PANCYCLICITY OF 3-CONNECTED GRAPHS: PAIRS OF FORBIDDEN SUBGRAPHS

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Abstract. We characterize all pairs of connected graphs $\{X, Y\}$ such that each 3-connected $\{X, Y\}$-free graph is pancyclic. In particular, we show that if each of the graphs in such a pair $\{X, Y\}$ has at least four vertices, then one of them is the claw $K_{1,3}$, while the other is a subgraph of one of six specified graphs.

1. Introduction

A graph G on n vertices is pancyclic if for each k, $3 \leq k \leq n$, a cycle of length k can be found in G. We say that G is $\{H_1, \ldots, H_\ell\}$-free, if it contains no induced copies of any of the graphs $H_1, \ldots, H_\ell$. For all terms not defined here we refer the reader to [1]. The problem of characterizing all families of $H_1, \ldots, H_\ell$ such that each "sufficiently connected" $\{H_1, \ldots, H_\ell\}$-free graph is pancyclic has been studied by a number of authors. In particular, the family of all pairs of graphs X, Y, such that each 2-connected $\{X, Y\}$-free graph $G \neq C_n$ on $n \geq 10$ vertices is pancyclic, has been characterized by Faudree and Gould in [2] (we refer the reader to this paper for further references to this problem). In this paper we characterize all graphs X, Y such that each 3-connected $\{X, Y\}$-free graph is pancyclic.

For any graph H, let $S(H)$ be the graph obtained from H through subdivision of every edge. Let $L(H)$ be the line graph of H.

Let $G_0 = L(S(K_4))$. Let G_1 be the graph obtained from G_0 by contraction of the two edges $x_1x_2, x_3x_4 \in E(G_0)$, where the edges x_1x_2 and x_3x_4 are selected in a way that $N(x_i) \cap N(x_j) = \emptyset$ for $1 \leq i < j \leq 4$ (see Figure 2). It is not hard to see that both G_0 and G_1 are 3-connected claw-free graphs. Furthermore, neither of them contains a cycle of length four.

Let $S_3(K_4)$ be the graph obtained from K_4 by a subdivision of each edge by three vertices of degree 2. Let H be the multigraph obtained from $S_3(K_4)$ by doubling each edge of $S_3(K_4)$ incident with a vertex

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of degree 3. Finally, let $G_2 = L(H)$. Alternatively, one can obtain G_2 through a replacement of each triangle of G_0 by the 9-vertex graph T pictured in Figure 1. Again, it is easy to see that G_2 is 3-connected, claw-free, and it contains no cycle of length $10 \le \ell \le 11$. Further, G_2 contains no induced cycles of length $4 \le \ell \le 9$.

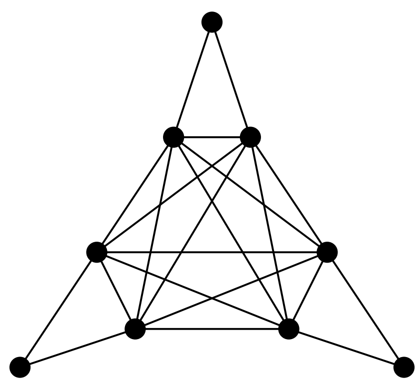

FIGURE 1. The graph T

By G_3 we denote the graph consisting of a K_{n-4} ($n \ge 7$) and four extra vertices x_1, x_2, x_3, x_4 with $N(x_1) = N(x_2) = N(x_3) = N(x_4)$ and $|N(x_1)| = 3$ (see Figure 2). Clearly, G_3 is 3-connected and not hamiltonian (and thus not pancyclic). Finally, G_4 is the point-line incidence graph of a projective plane of order seven, i.e., the vertices of G_4 correspond to the points and the lines of the plane, and two of them, v and w, are adjacent if v stands for a point and w for a line containing it. It is easy to check that G_4 is 3-connected, has girth six, and is thus not pancyclic.

Theorem 1. *For every connected graph X, $X \notin \{K_1, K_2\}$, the following two statements are equivalent:*

(i) *each X-free 3-connected graph G is pancyclic;*
(ii) $X = P_3$.

Proof. Any P_3-free connected graph is complete and therefore pancyclic. Thus, it is enough to show that (i) implies (ii).

