TOTAL EDGE IRREGULARITY STRENGTH OF LARGE GRAPHS

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ABSTRACT. Let $m := |E(G)|$ sufficiently large and $s := \lceil \frac{m-1}{3} \rceil$. We show that unless the maximum degree $\Delta > 2s$, there is a weighting $\hat{w} : E \cup V \to \{0, 1, \ldots, s\}$ so that $\hat{w}(uv) + \hat{w}(u) + \hat{w}(v) \neq \hat{w}(u'v') + \hat{w}(u') + \hat{w}(v')$ whenever $uv \neq u'v'$ (such a weighting is called *total edge irregular*). This validates a conjecture by Ivančo and Jendrol' for large graphs, extending a result by Brandt, Miškuf and Rautenbach.

1. Introduction

Let $G = (V, E)$ be a graph. In [1], Bača, Jendrol', Miller and Ryan define the notion of an *edge irregular total s-weighting* as a weighting

$$\hat{w} : E \cup V \to \{1, 2, \ldots, s\}$$

so that

$$\hat{w}(uv) + \hat{w}(u) + \hat{w}(v) \neq \hat{w}(u'v') + \hat{w}(u') + \hat{w}(v')$$

whenever $uv \neq u'v'$ are two different edges of G. They also define the *total edge irregularity strength* as the minimum s for which there exists such a weighting, denoted by $tes(G)$. If we denote by Δ the maximum degree of G and by m the number of edges they note that

$$tes(G) \geq \max\left\{\frac{m+2}{3}, \frac{\Delta+1}{2}\right\}.$$

After some more study of $tes(G)$, Ivančo and Jendrol' conjecture in [6] that this natural lower bound is sharp for all graphs other than the complete graph on 5 vertices (which has $tes(K^5) = 5$), i.e.,

Conjecture 1 (Ivančo and Jendrol' [6]). *For every graph G with $|E(G)| = m$ and maximum degree Δ which is different from K^5,*

$$tes(G) = \max\left\{\left\lceil\frac{m+2}{3}\right\rceil, \left\lceil\frac{\Delta+1}{2}\right\rceil\right\}.$$

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Conjecture 1 has been verified for trees in [6], for complete graphs and complete bipartite graphs by Jendrol', Miškuf and Soták in [7], and for graphs with a bound on Δ by Brandt, Miškuf and Rautenbach in [2] and [3]:

Theorem 2 (Brandt et.al [2] and [3])**.** *For every graph G with $|E(G)| = m$ and maximum degree Δ, where $\lceil \frac{\Delta+1}{2} \rceil \geq \frac{m+2}{3}$ or $\Delta \leq \frac{m}{111000}$,*

$$tes(G) = \max\left\{\left\lceil \frac{m+2}{3} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil\right\}.$$

In this paper, we show the conjecture for all sufficiently large graphs.

Theorem 3. *Let G be a graph with $m := |E(G)| \geq 7 \times 10^{10}$ and maximum degree Δ. Then*

$$tes(G) = \max\left\{\left\lceil \frac{m+2}{3} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil\right\}.$$

The proof of this Theorem will be presented in Section 3. With a similar proof, presented in Section 4, we can improve on Theorem 2 as follows.

Theorem 4. *Let G be a graph with $m := |E(G)|$, and $\Delta(G) \leq \frac{m}{4350}$. Then $tes(G) = \lceil \frac{m+2}{3} \rceil$.*

For notation not defined here, we refer the reader to Diestel's book [5]. In particular, if X and Y are subsets of the vertex set of a graph G and if $E' \subseteq E$ is a subset of its edges, we write $G[X]$ for the induced subgraph of G on X, and we write short $E'(X)$ for the edge set $E' \cap E(G[X])$, and $E'(X,Y)$ for all edges in E' from X to Y.

2. Preliminary Results

By Theorem 2, we only have to consider the case $\lceil \frac{\Delta+1}{2} \rceil < \frac{m+2}{3}$. Without loss of generality we may assume in the following that $m-1$ is divisible by 3, as otherwise we may just add one or two edges (and possibly vertices) and consider the larger graph, only increasing the difficulty of the assignment.

Let $s := \frac{m-1}{3}$ and $w : V \to \{0, 1, \ldots, s\}$ be a vertex weighting. For $e = xy$ we set $w(e) := w(x) + w(y)$. We call w *well guarded* if for all $0 \leq i \leq 2s$,

$$i + 1 \leq |\{e \in E \mid w(e) \leq i\}| \leq i + s + 1.$$

The following fact is immediate as well guarded weightings are easily extended to total edge irregular weightings and vice versa:

Fact 5. *A graph G has a total edge irregular weighting $\hat{w} : E \cup V \to \{1, 2, \ldots, s+1\}$ if and only if G has a well guarded weighting $w : V \to \{0, 1, \ldots, s\}$.*

Thus, we can restrict ourselves to vertex weightings in our quest for total edge irregular weightings. We will call an edge set $E' \subseteq E$ a *guarding set*, if for all $0 \le i \le 2s$,

$$i + 1 \le |\{e \in E' \mid w(e) \le i\}| \le i + s + 1 - |E \setminus E'|.$$

Clearly, w is well guarded if and only if a guarding set exists. The next lemma describes a set up where we can find a guarding set deterministically. In the proof of Theorem 3, we will encounter this set up several times.

