_1, x_2, ..., x_k)$ of k distinct vertices in G there exists a cycle that contains all the vertices of S in the designated order. A graph is *k-ordered hamiltonian* if for every sequence S of k vertices there exists a hamiltonian cycle which encounters the vertices in S in the designated order. We will always let $S = (x_1, x_2, ..., x_k)$ denote the ordered k-set. If we say a cycle C contains S, we mean C contains S in the designated order under some orientation. The neighborhood of a vertex v will be denoted by $N(v)$, the degree of v by $d(v)$, the degree of v to a subgraph H by $d_H(v)$, and the minimum degree of a graph G by $\delta(G)$. A graph on n vertices is said to be *k-linked* if $n \geq 2k$ and for every set $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ of $2k$ distinct vertices there are vertex disjoint paths $P_1, \ldots, P_k$ such that P_i joins x_i to y_i for all $i \in \{1, \ldots, k\}$. Clearly, a k-linked graph is also k-ordered.

In the process of finding bounds implying a graph to be k-ordered hamiltonian, Ng and Schultz [3] showed the following:

Proposition 1. *[3] Let G be a hamiltonian graph on n vertices, $n \geq 3$. If G is k-ordered, $3 \leq k \leq n$, then G is $(k-1)$-connected.*

Theorem 2. *[3] Let G be a graph of order $n \geq 3$ and let k be an integer with $3 \leq k \leq n$. If*

$$d(x) + d(y) \geq n + 2k - 6$$

for every pair x, y of nonadjacent vertices of G, then G is k-ordered hamiltonian.

Faudree *et al.*[4] improved the last bound as follows.

Theorem 3. *[4] Let G be a graph of sufficiently large order n. Let $k \geq 3$. If*

$$\delta(G) \geq \begin{cases} \frac{n+k-3}{2}, & \text{if } k \text{ is odd} \\ \frac{n+k-2}{2}, & \text{if } k \text{ is even,} \end{cases}$$

then G is k-ordered hamiltonian.

Theorem 4. *[4] Let G be a graph of sufficiently large order n. Let $k \geq 3$. If for any two nonadjacent vertices x and y,*

$$d(x) + d(y) \geq n + \frac{3k-9}{2},$$

then G is k-ordered hamiltonian.

Theorem 5. *[4] Let k be an integer, $k \geq 2$. Let G be a $(k+1)$-connected graph of sufficiently large order n. If*

$$|N(x) \cup N(y)| \geq \frac{n+k}{2}$$

for all pairs of distinct vertices $x, y \in V(G)$, then G is k-ordered hamiltonian.

Much like results for hamiltonicity, smaller bounds are possible if we restrict G to be a balanced bipartite graph. In fact, we get the following results:

Theorem 6. *Let $G(A \cup B, E)$ be a balanced bipartite graph of order $2n \geq 618$. Let $3 \leq k \leq \frac{n}{103}$. If $\delta(G) \geq 4k - 1$ and for any two nonadjacent vertices $x \in A$ and $y \in B$, $d(x) + d(y) \geq n + \frac{k-1}{2}$, then G is k-ordered hamiltonian.*

The bound on the degree sum is sharp, as will be shown later with an example. The upper bound on k comes out of the proof, the correct bound should be a lot larger and possibly as large as $n/4$.

Corollary 7. *Let G be a balanced bipartite graph of order $2n \geq 618$. Let $3 \leq k \leq \frac{n}{103}$. If*

$$\delta(G) \geq \frac{2n+k-1}{4}$$

then G is k-ordered hamiltonian.