As $K_{3,3}$ and the graph G_1 are not pancyclic, an induced copy of X must be contained in both $K_{3,3}$ and G_1. As G_1 does not contain a copy of C_4, X cannot contain a copy of C_4. As any induced subgraph of $K_{3,3}$ with diameter greater than two contains C_4, we know that X is a star $K_{1,r}$. As there are no induced copies of $K_{1,r}$ with $r \ge 3$ in G_1, we infer that $X = P_3$. □

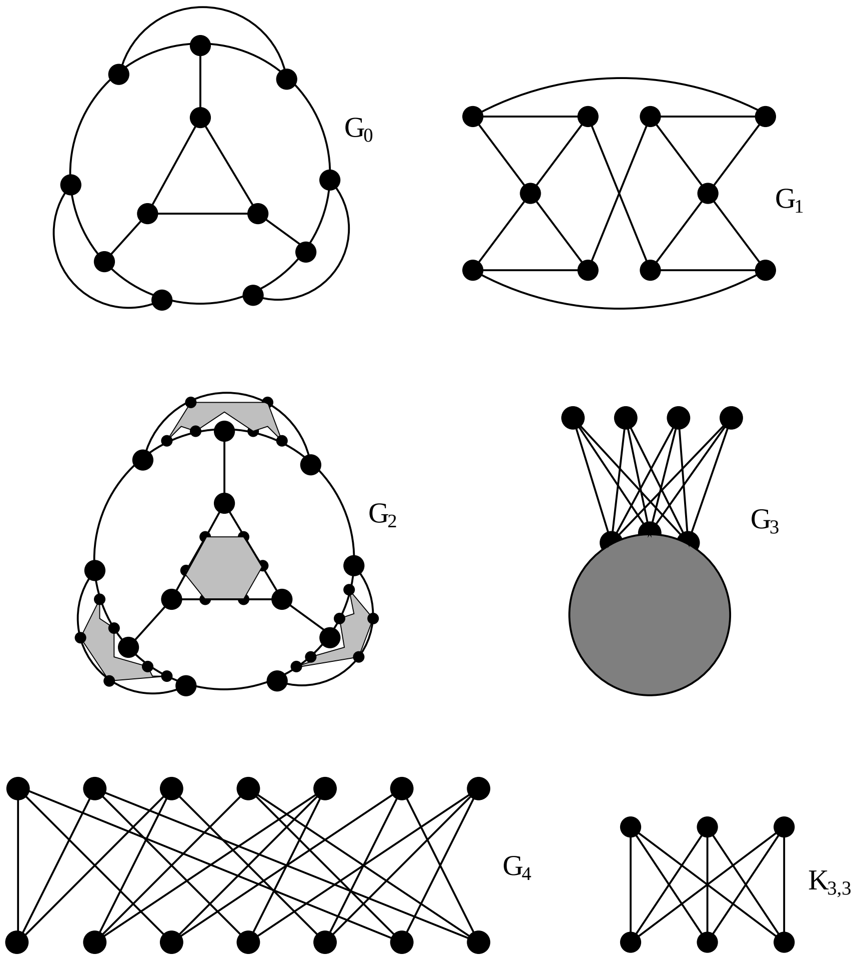

FIGURE 2. 3-connected non-pancyclic graphs

Lemma 2. *Let X and Y be connected graphs on at least three vertices and $X, Y \neq P_3$. If each $\{X, Y\}$-free 3-connected graph is pancyclic, then one of X, Y is $K_{1,3}$.*

Proof. Suppose that $X, Y \neq K_{1,3}$. As $K_{3,3}$ is not pancyclic, one of X and Y has to be an induced subgraph of $K_{3,3}$. Without loss of generality we may assume that X is an induced subgraph of $K_{3,3}$. As X is not $K_{1,3}$ or P_3, X contains C_4.

As C_4 is not a subgraph of G_4, Y is an induced subgraph of G_4, and thus Y has girth at least six and maximum degree at most three. Furthermore, G_3 contains no induced copies of C_4, so Y has to be an induced subgraph of G_3. But the only induced subgraphs of G_3 with girth larger than three and maximum degree at most three are $K_{1,3}$ and its subgraphs. This proves the lemma. □

Finally, each connected graph F which appears as an induced subgraph of all of G_0, G_1 and G_2, and is not contained in the claw $K_{1,3}$, is a subgraph of one of the following six subgraphs:

- P_7, the path on seven vertices,
- $Ł$, the graph which consists of two vertex-disjoint copies of K_3 and an edge joining them;
- $N_{4,0,0}$, $N_{3,1,0}$, $N_{2,2,0}$, $N_{2,1,1}$, where $N_{i,j,k}$ is the graph which consists of K_3 and vertex disjoint paths of length i, j, k rooted at its vertices.