Lemma 6. *Let $V(G) = A_1 \uplus A_2 \uplus C$ be a partition of the vertices of a graph G with $3s + 1 = m = |E(G)|$, let $E' = E(G) \setminus E(C)$, and let $\Delta_i := \max_{v \in C} |E(A_i, v)|$ for $i \in \{1, 2\}$. If*

(a) $|E(A_1)| \le |E'| - 2s - \Delta_1$,
(b) $|E(A_2)| \le |E'| - 2s - \Delta_2$,
(c) $|E(A_1, V)| \le |E'| - s + 1 - \Delta_2$,
(d) $|E(A_2, V)| \le |E'| - s + 1 - \Delta_1$,
(e) $\Delta_2 + \frac{s}{|E(A_1,C)| - \Delta_1} \Delta_1 \le s - |E \setminus E'|$.

then there exists a weighting such that E' is a guarding set.

Proof. Let $C = \{x_1, x_2, \ldots x_{|C|}\}$, where the exact order will be determined later. Let $C' = C - x_{|C|}$. Let

$$w(v) = \begin{cases} 0, & \text{for } v \in A_1, \\ s, & \text{for } v \in A_2, \\ \left\lceil \frac{s \cdot |E(A_1, \{x_1, \ldots, x_{i-1}\})|}{|E(A_1, C')|} \right\rceil, & \text{for } v = x_i \in C. \end{cases}$$

Then for $0 \le i < s$, we have

$$\begin{aligned} &|E(A_1)| + \left\lfloor i \cdot \tfrac{|E(A_1,C')|}{s} \right\rfloor + 1 \\ &\qquad\le |\{e \in E' \mid w(e) \le i\}| \\ &\qquad\qquad\le |E(A_1)| + i \cdot \tfrac{|E(A_1,C')|}{s} + \Delta_1, \end{aligned}$$

and therefore

$$i + 1 \le_{\text{(d)}} |\{e \in E' \mid w(e) \le i\}| \le_{\text{(a),(c)}} i + 1 + s - |E \setminus E'|,$$

regardless of the order of the vertices in C.

For $s \le i \le 2s$, we can now find a suitable ordering of C greedily to show the lemma. Pick $x_{|C|}$ first, so that $|E(A_2, x_{|C|})|$ is minimized under the condition that

$$|E(A_2)| + |E(A_2, x_{|C|})| \ge \Delta_2.$$

Now choose the other x_j, starting with an arbitrary x_1, such that for every j,

$$\begin{aligned} s + w(x_j) \le |E(A_1, V)| + |E(A_2, \{x_1, \dots, x_{j-1}\})| \\ \le 2s + 1 - |E \setminus E'| + w(x_j) - \Delta_2. \quad (1) \end{aligned}$$

This is always possible, as this inequality is true for x_1 (by (a) and (c)) and $x_{|C|}$ (by (b)), and at no point in the process there can be remaining $x, x' \in C$ such that setting $x_j = x$ violates the lower inequality, and setting $x_j = x'$ violates the upper inequality by (e). As

$$|\{e \in E' \mid w(e) \le i\}| = |E(A_1, V)| + |E(A_2, \{x_1, \dots, x_j\})|,$$

for $j \le |C|$ maximized such that $w(x_j) \le i - s$, this shows that E' is a guarding set. □

3. Proof of Theorem 3

Let $\varepsilon = 2.7 \times 10^{-5}$, and define the set of large degree vertices

$$B := \{v \in V \mid d(v) > \varepsilon m\}.$$

Then $m \ge |B|\varepsilon m - \frac{|B|^2}{2}$, and therefore $|E(B)| < \frac{|B|^2}{2} < 0.01m$.

Let $V' = V \setminus B$ and $m' = |E \setminus E(B)| > 0.99m$. Further, we partition B into B_0 and B_S as follows: Order the vertices in B by degree from large to small, and assign them in order to the set with fewer edges to V'. Let $e_0 := \frac{1}{m'}|E(B_0, V')|$ and $e_S := \frac{1}{m'}|E(B_S, V')|$. Observe the following fact.

Fact 7. *If $v \in B_0$, then $e_0 - e_S \le \frac{1}{m'}d(v)$.*

We will divide the proof into four cases. For the first three, we assume that $e_0 \ge e_S$.

Case 1. $e_0 \ge 0.52$ *and* $|B_0| = 1$.