Theorem 8. *Let $G(A \cup B, E)$ be a balanced bipartite graph of order $2n \geq 618$. Let $3 \leq k \leq min\{\frac{n}{103}, \frac{\sqrt{n}}{4}\}$. If for any two nonadjacent vertices $x \in A$ and $y \in B$, $d(x)+d(y) \geq n+k-2$, then G is k-ordered hamiltonian.*

The last bound is sharp, as the following graph G shows:
Let the vertex set $V := A_1 \cup A_2 \cup B_1 \cup B_2 \cup B_3$, with $|A_1| = |B_1| = k/2$, $|B_2| = k-1$, $|A_2| = n - k/2$, $|B_3| = n - 3k/2 + 1$. Let the edge set consist of all edges between A_1 and B_1 minus a k-cycle, all edges between A_1 and B_2, and all edges between A_2 and the B_i for $i \in \{1, 2, 3\}$. Then G has minimum degree $\delta(G) = 3k/2 - 3$, minimal degree sum $n+k-3$, and G is not k-ordered, as there is no cycle containing the vertices of $A_1 \cup B_1$ in the same order as the cycle whose edges were removed between A_1 and B_1. This example further suggests the following conjecture, strengthening Theorem 6 to a sharp result:

Conjecture 9. *Let $G(A \cup B, E)$ be a balanced bipartite graph of order $2n$. Let $k \geq 3$. If $\delta(G) \geq \frac{3k-1}{2} - 2$ and for any two nonadjacent vertices $x \in A$ and $y \in B$, $d(x) + d(y) \geq n + \frac{k-1}{2}$, then G is k-ordered hamiltonian.*

In some of the proofs the following theorem of Bollobás and Thomason[1] comes in handy.

Theorem 10. *[1] Every $22k$-connected graph is k-linked.*

2 Proofs

In this section we will prove Theorem 6 and Theorem 8.

From now on, A and B will always be the partite sets of the balanced bipartite graph G, and for a subgraph $H \subset G$, $H^A := H \cap A$ and $H^B := H \cap B$ will be its corresponding parts.

The following result and its corollary, which give sufficient conditions for k-ordered to imply k-ordered hamiltonian, will make the proofs easier.

Theorem 11. *Let $k \geq 3$ and let $G(A \cup B, E)$ be a balanced bipartite, k-ordered graph of order $2n$. If for every pair of nonadjacent vertices $x \in A$ and $y \in B$*

$$d(x) + d(y) \geq n + \frac{k-1}{2},$$

then G is k-ordered hamiltonian.

Proof: Let $S = \{x_1, x_2, \cdots, x_k\}$ be an ordered subset of the vertices of G. Let C be a cycle of maximum order $2c$ containing all vertices of S in appropriate order. Let $L := G - C$. Notice that L is balanced bipartite, since C is. Let $l := |L|/2 = |L^A| = |L^B|$.

Claim 1. *Either L is connected or L consists of the union of two complete balanced bipartite graphs.*

To prove the claim, it suffices to show that $d_L(u) + d_L(v) \geq l$ for all nonadjacent pairs $u \in L^A, v \in L^B$. Suppose the contrary, that is, there are two such vertices u, v with $d_L(u) + d_L(v) < l$. Since $d(u) + d(v) \geq n + (k-1)/2$, it follows that $d_C(u) + d_C(v) \geq c + (k+1)/2$. There are no common neighbors of u and v on C, hence there are at least $k+1$ edges on C with both endvertices adjacent to $\{u, v\}$. Fix a direction on C. Say there are r edges on C directed from a u-neighbor to a v-neighbor, and t edges from a v-neighbor to a u-neighbor. Without loss of generality, let $r \geq t$. On C, between any two of the $r \geq (k+1)/2$ edges of that type, there have to be at least two vertices of S, else C could be enlarged (see Figure 1). Thus $|S| \geq k+1$, a contradiction, which proves the claim. $\diamond$

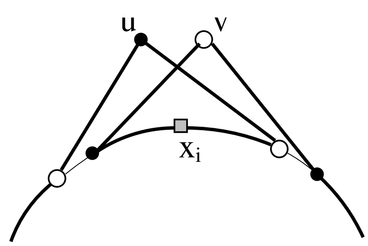

Figure 1:

In particular, the claim shows that there are no isolated vertices in L and that all of L's components are balanced.