To see this, observe first that F has at most $|V(G_1)| = 10$ vertices, and F cannot contain an induced cycle of length greater than 3 since F needs to be contained in G_2. If F contains at most one triangle, G_1 can be used to limit the possibilities to the graphs mentioned above. Further, if F contains more than one triangle, there are exactly two triangles, and they are at distance one from each other due to G_0. Finally, at most one vertex in each of the two triangles can have degree greater than 2; otherwise, such a triangle in an induced copy of F in G_2 has to be located in one of the K_6's in the center of one of the copies of T, but there is no other triangle in G_2 with distance 1 to such a triangle.

Let $\mathcal{F}$ denote the family which consists of the above six graphs (see Figure 3).

As we have already deduced from the properties of graphs G_0, G_1 and G_2, if each 3-connected $\{K_{1,3}, Y\}$-free graph is pancyclic, then Y is a subgraph of one of the graphs listed above. Our main result states that the inverse implication holds as well.

Theorem 3. *Let X and Y be connected graphs on at least three vertices such that $X, Y \neq P_3$ and $Y \neq K_{1,3}$. Then the following statements are equivalent:*

(i) *Every 3-connected $\{X, Y\}$-free graph G is pancyclic.*
(ii) *$X = K_{1,3}$ and Y is a subgraph of one of the graphs from the family $\mathcal{F} = \{P_7, Ł, N_{4,0,0}, N_{3,1,0}, N_{2,2,0}, N_{2,1,1}\}$.*

Since (i) implies (ii), it is enough to show that for each graph Y from $\mathcal{F}$ and each 3-connected $\{K_{1,3}, Y\}$-free graph G, G is pancyclic. Hence, the proof of Theorem 3 consists, in fact, of six statements, one for each graph from $\mathcal{F}$, which we show in the following sections of the paper.

In the proofs, for a cycle C we always distinguish one of the two possible orientation of C. By v^- and v^+ we denote the predecessor

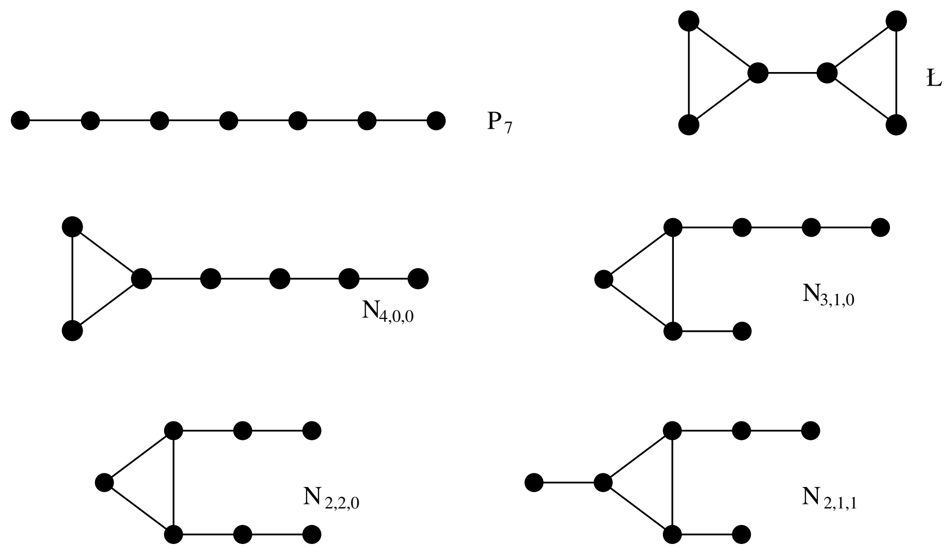

FIGURE 3. The family $\mathcal{F}$

and the successor of a vertex v on such a cycle, with respect to the orientation. We write vCw for the path from $v \in V(C)$ to $w \in V(C)$, following the direction of C, and by vC^-w we denote the path from v to w opposite to the direction of C. By $\langle x_1, \dots, x_k \rangle$ we mean the subgraph induced in G by vertices $x_1, \dots, x_k$.