Let v_1 be the vertex in B_0, then $d(v_1) = \Delta := \Delta(G) > 0.51m$, and let $H = G[V \setminus v_1]$. Note that in this case, we may assume that $V = \{v_1\} \cup N(v_1)$ (so $|V(H)| = \Delta$ and $|E(H)| = m - \Delta$). Otherwise, as H does not have enough edges to be connected, a vertex $u \in V(H) \setminus N(v_1)$ has distance at least 3 in G to a vertex v in another component of H. We can identify these two vertices and proceed with the smaller graph G', where $|E(G')| = |E(G)|$ and $tes(G') \ge tes(G)$.

Claim 1.1. *There exists* $X' \subseteq V'$ *with*

$$|X'| = \lfloor \tfrac{2}{3}(2|V(H)| - |E(H)|) \rfloor \text{ and } |E(X')| \le \tfrac{1}{2}|X'|.$$

Let $X' \subseteq V'$ with $|X'| = \lfloor \frac{2}{3}(2|V(H)| - |E(H)|) \rfloor$ and $|E(X')|$ minimal, and let $Y' = V' \smallsetminus X'$. If $|E(X')| > \frac{1}{2}|X'|$, then $|E(y, X')| \ge 2$ for all $y \in Y'$, as otherwise we could reduce $|E(X')|$ by a vertex switch. Thus,

$$\begin{aligned} |E(H)| \ge |E(X')| + |E(X', Y')| + |E(B, V)| - \Delta \\ > \frac{1}{2}|X'| + 2|Y'| + \tfrac{1}{2}(|B| - 1)\varepsilon m \\ \ge 2|V(H)| - \frac{3}{2}|X'| \ge |E(H)|, \end{aligned}$$

a contradiction showing the claim.

Claim 1.2. *There exists* $X \subseteq V'$ *with* $|X| \ge s+1$ *and* $|E(X)| \le 2s-\Delta+1$.

Use Claim 1.1 to find a vertex set $X' \subseteq V'$. Successively delete vertices of maximum degree in X' until we have a vertex set $X \subseteq X'$ with $|X| = s + 1$. Then either $|E(X)| = 0$ or

$$\begin{aligned} |E(X)| \le |E(X')| - (|X'| - |X|) \\ \le |X| - \lceil \tfrac{1}{2}|X'| \rceil \le s + 1 - \tfrac{1}{3}(2|V(H)| - |E(H)|) + \tfrac{1}{2} \\ \le s + 1 - (\Delta - s - \tfrac{1}{3}) + \tfrac{1}{2} = 2s - \Delta + \tfrac{11}{6}, \end{aligned}$$

showing the claim.

Now choose X according to Claim 1.2, maximizing $|X|$, and let $Y := V(H) \smallsetminus X$. We want to use Lemma 6 to show that $E' = E \smallsetminus E(X)$ is a guarding set: Let $A_1 = \{v_1\}$, $A_2 = Y$ and $C = X$. Then $\Delta_1 = 1$ and $\Delta_2 \le \varepsilon m$. Conditions (a), (d) and (e) are easily verified.

If (c) fails, say $|E(A_2)| + |E(A_2, C)| = s - 1 + \Delta_2 - \gamma$, note that X contains at least $|X| - s + \gamma$ vertices with no neighbors in Y. If (b) holds, we can use these vertices first in the proof of Lemma 6 until (1) is satisfied, and see that E' is a guarding set.

Finally, assume that (b) fails, i.e.,

$$|E(Y)| > |E'| - 2s - \Delta_2 \ge |E'| - 2s - \varepsilon m.$$

As every vertex in $Y \smallsetminus B$ has at least one neighbor in X by the maximality of $|X|$, we have

$$|Y| \le |E'| - \Delta - |E(Y)| + |B| < 2s - \Delta + \varepsilon m + \tfrac{2}{\varepsilon} \le 2s - \Delta + 3\varepsilon m.$$

Let $Y_1 := \{y \in Y : d(y) \ge 0.01s\}$. Then $0.48m \ge 0.01s|Y_1| - 0.5|Y_1|^2$, so $|Y_1| \le 160$ and $|E(Y_1)| < 0.02s$ as $s > 10^6$. Let $Y_2 = Y \smallsetminus Y_1$. Then more than $s - |E(X,Y)| - |E(Y_1)| > 0.47s > 2s - \Delta$ edges in $E(X \cup Y)$ are incident to Y_2, so we can greedily find some $Y_3 \subseteq Y_2$ with

$$2s - \Delta \leq |E(Y_3, X \cup Y)| < 2s - \Delta + 0.01s.$$

Let $a := |E(Y_3)|$, and $b := |E(Y_3, X \cup Y)| - a$.

Let $X = \{x_0, \dots, x_{|X|-1}\}$ with $|E(Y, x_i)| \leq |E(Y, x_j)|$ for $i \leq j$ and let $w(v_1) = 0$, $w(v) = s - c := \min\{s - b, \lceil \Delta/2 \rceil\}$ for $v \in Y_3$, $w(v) = s$ for $v \in Y \setminus Y_3$, and $w(x_i) = \min\{s, i\}$. We claim that E' is a guarding set for w, settling Case 1.