Suppose $l \geq 1$. Let L_1 be a component of L, $L_2 := L - L_1$, $l_1 := |L_1|/2$, and $l_2 := |L_2|/2$. The k vertices of S split the cycle C into k intervals: $[x_1, x_2]$, $[x_2, x_3], \ldots$, $[x_k, x_1]$. Assume there are vertices $x, y \in L_1$ ($x = y$ is possible) with distinct neighbors in one of the intervals of C determined by S, say $[x_i, x_{i+1}]$. Let z_1 and z_2 be the immediate successor and predecessor on C to the neighbors of x and y respectively according to the orientation of C. Observe that we can choose x and y and their neighbors in C such that none of the vertices on the interval $[z_1, z_2]$ have neighbors in L_1. We can also assume that $z_1 \neq z_2$, otherwise $x = y$ by the maximality of C, and bypassing z_1 through x would lead to a cycle of the same order, but the new outside component $L_1 - x$ would not be balanced, a contradiction to claim 1. Let z be either z_2 or its immediate predecessor such that z_1 and z are from different parts. Since x and y are in the same component of L, there is an x, y-path through L. Let $\bar{y}$ be either y or its immediate predecessor on the path such that x and $\bar{y}$ are from different parts. If $x = y$, let $\bar{y}$ be any neighbor of x in L. Let R be the path on C from z_1 to z_2 and $r := |R|$. Since C is maximal, the $x, \bar{y}$-path can't be inserted, and since neither x nor $\bar{y}$ have neighbors on R,

$$d(x)+d(\bar{y}) \leq 2l_1 + \frac{2c-r+1}{2}.$$

Further, the z_1, z-path can't be inserted anywhere on $C-R$, else C could be enlarged by inserting it and going through L instead (or in the case $x=y$ we would get a same length cycle with unbalanced outside components). Since z_1 and z have no neighbors in L_1, we get

$$d(z_1)+d(z) \leq 2l_2 + r + \frac{2c-r+1}{2}.$$

Hence

$$d(x)+d(\bar{y})+d(z_1)+d(z) \leq 2l_2+2l_1+2c+1 = 2n+1,$$

which contradicts (with $k \geq 3$) that

$$d(x)+d(z) \geq n + \frac{k-1}{2}$$

and

$$d(\bar{y})+d(z_1) \geq n + \frac{k-1}{2}.$$

Thus, there is no interval $[x_i, x_{i+1}]$ with two independent edges to L_1. By Proposition 1, G is $(k-1)$-connected, thus all but possibly one of the segments (x_i, x_{i+1}) have exactly one vertex with a neighbor in L_1.

Since $|N_C(L_1)| \leq k$, we assume without loss of generality that $|N_C(L_1^B)| \leq k/2$. Let $x \in L_1^B$ and let $|N_C(x)| = d \leq k/2$. Thus, for every $v \in C$ that is not adjacent to L_1 the degree sum condition implies:

$$d(v) \geq n + \frac{k-1}{2} - (l_1+d) = c + l_2 + (\frac{k}{2} - d - \frac{1}{2}).$$

On the other hand, we know $d(v) \leq c + l_2 - 1$. Thus, $d \geq 2$. Now we have shown that $N_{L_1}(C)$ includes vertices from both L_1^A and L_1^B. So, without loss of generality, assume L_1 has neighbors y and z in $(x_1 \ldots x_2)$ and $(x_2 \ldots x_3)$ respectively and such that y and z are in different partite sets.

Let y, z be the unique vertices in (x_1, x_2) and (x_2, x_3) respectively, which have neighbors in L_1. Since the successors of y and z are from different parts and not adjacent to L_1, they must be adjacent to each other. But now C can be extended, which is a contradiction.