2. FORBIDDING Ł

In this section we make the first step towards proving Theorem 3: we show the fact that each 3-connected claw-free graph which contains no induced copy of $Ł$ is pancyclic.

Theorem 4. *Every* 3*-connected* $\{K_{1,3}, Ł\}$*-free graph is pancyclic.*

Proof. Suppose that G is a minimal counterexample to the above statement, and that G contains a cycle C of length t but no cycles of length $t+1$ (the existence of triangles is obvious). Let H be a component of $G - C$. Note that for every vertex $x \in N(H) \cap V(C)$ and $v \in N(x) \cap V(H)$, we have that $vx^-, vx^+ \notin E$, and thus $x^-x^+ \in E$ to avoid a claw.

Claim 1. *No vertex from H has more than two neighbors on C.*

Proof. Suppose there is a vertex $v \in V(H)$ with $x, y, z \in N(v) \cap V(C)$. As $\langle v, x, y, z \rangle$ is not a claw, there is an extra edge, say $xy \in E$. As $\langle v, x, y, z, z^-, z^+ \rangle$ is not $Ł$, there is an extra edge between two of these vertices. We have $yz^+ \notin E$, otherwise $yz^+Cy^-y^+Czvy$ is a cycle of length $t+1$, a contradiction. A similar argument shows that none of the pairs yz^-, xz^-, xz^+, is an edge of G.

Therefore, either $yz \in E$, or $xz \in E$. If $xz \notin E$, then $\langle y, x, z, y^+ \rangle$ is a claw, thus $xz \in E$. Similarly, $yz \in E$, and so, by the previous argument $xy^\pm$, $x^\pm y$, $x^\pm z$, $y^\pm z \notin E$. Furthermore $x^+y^+ \notin E$, since otherwise $x^+y^+CxvyC^-x^+$ is a cycle of length $t+1$, contradicting the choice of G. Similarly, $x^-y^- \notin E$.

As $\langle x, x^-, x^+, y, y^-, y^+ \rangle$ is not Ł, either $x^+y^- \in E$, or $x^-y^+ \in E$. By symmetry we may assume $x^+y^- \in E$. Now $x^{++}y \notin E$, since otherwise the cycle $yx^{++}Cy^-y^+Cx^-x^+xvy$ has length $t+1$, while $C_{t+1} \not\subseteq G$. The edge $x^{++}v$ would lead to the cycle $vx^{++}Cx^-x^+xv$, thus $x^{++}v \notin E$. Finally, $x^{++}z \notin E$ to avoid the cycle $x^-xzvx^{++}Cz^-z^+Cx^-$.

Note that $x^{++}y^- \notin E$, since otherwise $\langle x^+, x^{++}, y^-, y, v, z \rangle$ is Ł. To avoid the claw $\langle x^+, x, x^{++}, y^- \rangle$, we have $xx^{++} \in E$. To avoid the claw $\langle x, x^{++}, x^-, v \rangle$, we have $x^{++}x^- \in E$. But now the cycle $x^-x^{++}Cy^-x^+xvyCx^-$ has length $t+1$ (see Figure 4), the contradiction establishing the claim. ◇

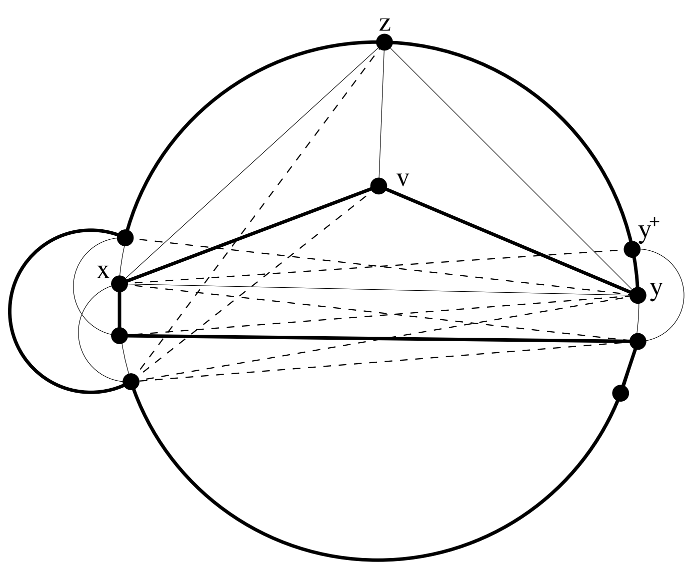

FIGURE 4

Claim 2. *Let $x, y \in V(C) \cap N(H)$. Then $xy \in E$ if and only if $N(x) \cap N(y) \cap V(H) \neq \emptyset$.*