For $0 \leq i \leq s - 1$, we have

$$\begin{aligned} i + 1 \leq |\{e \in E' \mid w(e) \leq i\}| &\leq i + 1 + |Y_3| \\ &< i + 1 + 2s - \Delta < i - s + \Delta = i + s + 1 - |E \setminus E'|. \end{aligned}$$

For $i = 2s$, we have

$$2s + 1 < \{e \in E' \mid w(e) \leq 2s\}| = |E'| = 3s + 1 - |E \setminus E'|.$$

For $s \leq i \leq 2s - 1$, consider first the lower bound. We have

$$|\{e \in E' \mid w(e) \leq i\}| \geq \left\{ \begin{array}{ll} \Delta, & \text{for } s \leq i < 2s - 2c, \\ \Delta + a, & \text{for } 2s - 2c \leq i < 2s - c, \\ \Delta + a + b, & \text{for } 2s - c \leq i < 2s, \end{array} \right\} \geq s + i + 1.$$

For the sake of analysis of the upper bound, define another weighting w', where $w'(v) = s$ for $v \in Y_3$ and $w' = w$ on all other vertices. Then for $s \leq i < 2s$,

$$\begin{aligned} \Delta \leq |\{e \in E' \mid w'(e) \leq i\}| &\leq \max\{\Delta, i + s - |X| - 1\} \\ &\leq \max\{\Delta, i + 3s - 2\Delta - 2\} < \max\{\Delta, i - 0.06s\} \end{aligned}$$

since $|\{e \in E' \mid w'(e) \leq i\}|$ is maximized if $\max_{x \in X} |E(x, Y)| = 1$.

Any edge $e \in E(X \cup Y)$ with $w'(e) < 2s$ has weight $w(e) \geq w'(e) - c$. Therefore, we have

$$
\begin{aligned}
&|\{e \in E' \mid w(e) \leq i\}| \\
&\quad \leq \begin{Bmatrix} |\{e \in E' \mid w'(e) \leq i + c\}|, & \text{for } s \leq i < 2s - 2c, \\ |\{e \in E' \mid w'(e) \leq i + c\}| + a, & \text{for } 2s - 2c \leq i < 2s - c, \\ |\{e \in E' \mid w'(e) \leq 2s - 1\}| + a + b, & \text{for } 2s - c \leq i < 2s, \end{Bmatrix} \\
&\qquad \leq \begin{Bmatrix} \max\{\Delta, i - 0.06s\}, & \text{for } s \leq i < 2s - 2c, \\ \max\{\Delta, i - 0.06s\} + a, & \text{for } 2s - 2c \leq i < 2s - c, \\ \max\{\Delta, i - 0.06s\} + a + b, & \text{for } 2s - c \leq i < 2s, \end{Bmatrix} \\
&\qquad = \begin{Bmatrix} \Delta, & \text{for } s \leq i < 2s - 2c, \\ \max\{\Delta, i - 0.06s\} + a, & \text{for } 2s - 2c \leq i < 2s - c, \\ \max\{\Delta, i - 0.06s\} + a + b, & \text{for } 2s - c \leq i < 2s, \end{Bmatrix} \\
&\qquad \leq \Delta + i - s = s + i + 1 - |E \setminus E'|.
\end{aligned}
$$

To see the last inequality, note that it is enough to check it for $i \in \{s, 2s - 2c, 2s - c\}$. For $i = 1$, the inequality is trivially true. For $i = 2s - 2c$, we have

$$
\begin{aligned}
\Delta + a < 2.01s - b &< \min\{2\Delta - 1.01s - b, 2\Delta - s\} \\
&\leq \min\{\Delta + s - 2b, 2\Delta - s\} \leq \Delta + i - s.
\end{aligned}
$$

For $i = 2s - c$, we have

$$
\begin{aligned}
\Delta + a + b < 2.01s &< \min\{2\Delta - 1.01s, \lceil \tfrac{3}{2}\Delta \rceil\} \\
&\leq \min\{\Delta + s - b, \lceil \tfrac{3}{2}\Delta \rceil\} = \Delta + i - s,
\end{aligned}
$$

and

$$
i - 0.06s + a + b = 1.94s - c + a + b < 3.95s - \Delta - c \leq \Delta + s - c = \Delta + i - s.
$$

This shows that E' is a guarding set, establishing Case 1.

Case 2. *$e_0 \geq 0.52$ and $|B_0| = 2$.*

Let $B_0 = \{v_2, v_3\}$ and $v_1 \in B_S$ with $d(v_1) = \Delta(G)$. Let $H = G[V \setminus \{v_1, v_2, v_3\}]$. Let $d_i := |E(v_i, V(H))|$ for $1 \leq i \leq 3$, and we may assume that $d_1 \geq d_2 \geq d_3$. Note that $d_2 + d_3 > 0.51m$. Then $|H| \geq d_1$, and $|E(H)| \leq m - d_1 - d_2 - d_3 < 0.24m$.