This proves that L has to be empty. Therefore C is hamiltonian. □

An immediate Corollary to Theorem 11 is the following:

Corollary 12. *Let $k \geq 3$ and let G be a k-ordered balanced bipartite graph of order $2n$. If $\delta(G) \geq \frac{n}{2} + \frac{k-1}{4}$, then G is k-ordered hamiltonian.*

To see that these bounds are sharp, consider the following graph $G(A \cup B, E)$:

$$A := A_1 \cup A_2, B := B_1 \cup B_2,$$

with

$$|A_1| = |B_1| = \left\lceil \frac{n}{2} + \frac{k-1}{4} \right\rceil - 1,$$
$$|A_2| = |B_2| = n - |A_1|,$$

and

$$E := \{ab | a \in A_1, b \in B\} \cup \{ab | a \in A, b \in B_1\}.$$

For n sufficiently large, G is obviously a k-connected, k-ordered, and balanced bipartite graph. The minimum degree is $\delta(G) = d(v) = |A_1|$ for any vertex $v \in B_2 \cup A_2$, thus the minimum degree condition is just missed. But G is not k-ordered hamiltonian, for if we consider $S = \{x_1, x_2, \ldots, x_k\}$, $\{x_1, x_3, \ldots\} \subseteq A_2$, $\{x_2, x_4, \ldots\} \subseteq B_2$. Let C be a cycle that picks up S in the designated order. Then $C \cap (A_1 \cup B_2)$ consists of at least $\lfloor k/2 \rfloor$ paths, all of which start and end in A_1. Therefore $|C \cap A_1| \geq |C \cap B_2| + (k-1)/2$. If C was hamiltonian, it would follow that $|A_1| \geq |B_2| + (k-1)/2$, which is not true.

The following easy lemmas will be useful.

Lemma 13. *Let G be a graph, let $k \geq 1$ be an integer and let $v \in V(G)$ with $d(v) \geq 2k-1$ for some k. If $G - v$ is k-linked, then G is k-linked.*

Proof: This is an easy exercise. □

Lemma 14. *Let G be a $2k$-connected graph with a k-linked subgraph $H \subset G$. Then G is k-linked.*

Proof: Let $S := \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ be a set of $2k$ vertices in G, not necessarily disjoint from H. Since G is $2k$-connected, there are $2k$ disjoint paths from S to H, including the possibility of one-vertex paths. Since H is k-linked, those paths can be joined in a way that k paths arise which connect x_i with y_i for $1 \leq i \leq k$. □

Lemma 15. *Let $k \geq 1$. Let $G(A \cup B, E)$ be a bipartite graph with $d(v) \geq \frac{|B|}{2} + \frac{3k}{2}$ for all $v \in A$, and $d(w) \geq 2k$ for all $w \in B$. Then G is k-linked.*

Proof: Let $S := \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ be a set of $2k$ vertices in G. Pick a set $S' := \{x'_1, \ldots, x'_k, y'_1, \ldots, y'_k\} \subset A$ as follows: If $x_i \in A$ set $x'_i = x_i$. Otherwise let x'_i be a neighbor of x_i not in S. Similarly pick the y'_i. It is possible to pick $2k$ different vertices for S' since $d(w) \geq 2k$ for all $w \in B$.

Now find disjoint paths of length 2 between x'_i and y'_i avoiding all the other vertices of S for $1 \leq i \leq k$. This is possible since $|N(x'_i) \cap N(y'_i)| \geq d(x'_i) + d(y'_i) - |B| \geq 3k$. □

Proof of Theorem 6: By Theorem 11, it suffices to show that G is k-ordered.

Let K be a minimal cutset. If $|K| \geq 22k$, then G is k-linked by Theorem 10. Therefore it is k-ordered. Assume now that $|K| < 22k$. We have to deal with two cases.

Case 1. *There is an isolated vertex $v \in G - K$.*

Since $|K| = |N(v)| \geq \delta(G) \geq 4k - 1$, G is $2k$-connected, thus by Lemma 14 it suffices to find a k-linked subgraph. Without loss of generality, let $v \in B$. Let $R = G - K - v$. Then $d(w) > n - 22k$ for all $w \in R^A$. So there are at least $(n - 22k)^2$ edges in R, resulting in less than $23k$ vertices $u \in R^B$ with $d_R(u) < 2k$. Let H be the subgraph of R induced by R^A and the vertices of R^B with $d_R(u) \geq 2k$. For $w \in R^A$, we have $d_H(w) \geq n - 45k \geq \frac{|H^B|}{2} + \frac{3k}{2}$, since $n > 100k$. By Lemma 15, H is k-linked.