Proof. For one direction, suppose $z \in N(x) \cap N(y) \cap V(H)$. Let P be a shortest path from z to C in $G - \{x, y\}$. Let v be the first internal vertex on this path. By Claim 1, $v \notin V(C)$. If $v \in N(x) \cap N(y)$, start over with $z' = v$ and $P' = P - x$. So assume that $v \notin N(x) \cap N(y)$, say $vx \notin E$. If $vy \notin E$, then $xy \in E$ to avoid a claw, and we are done. Assume that $xy \notin E$, and thus $vy \in E$. We know that $vx^-, vx^+ \notin E$, otherwise we can expand C by including vertices v and z and omitting y to get a cycle of length $t+1$. Moreover, $yx^-, yx^+ \notin E$, since otherwise we can replace y^-yy^+ by y^-y^+, and insert y and z between x and x^+ or between x^- and x, respectively, to increase the length of the cycle by one. But now $\langle z, y, v, x, x^-, x^+\rangle$ is Ł, a contradiction.

For the other direction, let P be a shortest x–y path through H not using xy. By symmetry, we may assume that $y \neq x^+$. Let x_1 be the successor of x on P, let y_1 be the predecessor of y on P. If $x_1 = y_1$ we are done, so let $x_1 \neq y_1$. To avoid the claw $\langle x, x^+, x_1, y\rangle$, $x^+y \in E$. If $x_1y_1 \in E$, then we can extend C through $xx_1y_1yx^+$ and skip y and another vertex in $N(H) \cap V(C)$ to get a cycle of length $t+1$. So assume $x_1y_1 \notin E$.

Let x_2 be another neighbor of x_1 not on P, and let y_2 denote another neighbor of y_1 not on P. We know that $N(x_2) \cap \{x^-, x^+\} = N(y_2) \cap \{y^-, y^+\} = \emptyset$, as otherwise a cycle of length $t+1$ can be found. Now $xx_2, yy_2 \in E$ to avoid claws and Ł's around x_1 and y_1. If $x_2, y_2 \in V(H)$ we get the Ł $= \langle x, x_1, x_2, y, y_1, y_2\rangle$, as P is shortest. Thus, we may assume that $x_2 \in V(C)$, and $N(x_2) \cap \{y, y_1, y_2\} \neq \emptyset$. By the first part of the claim this implies that $x_2y \in E$ or $x_2y_2 \in E$ and $y_2 \in V(C)$.

If $x_2y \in E$, then the cycle $xx_1x_2yx^+Cx_2^-x_2^+Cy^-y^+Cx$ has length $t+1$ (see Figure 5).

If $x_2y_2 \in E$ and $y_2 \in V(C)$, and $x_2y_2 \notin E(C)$, then the cycle $xx_1x_2y_2yx^+Cx_2^-x_2^+Cy_2^-y_2^+Cy^-y^+Cx$ has length $t+1$.

Finally, if $x_2y_2 \in E(C)$, say $y_2 = x_2^+$, then $x_2^-y_2^+ \in E$ to avoid the claw $\langle x_2, x_1, x_2^-, y_2^+\rangle$. But now the cycle

$$xx_1x_2y_2yx^+C(x_2)^-(y_2)^+Cy^-y^+Cx$$

has length $t+1$. ◇

Note that, as a consequence of Claim 2, $N(H)$ does not include two consecutive vertices on C.