Claim 2.1. *There is a set $X' \subseteq V'$ such that*

$$
|E(X', \{v_2, v_3\})| \geq \min\{d_2 + d_3 - |B|, \tfrac{4}{3}(d_1 + 2d_2 + 2d_3 - m) - 2\},
$$

and $|E(X')| \leq 0.25|E(X', \{v_2, v_3\})|$.

Let $X' \subseteq V' \cap (N(v_2), N(v_3))$ with

$$\begin{aligned}\min\{d_2 + d_3 - |B|, \tfrac{4}{3}(d_1 + 2d_2 + 2d_3 - m - 2)\} &\le |E(X', \{v_2, v_3\})| \\ &< \tfrac{4}{3}(d(v_1) + 2d(v_2) + 2d(v_3) - m),\end{aligned}$$

such that $|E(X')|$ is minimal. Let $Y' = V' \setminus X'$. If $|E(X')| > 0.25|E(X', \{v_2, v_3\})|$, then $|E(y, X')| \ge |E(y, \{v_2, v_3\})|$ for all $y \in Y'$, as otherwise we could reduce $|E(X')|$ by a vertex switch. Thus,

$$\begin{aligned}m - d_1 - d_2 - d_3 &\ge |E(H)| \ge |E(X')| + |E(X', Y')| + \tfrac{1}{2}(|B| - 3)\varepsilon m \\ &> 0.25|E(X', \{v_2, v_3\})| + |E(Y', \{v_2, v_3\})| + \tfrac{1}{2}(|B| - 3)\varepsilon m \\ &\ge d_2 + d_3 - 0.75|E(X', \{v_2, v_3\})| \\ &> m - d_1 - d_2 - d_3\end{aligned}$$

a contradiction showing the claim.

Claim 2.2. *There is a set $X \subseteq V'$ such that $|E(X, \{v_2, v_3\})| \ge s + 2$, and*

$$|E(X)| \le \max\{0.5s - \tfrac{1}{4}(d_2 + d_3 - |B|) + 1.5, 1.5s - \tfrac{1}{3}(d_1 + 2d_2 + 2d_3) + 2.5\}.$$

Start with a set X' from Claim 2.1, and succesively delete vertices maximizing $\frac{|E(v,X)|}{|E(v,\{v_2,v_3\})|}$ until $|E(X, \{v_2, v_3\})| \le s + 3$. Then either $|E(X)| = 0$ or

$$\begin{aligned}|E(X)| &\le |E(X')| - 0.5(|E(X', \{v_2, v_3\})| - s - 3) \\ &\le -0.25|E(X', \{v_2, v_3\})| + 0.5(s + 3) \\ &\le 0.5s + 1.5 - \min\{\tfrac{1}{4}(d_2 + d_3 - |B|), \tfrac{1}{3}(d_1 + 2d_2 + 2d_3 - m - 2)\} \\ &\le \max\{0.5s - \tfrac{1}{4}(d_2 + d_3 - |B|) + 1.5, 1.5s - \tfrac{1}{3}(d_1 + 2d_2 + 2d_3) + 2.5\},\end{aligned}$$

showing the claim.

We want to apply Lemma 6 to this situation with $A_1 = \{v_2, v_3\}$, $C = X$ for a maximal X, and $A_2 = V \setminus (A_1 \cup C)$. Conditions (a), (d) and (e) are clearly satisfied. For condition (b) note that by the maximality of $|X|$, every vertex in $A_2 \setminus B$ has a neighbor in X, so in particular, $|E(A_2, C)| \ge d_1 - |B| \ge d_1 - 2\varepsilon m$, and so

$$|E(A_1, V)| + |E(A_2, C)| \ge d_1 + d_2 + d_3 - 2 - 2\varepsilon m > 0.7m > 2s + \Delta_2.$$

If condition (c) holds, we are done by Lemma 6. Finally, if condition (c) fails, we have

$$\begin{aligned} d_2 + d_3 &\geq |E(A_1, V)| \\ &> |E'| - s + 1 - \Delta_2 \\ &= 2s + 2 - |E(X)| - \Delta_2 \\ &\geq \min\{1.49s + \tfrac{1}{4}(d_2 + d_3), 0.49s + \tfrac{5}{6}(d_2 + d_3)\}, \end{aligned}$$

and therefore

$$d_2 + d_3 > 1.98s, \; d_1 + d_2 + d_3 > 2.97s, \text{ and } d_3 > 0.96s.$$

Now it is easy to construct a weighting with guarding set $E(\{v_1, v_2, v_3\}, V)$, starting with $w(v_1) = 0$, $w(v_2) = s$ and $w(v_3) = \lceil 0.5s \rceil$.

Case 3. $|B_0| \geq 3$ *and* $e_0 + e_S \geq 0.86$.