Case 2. *There are no isolated vertices in $G - K$.*

First, observe that $G - K$ has exactly two components. Otherwise, for the three components C_1, C_2, C_3 choose vertices $v_i \in C_i^A, w_i \in C_i^B, 1 \leq i \leq 3$.
Then we can bound their degree sum as follows:

$$\begin{aligned}
2n + 2|K| &\geq (|C_1| + |K|) + (|C_2| + |K|) + (|C_3| + |K|) \\
&\geq (d(v_1) + d(w_1)) + (d(v_2) + d(w_2)) + (d(v_3) + d(w_3)) \\
&= (d(v_1) + d(w_2)) + (d(v_2) + d(w_3)) + (d(v_3) + d(w_1)) \\
&\geq 3(n + \tfrac{k-1}{2}),
\end{aligned}$$

a contradiction.

Call the two components L and R. Without loss of generality, let $|R| \geq |L|$ and $|L^A| \geq |L^B|$. Let $v \in L^A, w \in L^B, x \in R^A, y \in R^B$. Then

$$|L^A| + |R^A| + |K^A| = |L^B| + |R^B| + |K^B| = n,$$

$$|L^B| + |R^A| + |K| \geq d(w) + d(x) \geq n + \frac{k-1}{2},$$

$$|L^A| + |R^B| + |K| \geq d(v) + d(y) \geq n + \frac{k-1}{2}.$$

Thus, the inequalities above imply the parts of the components are of similar size:

$$|L^A| - |L^B| \leq |K^B| - \frac{k-1}{2},$$

$$|R^A| - |R^B| \leq |K^B| - \frac{k-1}{2},$$

$$|R^B| - |R^A| \leq |K^A| - \frac{k-1}{2}.$$

Further, we get the following bounds for the degrees inside the components:

$$\begin{aligned}
d_R(y) &\geq n + \tfrac{k-1}{2} - d(v) - |K^A| \\
&\geq n + \tfrac{k-1}{2} - |L^B| - |K^B| - |K^A| \\
&= |R^B| - (|K^A| - \tfrac{k-1}{2}), \\
d_R(x) &\geq |R^A| - (|K^B| - \tfrac{k-1}{2}), \\
d_L(w) &\geq |L^B| - (|K^A| - \tfrac{k-1}{2}), \\
d_L(v) &\geq |L^A| - (|K^B| - \tfrac{k-1}{2}).
\end{aligned}$$

Claim 1. *R is k-linked.*

By symmetry of the argument, we may assume that $|R^B| \geq |R^A|$, thus

$$|R^B| \geq \frac{|R|}{2} \geq \frac{2n - |K| - |L|}{2} \geq \frac{n}{2} - \frac{|K|}{4}.$$

Now,

$$\begin{array}{rllll} d_R(y) & \geq & |R^B| - (|K^A| - \frac{k-1}{2}) & \geq & \frac{|R^A|}{2} + \frac{|R^B|}{2} - |K| + \frac{k-1}{2} \\ & \geq & \frac{|R^A|}{2} + \frac{n}{4} - \frac{9|K|}{8} + \frac{k-1}{2} & \geq & \frac{|R^A|}{2} + \frac{103k}{4} - \frac{9(22k-1)}{8} + \frac{k-1}{2} \\ & > & \frac{|R^A|}{2} + \frac{3k}{2}. & & \end{array}$$

Further,

$$d_R(x) \geq |R^A| - (|K^B| - \frac{k-1}{2}) \geq |R^B| - |K| + \frac{k-1}{2} > 2k.$$

Hence, the conditions of Lemma 15 are satisfied for R, and R is k-linked. $\diamond$

If $|K| \geq 2k$, then G is k-linked by Lemma 14 and we are done. So assume from now on $|K| < 2k$.