Claim 3. *If $x, y \in N(H) \cap V(C)$ and $xy \in E$, then $xy^-, xy^+ \notin E$.*

Proof. Suppose $xy^- \in E$. By Claim 2, there is a vertex $z \in N(x) \cap N(y) \cap V(H)$. Now the cycle $xzyCx^-x^+Cy^-x$ has length $t+1$, a contradiction. The symmetric case $xy^+ \in E$ can be treated in the same way. ◇

Claim 4. *If $x, y, z \in N(H) \cap V(C)$ and $xz, yz \in E$, then $xy \in E$.*

Proof. Otherwise, $\langle z, z^+, x, y\rangle$ is a claw by Claim 3. ◇

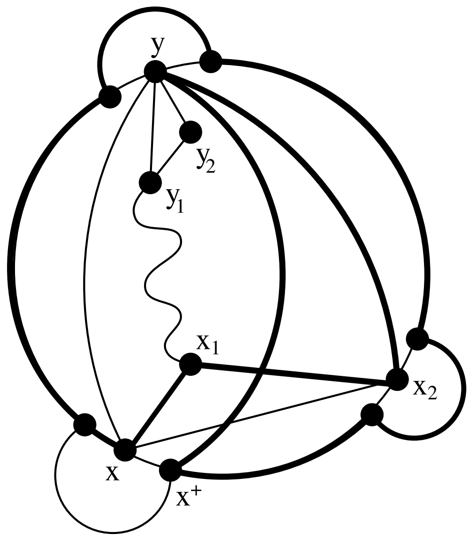

FIGURE 5

Claim 5. $\langle N(H) \cap V(C)\rangle$ *is complete.*

Proof. Suppose the claim is false. Then there are two vertices $x, y \in N(H) \cap V(C)$ with $xy \notin E$. Let P be a shortest x–y path through H. We may assume that x and y were chosen such that P is shortest. Let $P = v_0(= x)v_1 \ldots v_{k-1}v_k(= y)$. By Claim 2, $k + 1 = |V(P)| \geq 4$. Let $R = R(P)$ be a shortest path in $G - \{v_0, v_2\}$ from v_1 to C. We may assume that P is chosen such that R is shortest.

Suppose that $k = 3$. Suppose there is a vertex $z \in N(v_1) \cap N(v_2)$. Then, one of the pairs xz, yz is not an edge, otherwise, either $z \in V(C)$ and $xy \in E$ by Claim 4, or $z \notin V(C)$ and $xy \in E$ by Claim 2. Say $xz \notin E$. By Claim 2, $z \notin V(C)$. But now we can find a copy of Ł at $\langle v_1, v_2, z, x, x^+, x^-\rangle$, a contradiction showing that $N(v_1) \cap N(v_2) = \emptyset$.

Let z_1 be the first vertex on R following v_1 and let $z_2 \in N(v_2)\setminus V(P)$. To avoid claws, $xz_1, yz_2 \in E$. If one of the pairs yz_1, xz_2 is an edge, then Claim 2 and Claim 4 imply that $xy \in E$, a contradiction. Furthermore, $z_1z_2 \notin E$, for otherwise $P' = xz_1z_2y$ would allow a shorter R. But now $\langle z_1, v_1, x, z_2, v_2, y\rangle$ is a copy of Ł, a contradiction showing that $k > 3$.

Just like above, let z_1 be the first vertex on R following v_1 and let $z_2 \in N(v_2) \setminus V(P)$. If $z_2 \in V(C)$, then $xz_2, yz_2 \in E$ as P is shortest, implying that $xy \in E$ by Claim 4. Thus, $z_2 \notin V(C)$. If $v_1z_2 \in E$, then $xz_2 \in E$ to avoid a copy of Ł at $\langle v_1, v_2, z_2, x, x^+, x^-\rangle$. By the same argument, if $v_2z_1 \in E$, then $z_1 \notin V(C)$ and $xz_1 \in E$. But, as before, this is impossible since R is shortest. Thus, $v_2z_1 \notin E$ and $xz_1 \in E$ to avoid the claw $\langle v_1, v_2, x, z_1\rangle$.

If $v_1z_2 \notin E$, then $v_3z_2 \in E$ to avoid the claw $\langle v_2, v_1, v_3, z_2\rangle$. If $z_1 \in V(C)$, then $z_1z_2 \notin E$, otherwise $yz_1 \in E$ as P is shortest, and thus $xy \in E$ by Claim 4. If $z_1 \notin V(C)$, then $z_1z_2 \notin E$ as R is shortest. But now $\langle v_2, v_3, z_2, v_1, x, z_1\rangle$ is a copy of $\L$. Thus, $v_1z_2, xz_2 \in E$.