Since $|B_0| \geq 3$, we have $e_S \geq \frac{2}{3}e_0$. Thus, this case covers all remaining situations with $e_0 \geq 0.52$. Let $A_1 = B_0$, $A_2 = B_S$, and $C = V \setminus (A_1 \cup A_2)$. Then $|E'| \geq (e_0+e_S)m' > 2.52s$ and $\Delta_i < \varepsilon m$. All conditions of Lemma 6 apply but possibly (c). If (c) fails, we have

$$s - 1 + \Delta_2 > |E(B_S)| + |E(B_S, C)| \geq e_s m' > 0.344m' > s + \Delta_2,$$

a contradiction finishing the case.

For the last case, we will drop the assumption of $e_0 \geq e_S$ to be able to use symmetry in a different place.

Case 4. $\max\{e_0, e_S\} \leq 0.52$.

Let $w(v) = 0$ for $v \in B_0$, $w(v) = s$ for $v \in B_S$, and determine $w(v)$ for all other vertices independently at random with

$$P_i := \mathbb{P}\left(w(v) = s\frac{i}{20}\right) = \begin{cases} (19 - 30e_0)\beta^{-1} & i = 0 \\ \beta^{-1} & 0 < i < 20 \\ (19 - 30e_S)\beta^{-1} & i = 20 \\ 0 & \text{otherwise} \end{cases},$$

where $\beta = 57 - 30(e_0 + e_S)$ and $i \in \mathbb{Z}$.

The set $E' = E \setminus E(B)$ is guarding for the resulting weighting, if for $0 \leq i \leq 39$,

$$\tfrac{i+1}{20}s \leq |\{e \in E' \mid w(e) \leq s\tfrac{i}{20}\}| \leq \tfrac{i}{20}s + s + 1 - |E(B)|.$$

To show that E' has a positive probability of being a guarding set, we will use Azuma's inequality. For this, let us first consider the expected number of edges of the particular weights

$$X_i := |\{e \in E' \mid w(e) \leq s\tfrac{i}{20}\}|$$

and find values $\delta_i, \hat{\delta}_i \in (0, 0.01)$ such that for $0 \le i \le 39$,

$$\mathbb{E}(X_i) \ge \begin{cases} \frac{i+1}{20}s + \delta_i m', & \text{if } e_0 \ge e_S, \\ \frac{i+1}{20}s + \hat{\delta}_i m', & \text{if } e_0 \le e_S, \\ \frac{i+1}{20}s + 0.2m', & \text{if } 20 \le i \le 39, \end{cases} \tag{2}$$

and

$$\mathbb{E}(X_i) \le \begin{cases} \frac{i}{20}s + s + 1 - |E(B)| - \delta_i m', & \text{if } e_0 \ge e_S, \\ \frac{i}{20}s + s + 1 - |E(B)| - \hat{\delta}_i m', & \text{if } e_0 \le e_S, \\ \frac{i}{20}s + s + 1 - |E(B)| - 0.2m', & \text{if } 0 \le i \le 19. \end{cases} \tag{3}$$

By symmetry (change the sets B_0 and B_S), we can set $\delta_i = \hat{\delta}_{39-i}$, so we only have to treat the cases $0 \le i \le 19$. For every edge $uv \in E'$, we get

$$\mathbb{P}(w(uv) = s\tfrac{i}{20}) = \sum P_k P_{i-k} = 2P_0 P_i + (i-1)P_i^2.$$

For $uv \in E'(B_0, V)$, we get $\mathbb{P}(w(uv) = s\frac{i}{20}) = P_i$, and for $uv \in E'(B_S, V)$, we get $\mathbb{P}(w(uv) = s\frac{i}{20}) = P_{i-20} = 0$. Thus,

$$\begin{aligned} \tfrac{1}{m'}\mathbb{E}(X_i) &= \sum_{j=0}^{i} (e_0 P_j + (1 - e_0 - e_S)(2P_0 P_j + (j-1)P_j^2)) \\ &= e_0(P_0 + iP_1) + (1 - e_0 - e_S)(P_0^2 + 2iP_0P_1 + \tfrac{i^2 - i}{2}P_1^2). \end{aligned}$$

Fixing i and taking the partial derivatives, your favorite computer algebra program tells you that $\frac{d}{de_0}\mathbb{E}(X_i) = 0$ if and only if

$$e_0 = f_1(e_S) := -\frac{p_0 + p_1 e_S + p_2 e_S^2}{p_3 + p_4 e_S},$$

and $\frac{d}{de_S}\mathbb{E}(X_i) = 0$ if and only if

$$e_0 = f_2(e_S) := \frac{p_5 + p_6 e_S - \sqrt{p_7 + p_8 e_S + p_9 e_S^2}}{p_{10}},$$

where the p_j are polynomials in i (see Appendix B).