Claim 2. *L is k-linked.*

If $|L| > n - 2k$, the proof is similar to the last case:

$$d_L(v) \geq |L^A| - |K^B| + \frac{k-1}{2} > \frac{|L^B|}{2} + \frac{n-2k}{4} - 2k + \frac{k-1}{2} > \frac{|L^B|}{2} + \frac{3k}{2},$$

and

$$d_L(w) \geq |L^A| - (|K^B| - \frac{k-1}{2}) > |L^B| - |K| > 2k.$$

Applying Lemma 15 to L gives the result.

If $|L| \leq n - 2k$, L is complete bipartite from the degree sum condition. Further, $|L^A| \geq |L^B| \geq d(v) - |K^B| \geq 2k$ from the minimum degree condition, hence L is k-linked. $\diamond$

Let $S := \{x_1, x_2, \ldots, x_k\}$ be a set in $V(G)$. We want to find a cycle passing through S in the prescribed order. Note that the minimum degree condition forces $|R| \geq |L| \geq |K|$. Assume $|K| = \kappa(G) = k + t$ where $t \geq -1$. Using the fact that K is a minimal cut set, by Hall's Theorem (see for instance [2]) there is a matching of K into L and respectively K into R, which together produce $k + t$ pairwise disjoint P_3's. Of all such matchings, pick one on either side with the fewest intersections with the set S.

Observe that a vertex $s \in K^B$ is either adjacent to every vertex of L^A or $d(s) > n/4$. Otherwise there would be a vertex $v \in L^A$ not connected to s, and $d(v) + d(s) \leq |L^B| + |K^B| + n/4 \leq n/2 - k + 2k + n/4$, a contradiction. A similar argument shows that the analog statement is true for $s \in K^A$, since $|L^A|$ and $|L^B|$ differ by less than $|K| < 2k$. Hence, each vertex $s \in K$ has large degree to at least one of L or R, in fact large enough that either $(L \cup \{s\})$ or $(R \cup \{s\})$ is k-linked.

Assign every vertex of K one by one to either L or R such that the new subgraphs $\bar{L}$ and $\bar{R}$ are still k-linked, applying Lemma 13 repeatedly. Left over from the P_3's is now one matching with $k+t$ edges between $\bar{L}$ and $\bar{R}$. We call an edge of this matching a *double* if both its endvertices are in S and a *single* if exactly one endvertex is in S. If an edge is disjoint from S, we call it *free*.

We claim that the number of doubles is at most t if k is even and at most $t+1$ if k is odd. Let l^A (and respectively r^A) be the number of doubles which are edges between L^A and K^B (respectively between R^A and K^B). Define l^B and r^B similarly. Note that this means $d := l^A + l^B + r^A + r^B$ is the number of doubles. Let $v \in L^A - S, w \in L^B - S, x \in R^A - S$ and $y \in R^B - S$ such that none of those vertices are on an edge of the matching (this is possible since $|L^A| - |K^B| \geq 2k$, $|L^B| - |K^A| \geq 2k$ from the minimum degree condition). Then

$$2n + 2\left\lceil \frac{k-1}{2} \right\rceil \leq d(v) + d(w) + d(x) + d(y) \leq 2n + k + t - l^A - l^B - r^A - r^B.$$

If $d \geq t+1$ for k even or $t+2$ for k odd, we obtain a contradiction to the above inequality.

Let c be the number of elements of S that are not vertices on any of the $k+t$ edges of the matching. Then $t+d+c$ of the edges are free. We are now prepared to construct the cycle containing the set $\{x_1, x_2, \cdots, x_k\}$ by constructing a set of disjoint x_i, x_{i+1}-paths, using that $\bar{L}$ and $\bar{R}$ are k-linked. Note that in constructing each x_i, x_{i+1}-path, using a free edge is only necessary if (1) x_i is not on a single and (2) x_i and x_{i+1} are on different sides. If k is even, these two conditions can occur at most $2d + c$ times. If k is odd, these two conditions can occur at most $2d - 1 + c$ times (because of the parity, condition 2 cannot occur for every vertex). But neither ever exceeds $t + d + c$, the number of free edges. Hence, we may form a cycle containing the elements of S in the appropriate order. □

Proof of Theorem 8: By Theorem 11 it suffices to show that G is k-ordered.