Let $z_3 \in N(v_3) \setminus V(P)$. If $xz_3 \in E$, then $z_3 \in V(C)$ as P is shortest. But then $yz_3 \in E$ as $z_3v_3v_4 \dots v_k$ is shorter than P, and so $xy \in E$ by Claim 4. Thus, $xz_3 \notin E$. If $v_2z_3 \in E$, then $xz_3 \in E$ by the above argument, a contradiction. Thus, $v_2z_3 \notin E$, and therefore $v_4z_3 \in E$ to avoid the claw $\langle v_3, v_2, v_4, z_3\rangle$. Moreover, $z_2z_3 \notin E$, since otherwise $\langle z_2, v_2, x, z_3\rangle$ is a claw. But now, $\langle v_2, v_1, z_2, v_3, v_4, z_3\rangle$ is a copy of $\L$, the final contradiction establishing the claim. $\diamond$

Now we are ready to complete the proof of the theorem. By Claim 1, $|V(H)| \geq 2$. Contract H to a single vertex h in the new graph G'. As $\langle N(H) \cap V(C)\rangle$ is complete by Claim 5, G' is 3-connected and claw-free. Since $N(h)$ induces a complete graph G' contains no copies of $\L$ involving h as one of the center vertices. If there was $\L$ with h as a corner vertex of a triangle xyh, there would be $\L$ in G with the triangle xyz, where z is a vertex in $N(x) \cap N(y) \cap V(H)$ whose existence is guaranteed by Claim 2. Consequently, G' is a 3-connected $\{K_{1,3}, \L\}$-free graph smaller than G. Thus, G' is pancyclic and contains a cycle C' of length $t+1$. If $h \notin V(C')$, then C' is a cycle of length $t+1$ contained in G. If h appears on C' between x and y, replace it with $z \in N(x) \cap N(y) \cap V(H)$ from Claim 2, again forming a cycle of length $t+1$, a contradiction proving the theorem. $\square$

3. Forbidding $N_{2,2,1}$

In this section we deal with 3-connected claw-free graphs, which contain no induced copy of the graph $N_{2,2,1}$, a common supergraph of both $N_{2,2,0}$ and $N_{2,1,1}$.

Here and below a hop is a chord of a cycle C of type vv^{++}.

Lemma 5. *Let G be a claw-free graph with minimum degree $\delta(G) \geq 3$, and let C be a cycle of length t without hops, for some $t \geq 5$. Set*

$$X = \{v \in V(C) \mid \text{ there is no chord incident to } v\},$$

and suppose for some chord xy of C we have $|X \cap V(xCy)| \leq 2$. Then G contains cycles C' and C'' of lengths $t-1$ and $t-2$, respectively.

Proof. Let us choose a chord xy such that $|X \cap V(xCy)|$ is minimal, and among those such that $|V(xCy)|$ is minimal. Consider the cycle $\bar{C} = xyCx$. As C has no hops, $|V(\bar{C})| \leq t-2$. A vertex

$v \in V(x^+Cy^-) \setminus X$ has a neighbor $w \in V(y^+Cx^-)$ as $|V(xCy)|$ is minimal. To avoid the claw $\langle w, w^+, w^-, v\rangle$, one of the edges vw^+, vw^- appears in G, thus v can be inserted into $\bar{C}$, i.e., $\bar{C}$ can be extended to the cycle $xyCwvw^+Cx$ or $xyCw^-vwCx$. This way we can append all the vertices from $V(x^+Cy^-) \setminus X$ to $\bar{C}$ one-by-one. The only possible problem in this process occurs if we want to insert a second vertex $v' \in V(x^+Cy^-) \setminus X$ at the same spot. But as G is claw-free and there are no chords inside x^+Cy^-, $\langle N(w) \cap V(x^+Cy^-)\rangle$ consists of at most two complete subgraphs of size at most two each, where one of them is a subset of $N(w) \cap N(w^+)$, the other one a subset of $N(w) \cap N(w^-)$. Therefore, we can insert any number of vertices in $N(w) \cap V(x^+Cy^-)$ into $\bar{C}$. So we conclude that we can transfer any number of vertices from $V(x^+Cy^-) \setminus X$ into $\bar{C}$.

