Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs

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Abstract

A *rainbow matching* in an edge-colored graph is a matching in which all the edges have distinct colors. Wang asked if there is a function $f(\delta)$ such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must have a rainbow matching of size δ. We answer this question in the affirmative; $f(\delta) = 6.5\delta$ suffices. Furthermore, the proof provides a $O(\delta(G)|V(G)|^2)$-time algorithm that generates such a matching.

Keywords: Rainbow matching, properly edge-colored graphs

1 Introduction

All graphs under consideration in this paper are simple, and we let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a graph G, respectively. A *rainbow subgraph* in an edge-colored graph is a subgraph in which all edges have distinct colors. Rainbow matchings are of particular interest given their connection to transversals of Latin squares: each Latin square can be converted to a properly edge-colored complete bipartite graph, and a transversal of the Latin square is a perfect rainbow matching in the graph. Ryser's conjecture [2] that every Latin square of odd order has a transversal can be seen as the beginning of the study of rainbow matchings. Stein [5] later conjectured that every Latin square of order n has a transversal of size $n-1$; equivalently every properly edge-colored $K_{n,n}$ has a rainbow

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matching of size $n-1$. The connection between Latin transversals and rainbow matchings in $K_{n,n}$ has inspired additional interest in the study of rainbow matchings in triangle-free graphs.

Several results have been attained for rainbow matchings in arbitrarily edge-colored graphs. The *color degree* of a vertex v in an edge-colored graph G, written $\hat{d}(v)$, is the number of different colors on edges incident to v. We let $\hat{\delta}(G)$ denote the minimum color degree among the vertices in G. Wang and Li [7] proved that every edge-colored graph G contains a rainbow matching of size at least $\left\lceil \frac{5\hat{\delta}(G)-3}{12} \right\rceil$, and conjectured that $\left\lceil \hat{\delta}(G)/2 \right\rceil$ could be guaranteed when $\hat{\delta}(G) \geq 4$. LeSaulnier et al. [4] then proved that every edge-colored graph G contains a rainbow matching of size $\left\lfloor \hat{\delta}(G)/2 \right\rfloor$. Finally, Kostochka and Yancey [3] proved the conjecture of Wang and Li in full, and also that triangle-free graphs have rainbow matchings of size $\left\lceil \frac{2\hat{\delta}(G)}{3} \right\rceil$.

Since the edge-colored graphs generated by Latin squares are properly edge-colored, it is of interest to consider rainbow matchings in properly edge-colored graphs. In this direction, LeSaulnier et al. proved that a properly edge-colored graph G satisfying $|V(G)| \neq \delta(G)+2$ that is not K_4 has a rainbow matching of size $\lceil \delta(G)/2 \rceil$. Wang then asked if there is a function f such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must contain a rainbow matching of size δ [6]. As a first step towards answering this question, Wang showed that a graph G with order at least $\frac{8\delta}{5}$ has a rainbow matching of size $\left\lfloor \frac{3\delta(G)}{5} \right\rfloor$.

In this paper we answer Wang's question from [6] in the affirmative.

Theorem 1. *If G is a properly edge-colored graph satisfying $|V(G)| > \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$, then G contains a rainbow matching of size $\delta(G)$.*

If G is triangle-free, a smaller order suffices.

Theorem 2. *If G is a triangle-free properly edge-colored graph satisfying $|V(G)| > \frac{49}{8}\delta - \frac{21}{2} + \frac{9}{2\delta}$, then G contains a rainbow matching of size $\delta(G)$.*

The proofs of Theorems 1 and 2 depend on the implementation of a greedy algorithm, a significantly different approach than those found in [3], [4], [6], and [7]. This algorithm generates a rainbow matching in a properly edge-colored graph G in $O(\delta(G)|V(G)|^2)$-time.

Since there are $n \times n$ Latin squares with no transversals (see [1]) when n is even, it is clear that $f(\delta) > 2\delta$ when δ is even. Furthermore, since maximum matchings in $K_{\delta,n-\delta}$ have only δ edges (provided $n \geq 2\delta$), there is no function for the order of G depending on $\delta(G)$ that can guarantee a rainbow matching of size greater than δ.

2 Proof of the Main Results

Proof of Theorem 1. We proceed by induction on $\delta(G)$. The result is trivial if $\delta(G) = 1$. We assume that G is a graph with minimum degree δ and order greater than $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$.

Lemma 3. *If G satisfies $\Delta(G) > 3\delta - 3$, then G has a rainbow matching of size δ.*

Proof. Let v be a vertex of maximum degree in G. By induction, there is a rainbow matching M of size $\delta - 1$ in $G - v$. Since v is incident to at least $3\delta - 2$ edges with distinct colors, there is an edge incident to v that is not incident to any edge in M and also has a color that does not appear in M. Thus there is a rainbow matching of size δ in G. $\square$

Lemma 4. *If G has a color class containing at least $2\delta - 1$ edges, then G has a rainbow matching of size δ.*

Proof. Let C be a color class with at least $2\delta - 1$ edges. By induction, there is a rainbow matching M of size $\delta - 1$ in $G - C$. There are $2\delta - 2$ vertices covered by the edges in M, thus one of the edges in C has no endpoint covered by M, and the matching can be extended. $\square$

The proof of Theorem 1 relies on the implementation of a greedy algorithm. We begin by preprocessing the graph so that each edge is incident to at least one vertex with degree δ. To achieve this, we order the edges in G and process them in order. If both endpoints of an edge have degree greater than δ when it is processed, delete that edge. In the resulting graph, every edge is incident to a vertex with degree δ. Furthermore, by Lemma 3 we may assume that $\Delta(G) \leq 3\delta - 3$; thus the degree sum of the endpoints of any edge is bounded above by $4\delta - 3$. After preprocessing, we begin the greedy algorithm.