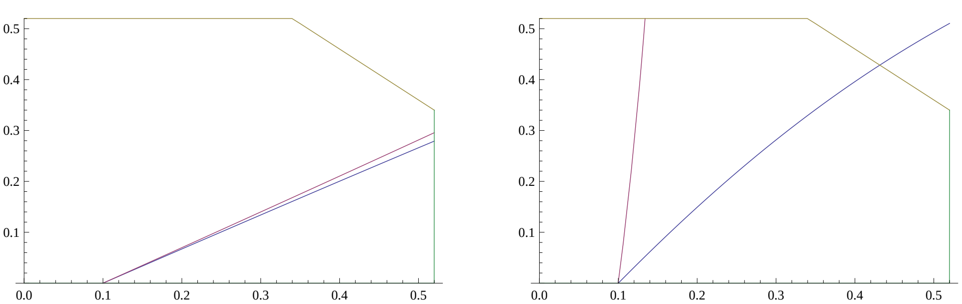

FIGURE 1. Plots of f_1 and f_2 for $i = 3$ and $i = 19$

We have $f_1 \geq f_2$ and

$$(\tfrac{d}{de_0}, \tfrac{d}{de_S})\mathbb{E}(X_i) = \begin{cases} (<0, >0), & \text{for } e_0 > f_1(e_S), \\ (>0, >0), & \text{for } f_2(e_S) < e_0 < f_1(e_S), \\ (>0, <0), & \text{for } e_0 < f_2(e_S). \end{cases}$$

We conclude that the minimum of $\mathbb{E}(X_i)$ in the considered area with $e_0 \leq e_S$ occurs in $(e_0, e_S) = (0, 0.52)$ or on the line $e_0 = e_S$, and similarly, the minimum of $\mathbb{E}(X_i)$ in the considered area with $e_0 \geq e_S$ occurs in $(e_0, e_S) = (0.52, 0)$ or on the line $e_0 = e_S$. On this line, i.e., $0 \leq e_0 = e_S \leq 0.43$, the minimum is attained at

$$e_0 = e_S = \begin{cases} 0.43, & \text{for } 0 \leq i \leq 6, \\ 0, & \text{for } 7 \leq i \leq 19. \end{cases}$$

Similarly, the maximum on the line $e_0 = 0.86 - e_S$ is an upper bound for the maximum in the considered area, and this maximum is attained at

$$(e_0, e_S) = \left(\tfrac{361+19i}{1350}, \tfrac{800-19i}{1350}\right).$$

Computing the four values for each i with $0 \leq i \leq 19$, we find that (see Appendix A) we can choose δ_i and $\hat{\delta}_i$ as follows:

i	0	1	2	3	4	5	6	7	8	9
$1000\ \delta_i$	29	26	24	22	21	20	19	18	17	17
$1000\ \delta_{i+10}$	18	18	19	20	21	22	24	26	29	31
$1000\ \hat{\delta}_i$	72	71	70	66	61	56	51	46	42	38
$1000\ \hat{\delta}_{i+10}$	34	31	28	25	23	20	18	17	15	14

satisfying (2) and (3). For this, note that (see Appendix A)

$$\mathbb{E}(X_i) \leq \tfrac{i}{20}s + s - 0.22m' < \tfrac{i}{20}s + s - |E(B)| - 0.2m'.$$

Now we are ready to use Azuma's inequality (cf. [8]):

Theorem 8. *(Azuma's inequality) Let X be a random variable determined by n trials $T_1, \ldots, T_n$, such that for each j, and any two possible sequences of outcomes $t_1, \ldots, t_j$ and $t_1, \ldots, t_{j-1}, t'_j$:*

$$|\mathbb{E}(X|T_1 = t_1, \ldots, T_j = t_j) - \mathbb{E}(X|T_1 = t_1, \ldots, T_j = t'_j)| \leq c_j,$$

then for all $\bar{t}, \underline{t} > 0$

$$\mathbb{P}(X - \mathbb{E}(X) \geq \bar{t}) + \mathbb{P}(\mathbb{E}(X) - X \geq \underline{t}) \leq e^{-\bar{t}^2/(2\sum c_j^2)} + e^{-\underline{t}^2/(2\sum c_j^2)}.$$

In our application, T_j is the weight of the j^{th} vertex in V', and $X = X_i$. As the weight of one vertex in $v \in V'$ changes the value (and thus the expectation) of an X_i by at most $d(v) \le \varepsilon m$, we have

$$\begin{aligned}\mathbb{P}(X_i - \mathbb{E}(X_i) \ge 0.2m') + \mathbb{P}(\mathbb{E}(X_i) - X_i \ge tm') \\ \le e^{-(0.2m')^2/(2\sum_{v\in V'} d(v)^2)} + e^{-(tm')^2/(2\sum_{v\in V'} d(v)^2)} \\ \le e^{-(0.2m')^2/(2\varepsilon m \sum_{v\in V'} d(v))} + e^{-(tm')^2/(2\varepsilon m\sum_{v\in V'} d(v))} \\ \le e^{-(0.04m')(0.99m)/4\varepsilon mm'} + e^{-(t^2m')(0.99m)/4\varepsilon mm'} \\ = e^{-0.0099/\varepsilon} + e^{-0.99t^2/4\varepsilon}.\end{aligned}$$

Thus, as $\varepsilon = 2.7 \times 10^{-5}$,

$$\begin{aligned}\mathbb{P}(\text{inequality (2) or (3) fails for some } i) \\ \le 40e^{-0.0099/\varepsilon} + \sum_{i=0}^{19}\left(e^{-0.99\delta_i^2/4\varepsilon} + e^{-0.99\hat{\delta}_i^2/4\varepsilon}\right) < 1.\end{aligned}$$

Therefore, there is a choice of the T_j such that none of the X_i falls out of the given range. This yields a well guarded vertex weighting. □

4. Graphs with small maximum degree

With the same methods as above we can improve on the bound in Theorem 2 as stated in Theorem 4. Here we give only a proof sketch.