If the minimum degree $\delta(G) \geq 4k-1$, then we are done by Theorem 6. Thus, suppose that $s \in A$ is a vertex with $d(s) < 4k - 1$. Let R be the induced subgraph of G on the following vertex set:

$$R^B := \{v \in B : sv \notin E\},$$
$$R^A := \{w \in A : d_{R^B} \geq 2k\}.$$

The degree sum condition guarantees $d(v) \geq n - 3k$ for all $v \in R^B$. Further, $|R^B| = n - d(s) \geq n - 4k + 2$. It is easy to see that $|R^A| > n - 4k$ and that all the conditions for Lemma 15 are satisfied. Hence, R is k-linked.

Let H be the biggest k-linked subgraph of G. If $G = H$, we are done. Otherwise, let $L := G - H$. The size of L is $|L| = 2n - |H| \leq 2n - |R| \leq 8k$. Observe that no vertex $v \in L$ has $d_H(v) > 2k-2$, otherwise $V(H) \cup \{v\}$ would induce a bigger k-linked subgraph by Lemma 13. Hence, no vertex in L has degree greater than $10k$, and therefore, L is complete bipartite.

Define

$$\alpha := min\{\{d_H(v) | v \in L^A\} \cup \{2k\}\},$$

$$\beta := min\{\{d_H(v)|v \in L^B\} \cup \{2k\}\}.$$

Since L is small, there are vertices $x \in H^A, y \in H^B$, with $N(x) \cup N(y) \subset H$. If $L^A = \emptyset$, then $\alpha = 2k$, and if $L^B = \emptyset$, then $\beta = 2k$. Either way, we get $\alpha + \beta \geq 2k$.

Now assume that $L^A \neq \emptyset$ and $L^B \neq \emptyset$. Let $v \in L^A$ such that $d_H(v) = \alpha$. Then

$$n + k - 2 \leq d(v) + d(y) \leq d(v) + |H^A| = d(v) + n - |L^A|.$$

Thus, $d(v) \geq |L^A| + k - 2$, and

$$|L^B| + \alpha = d(v) \geq |L^A| + k - 2.$$

Analogously, let $w \in L^B$ with $d_H(w) = \beta$, then

$$n + k - 2 \leq d(w) + d(x) \leq d(w) + |H^B| = d(w) + n - |L^B|,$$

and thus $d(w) \geq |L^B| + k - 2$ and

$$|L^A| + \beta = d(w) \geq |L^B| + k - 2.$$

Therefore,

$$\alpha + \beta \geq 2k - 4.$$

Let $S := \{x_1, x_2, \ldots, x_k\}$ be a set in $V(G)$. From now on, all the indices are modulo k. To build the cycle, we need to find paths from x_i to x_{i+1} for all $1 \leq i \leq k$.

If x_i and x_{i+1} are neighbors, just use the connecting edge as path. Now, for all other $x_i \in L$ we find two neighbors y_i and z_i not in S. If x_i and x_{i+i} have a common neighbor v which is not already used, set $z_i = y_{i+1} = v$. Afterwards, we can find distinct y_i and z_i by the following count: Suppose $x_i \in L^A$, so we need to find $y_i, z_i \in N(x_i) - U_i$, where

$$U_i := N(x_i) \cap \{\{x_j, y_j, z_j : |i - j| > 1\} \cup \{z_{i+1}, y_{i-1}\}\}.$$

For every $x_j \in L^A, |i-j| > 1$, there can be at most two vertices in U_i. For $x_j \in L^A, |i-j| = 1$, there can be at most one vertex in U_i. For $x_j \in B, |i - j| > 1$, there can be at most one vertex in U_i. Hence,

$$|U_i| \leq 2|L^A \cap S - \{x_{i-1}, x_i, x_{i+1}\}| + 2 + |B \cap S - \{x_{i-1}, x_i, x_{i+1}\}| \leq |L^A| + k - 4,$$

and since $d(x_i) \geq |L^A| + k - 2$, we can pick y_i and z_i.