In the ith step of the algorithm, a smallest color class is chosen (without loss of generality, color i), and then an edge e_i of color i is chosen such that the degree sum of the endpoints is minimum. All the remaining edges of color i and all edges incident with an endpoint of e_i are deleted. The algorithm terminates when there are no edges in the graph.

We assume that the algorithm fails to produce a matching of size δ in G; suppose that the rainbow matching M generated by the algorithm has size k. We let R denote the set of vertices that are not covered by M.

Let c_i denote the size of the smallest color class at step i. Since at most two edges of color $i + 1$ are deleted in step i (one at each endpoint of e_i), we observe that $c_{i+1} + 2 \geq c_i$. Otherwise, at step i color class $i + 1$ has fewer edges. Let step h be the last step in the algorithm in which a color class that does not appear in M is completely removed from G.

It then follows that $c_h \leq 2$, and in general $c_i \leq 2(h-i+1)$ for $i \in [h]$. Let f_i denote the number of edges of color i deleted in step i with both endpoints in R. Since $f_i < c_i$, we have $f_i \leq 2(h-i)+1$ for $i \in [h]$. Note that after step h, there are exactly $k-h$ colors remaining in G. By Lemma 4, color classes contain at most $2\delta - 2$ edges, and therefore the last $k-h$ steps remove at most $(k-h)(2\delta-2)$ edges. Furthermore, for $i > h$, the degree sum of the endpoints of e_i is at most $2(\delta-1)$.

For $i \in [h]$, let x_i and y_i be the endpoints of e_i, and let $d_i(v)$ denote the degree of a vertex v at the beginning of step i. Let $\mu_i = \max\{0, d_i(x_i) + d_i(y_i) - 2\delta\}$; note that $2\delta \leq 2\delta + \mu_i \leq 4\delta - 3$. Thus, at step i, at most $2\delta + \mu_i + f_i - 1$ edges are removed from the graph. Since the algorithm removes every edge from the graph, we conclude that

$$|E(G)| \leq (k-h)(2\delta-2) + \sum_{i=1}^{h}(2\delta + \mu_i + f_i - 1). \tag{1}$$

We now compute a lower bound for the number of edges in G. Since the degree sum of the endpoints of e_i in G is at least $2\delta + \mu_i$, we immediately obtain the following inequality:

$$\frac{n\delta + \sum_{i \in [h]} \mu_i}{2} \leq |E(G)|.$$

If $f_i > 0$ and $\mu_i > 0$, then there is an edge with color i having both endpoints in R. Since this edge was not chosen in step i by the algorithm, the degree sum of its endpoints is at least $2\delta + \mu_i$, and one of its endpoints has degree at least $\delta + \mu_i$. For each value of i satisfying $f_i > 0$, we wish to choose a representative vertex in R with degree at least $\delta + \mu_i$. Since there are f_i edges with color i with both endpoints in R, there are f_i possible representatives for color i. Since a vertex in R with high degree may be the representative for multiple colors, we wish to select the largest system of distinct representatives.

Suppose that the largest system of distinct representatives has size t, and let T be the set of indices of the colors that have representatives. For each color $i \in T$ there is a distinct vertex in R with degree at least $\delta + \mu_i$. Thus we may increase the edge count of G as follows:

$$\frac{n\delta + \sum_{i \in [h]} \mu_i + \sum_{i \in T} \mu_i}{2} \leq |E(G)|. \tag{2}$$

We let $\{f_i^{\downarrow}\}$ denote the sequence $\{f_i\}_{i \in [h]}$ sorted in nonincreasing order. Since $f_i \leq 2(h-i)+1$, we conclude that $f_i^{\downarrow} \leq 2(h-i)+1$. Because there is no system of distinct representatives of size $t+1$, the sequence $\{f_i^{\downarrow}\}$ cannot majorize the sequence $\{t+1, t, t-1, \ldots, 1\}$. Hence there is a smallest value $p \in [t+1]$ such that $f_p^{\downarrow} \leq t+1-p$. Therefore, the maximum value of $\sum_{i=1}^{h} f_i^{\downarrow}$ is bounded by the sum of the sequence $\{2h-1, 2h-3, \ldots, 2(h-p)+3, t+1-p, \ldots, t+1-p\}$. Summing we attain

$$\sum_{i\in[h]} f_i \leq (p-1)(2h-p+1) + (h-p+1)(t+1-p).$$

Over p, this value is maximized when $p = t+1$, yielding $\sum_{i\in[h]} f_i \leq t(2h-t)$. Since $h \leq \delta-1$, we then have $\sum_{i\in[h]} f_i \leq 2(\delta-1)t - t^2$.