Proof. Let $\varepsilon = 2.3 \times 10^{-4} > \frac{1}{4350}$. We proceed as in Case 4 of the proof of Theorem 3 and note that we have $B = \varnothing$, $e_0 = e_S = 0$ and $m = m'$. This yields with the same calculations as above, that we can choose $\delta_i = \hat{\delta}_i$ as follows:

i	0	1	2	3	4	5	6	7	8	9
1000 δ_i	94	89	84	80	76	72	69	66	63	60
1000 δ_{i+10}	58	56	55	53	52	52	51	51	52	52

satisfying (2) and (3). The same calculation as above involving Azuma's inequality yields the theorem. □

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Appendix A. Expected numbers of edges with certain weights

Let $e^* = \frac{-361+18i+i^2}{20(152-36i+i^2)}$ and $\bar{e} = \frac{361+19i}{1350}$. For given (e_0, e_S), let $Y_i := \frac{1}{m'} X_i$. Then we calculate the following values for $Y_i - \frac{i+1}{60\times 0.99}$ (and in the last column for $\frac{i+20}{60} - Y_i$):

i	(e_0, e_S)					
	$(0, 0.52)$	$(0, 0)$	(e^*, e^*)	$(0.43, 0.43)$	$(0.52, 0)$	$(\bar{e}, 0.86 - \bar{e})$
0	0.0842642	0.0942761	0.0945847	**0.072**5870	**0.029**1077	0.221914
1	0.0780712	0.0891370	0.0894879	**0.071**2887	**0.026**7374	0.226687
2	0.0721583	0.0843057	0.0847193	**0.070**1341	**0.024**6472	0.230987
3	**0.066**5254	0.0797821	0.0803057	0.0691234	**0.022**8371	0.234814
4	**0.061**1725	0.0755664	0.0763530	0.0682565	**0.021**3070	0.238167
5	**0.056**0997	0.0716584	0.0741222	0.0675334	**0.020**0569	0.241046
6	**0.051**3070	0.0680582	0.0669080	0.0669542	**0.019**0869	0.243452
7	**0.046**7943	0.0647658	0.0644259	0.0665187	**0.018**3969	0.245385
8	**0.042**5617	0.0617812	0.0616330	0.0662271	**0.017**9871	0.246844
9	**0.038**6092	0.0591044	0.0590379	0.0660793	**0.017**8572	0.247830
10	**0.034**9366	0.0567354	0.0567098	0.0660753	**0.018**0074	0.248342
11	**0.031**5442	0.0546742	0.0546683	0.0662151	**0.018**4377	0.248381
12	**0.028**4318	0.0529207	0.0529207	0.0664988	**0.019**1480	0.247946
13	**0.025**5994	0.0514750	0.0514703	0.0669263	**0.020**1384	0.247038
14	**0.023**0471	0.0503372	0.0503181	0.0674976	**0.021**4088	0.245657
15	**0.020**7749	0.0495071	0.0494643	0.0682127	**0.022**9593	0.243802
16	**0.018**7827	0.0489848	0.0489084	0.0690716	**0.024**7898	0.241473
17	**0.017**0705	0.0487703	0.0486493	0.0700744	**0.026**9004	0.238671
18	**0.015**6384	0.0488635	0.0486852	0.0712209	**0.029**2910	0.235396
19	**0.014**4864	0.0492646	0.0490139	0.0725113	**0.031**9617	0.231647

Appendix B. Polynomials used in Case 4 of the proof

$$
\begin{aligned}
p_0(i) &= 1444 + 39i + i^2\\
p_1(i) &= 10\ (-1558 - 45i + i^2)\\
p_2(i) &= 600\ (19 + i)\\
p_3(i) &= 10\ (2660 - 147i + i^2)\\
p_4(i) &= 600\ (-35 + i)\\
p_5(i) &= 10\ (1216 + 27i - i^2)\\
p_6(i) &= 600\ (19 + i)\\
p_7(i) &= 100\ (2085136 + 111336i - 2663i^2 - 78i^3 + i^4)\\
p_8(i) &= 12000\ (-27436 - 2077i + 88i^2 + i^3)\\
p_9(i) &= 360000\ (361 + 38i + i^2)\\
p_{10}(i) &= 1200\ (35 - i)
\end{aligned}
$$