Try to choose as few y_i, z_i out of L as possible (i.e. pick as many as possible in H). Now for all y_i, z_j, where $y_i \neq z_{i-1}, z_j \neq y_{j+1}$, choose vertices $y'_i, z'_i \in H$ as follows: If $y_i \in H$, let $y'_i = y_i$, if $z_i \in H$, let $z'_i = z_i$. Otherwise, let y'_i be a neighbor of y_i in H, and let z'_i be a neighbor of z_i in H, which is not already used. We need to check if there is a vertex in $N(y_i) \cap H$ available.

Let $O_i = (N(x_i) \cup N(y_i)) \cap H$. We know that

$$|O_i| = d_H(x_i) + d_H(y_i) \geq \alpha + \beta \geq 2k - 4.$$

For every $j \notin \{i-1, i, i+1\}$, $|O_i \cap \{x_j, y_j, z_j, y'_j, z'_j\}| \leq 2$, and for $j = i+1$, $|O_i \cap \{x_j, y_j, y'_j\}| \leq 1$. This is a total count of at most $2k-5$, at least one is left over for y'_i. Observe that $y'_i \notin N(x_i)$, otherwise we would have chosen it to be y_i, so in fact $y'_i \in N(y_i)$. A similar count shows the availability of a vertex for z'_i, with one possible exception: The one vertex left over could be y'_i. This is only a problem if the count for y'_i gave us exactly one available vertex, otherwise we can just pick a different y'_i. But now we can switch the vertices y_i and z_i, and choose y'_i from $\{x_{i+1}, y_{i+1}, y'_{i+1}\}$ (one of those is in $N(x_i) \cup N(y_i)$, since the count of used vertices gave exactly $2k-5$), and choose z'_i from $\{x_{i-1}, y_{i-1}, y'_{i-1}\}$.

For all $x_i \in H$, set $y'_i = z'_i = x_i$. Since H is k-linked, we can now find z'_i, y'_{i+1}-paths inside H for all needed indices to complete the cycle. □

3 Further Results

We also looked at the following closely related property:

Definition 1. *We say a graph G is* k-ordered connected *if for every sequence $S = (x_1, x_2, ..., x_k)$ of k distinct vertices in G, there exists a path from x_1 to x_k that contains all the vertices of S in the given order. A graph is* k-ordered hamiltonian connected *if there is always a hamiltonian path from x_1 to x_k which encounters S in the designated order.*

Along the lines of the proofs in [4], you can show the following theorems for this property:

Theorem 16. *Let G be a graph of sufficiently large order n. Let $k \geq 3$. If*

$$\delta(G) \geq \frac{n+k-3}{2},$$

then G is k-ordered hamiltonian connected.

Theorem 17. *Let G be a graph of sufficiently large order n. Let $k \geq 3$. If for any two nonadjacent vertices x and y, $d(x) + d(y) \geq n + \frac{3k-6}{2}$, then G is k-ordered hamiltonian connected.*

The proofs do not give any new insights, so we will not present them here.

References

[1] B. Bollobás, C. Thomason, Highly Linked Graphs, *Combinatorica* **16** (1996), no.3, 313–320.

[2] G. Chartrand, L. Lesniak, "Graphs & Digraphs", Chapman and Hall, London, 1996.

[3] L. Ng, M. Schultz, k-Ordered Hamiltonian Graphs, *J. Graph Theory* **1** (1997), 45–57.

[4] J.Faudree, R.Faudree, R.Gould, M.Jacobson, L.Lesniak, On k-Ordered Graphs, *J. Graph Theory* **35** (2000), no.2, 69–82.