We now combine bounds (1) and (2):

$$\frac{n\delta + \sum_{i\in[h]}\mu_i + \sum_{i\in T}\mu_i}{2} \leq (k-h)(2\delta-2) + \sum_{i=1}^{h}(2\delta + \mu_i + f_i - 1).$$

Hence, since $k \leq \delta - 1$,

$$\begin{aligned}
\frac{n\delta}{2} &\leq (2\delta-1)(\delta-1) + \frac{1}{2}\sum_{[h]\setminus T}\mu_i + \sum_{i\in[h]} f_i \\
&\leq (2\delta-1)(\delta-1) + (\delta-1-t)(\delta-3/2) + 2(\delta-1)t - t^2 \\
&\leq 3\delta^2 - \frac{11}{2}\delta + \frac{5}{2} + \left(\delta - \frac{1}{2}\right)t - t^2.
\end{aligned}$$

This bound is maximized when $t = (\delta - \frac{1}{2})/2$. Thus

$$n \leq \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta},$$

contradicting our choice for the order of G. □

Sketch of Proof of Theorem 2. When G is triangle-free, Lemma 3 can be improved. In particular, $\Delta(G) \leq 2\delta - 2$ since there is at most one edge joining a vertex of maximum degree to each edge in a matching of size $\delta - 1$. Since $\Delta(G)$ is used to bound the value of μ_i in the proof of Theorem 1, the same argument yields the following inequality:

$$\begin{aligned}
\frac{n\delta}{2} &\leq (2\delta-1)(\delta-1) + \frac{1}{2}\sum_{[h]\setminus T}\mu_i + \sum_{i\in[h]} f_i \\
&\leq (2\delta-1)(\delta-1) + \frac{1}{2}(\delta-1-t)(\delta-2) + 2(\delta-1)t - t^2 \\
&\leq \frac{5}{2}\delta^2 - \frac{9}{2}\delta + 2 + \left(\frac{3}{2}\delta - 1\right)t - t^2.
\end{aligned}$$

This upper bound is maximized when $t = (\frac{3}{2}\delta - 1)/2$, yielding

$$n \leq \frac{49}{8}\delta - \frac{21}{2} + \frac{9}{2\delta}.$$

□

3 Conclusion

The proof of Theorem 1 provides the framework of a $O(\delta(G)|V(G)|^2)$-time algorithm that generates a rainbow matching of size $\delta(G)$ in a properly edge-colored graph G. Given such a G, we create a sequence of graphs $\{G_i\}$ as follows, letting $G = G_0$, $\delta = \delta(G)$, and $n = |V(G)|$. First, determine $\delta(G_i)$, $\Delta(G_i)$, and the maximum size of a color class in G_i; this process takes $O(n^2)$-time. If $\Delta(G_i) \leq 3\delta(G_i)-3$ and the maximum color class has at most $2\delta(G_i)-2$ edges, then terminate the sequence and set $G_i = G'$. If $\Delta(G_i) > 3\delta(G_i) - 3$, then delete a vertex v of maximum degree and then process the edges of $G_i - v$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1}. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ but a maximum color class C has at least $2\delta(G_i) - 1$ edges, then delete C and then process the edges of $G_i - C$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1}. Note that $\delta(G_{i+1}) = \delta(G_i) - 1$. If this process generates G_δ, we set $G' = G_\delta$ and terminate. Generating the sequence $\{G_i\}$ consists of at most δ steps, each taking $O(n^2)$-time.

Given that $G' = G_i$, the algorithm from the proof of Theorem 1 takes $O(\delta n^2)$-time to generate a matching of size $\delta-i$ in G'. The preprocessing step and the process of determining a smallest color class and choosing an edge in that class whose endpoints have minimum degree sum both take $O(n^2)$-time. This process is repeated at most δ times.

A matching of size $\delta - (i+1)$ in G_{i+1} is easily extended in G_i to a matching of size $\delta - i$ using the vertex of maximum degree or maximum color class. The process of extending the matching takes $O(\delta)$-time. Thus the total run-time of the algorithm generating the rainbow matching of size δ in G is $O(\delta n^2)$.

It is worth noting that the analysis of the greedy algorithm used in the proof of Theorem 1 could be improved. In particular, the bound $c_{i+1} \geq c_i - 2$ is sharp only if at step i there are an equal number of edges of color i and $i+1$ and both endpoints of e_i are incident to edges with color $i+1$. However, since one of the endpoints of e_i has degree at most δ, at most $\delta - 1$ color classes can lose two edges in step i. Since the maximum size of a color class in G is at most $2\delta - 2$, if G has order at least 6δ, then there are at least $3\delta/2$ color classes. Thus, for small values of i, the bound $c_i \leq 2(k - i + 1)$ can likely be improved. However, we doubt that such analysis of this algorithm can be improved to yield a bound on $|V(G)|$ better than 6δ.

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