As $|X \cap V(xCy)| \leq 2$, we can build in this way a cycle C'' of length $t-2$. Using this procedure we can also construct a cycle of length $t-1$ unless $|X \cap V(xCy)| = 2$. But then $|X \cap V(yCx)| \geq 2$ by the minimality of $|X \cap V(xCy)|$, and we can extend C'' through a vertex $z' \in N(z) \setminus V(C)$, where $z \in X \cap V(yCx)$ (observe that one of $z'z^+, z'z^-$ is an edge to avoid a claw at z, and no vertex of $V(xCy)$ was inserted next to z as z is not an end of a chord). □

Fact 6. *Let G be a 3-connected claw-free graph which contains no cycles of length t, for some $4 \leq t \leq n$. Let C be a cycle of length $t-1$ in G and $x \in V(G) \setminus V(C)$ be adjacent to vertices $v, w \in V(C)$, which are themselves adjacent in G. Then, G contains an induced copy of $N_{2,2,1}$.*

Proof. Let P be a shortest path from x to C in $G - \{v, w\}$. We may assume that x was chosen from $N(v) \cap N(w) \setminus V(C)$ such that P is shortest.

To avoid claws, $v^-v^+, w^-w^+ \in E$. Note that $wv^-, vw^- \notin E$, otherwise C could be extended through x. Let $v_2 \in V(v^+Cw)$ be the vertex closest to v on C with $vv_2 \notin E$, let $v_1 = v_2^-$. Let $w_2 \in V(w^+Cv)$ be the vertex closest to w on C with $ww_2 \notin E$, let $w_1 = w_2^-$.

First, we want to show that $\langle x, v, v_1, v_2, w, w_1, w_2\rangle$ is an induced copy of $N_{2,2,0}$. If $xw_i \in E$ for $i \in \{1, 2\}$, then the cycle $wxw_iCw^-w^+Cw_i^-w$ has length t. Thus, $xw_i \notin E$ for $i \in \{1, 2\}$ and, by symmetry, $xv_i \notin E$ for $i \in \{1, 2\}$.

If $v_iw_j \in E$ for $i, j \in \{1, 2\}$, then

$$v_iw_jCv^-v^+Cv_i^-vxww_j^-C^-w^+w^-C^-v_i$$

is a cycle of length t. Thus, $v_iw_j \notin E$ for $i, j \in \{1, 2\}$, and $\langle x, v, v_1, v_2, w, w_1, w_2\rangle$ is an induced copy of $N_{2,2,0}$.

Now consider the vertex x_1, the unique neighbor of x on P. If $x_1v \in E$, then also $x_1w \in E$ as otherwise $\langle v, x_1, w, v^- \rangle$ is a claw (if $x_1v^- \in E$, C can be extended through x_1 to form a cycle of length t unless $x_1 \in V(C)$. But then, the cycle $v^-x_1xvCx_1^-x_1^+Cv^-$ contains t vertices). Consequently, since P is shortest, $x_1 \in V(C)$. Now one can mimic the argument we have used above to show that $\langle x_1, x_1^+, v, v_1, v_2, w, w_1, w_2 \rangle$ is an induced copy of $N_{2,2,1}$.

So assume that $x_1v, x_1w \notin E$. If $x_1v_i \in E$ for some $i \in \{1, 2\}$, then we can again extend C through x and x_1, possibly skipping a third neighbor of $V(G) \setminus V(C)$ on the cycle to create a C_t. Thus, $x_1v_i, x_1w_i \notin E$ for $i \in \{1, 2\}$, and $\langle x, x_1, v, v_1, v_2, w, w_1, w_2 \rangle$ is an induced copy of $N_{2,2,1}$, finishing the proof. □

Lemma 7. *Let G be a 3-connected claw-free graph such that for some $6 \le t \le n$, G contains a cycle C of length $t-1$ but contains no cycles of length t. Then, G contains an induced copy of $N_{2,2,1}$.*

Proof. Suppose, for the sake of contradiction, that G contains no induced copy of $N_{2,2,1}$. Let H be a component of $\langle V(G) \setminus V(C) \rangle$, and let $u, v, w \in N(H) \cap V(C)$. Let $x \in V(H)$, and let P_u, P_v and P_w be shortest paths through H from x to u, v and w, respectively. Let $S = V(P_u) \cup V(P_v) \cup V(P_w)$. We may assume that H, u, v, w and x are chosen in a way that $|S|$ is minimal and that x has degree three in $\langle S \rangle$. To avoid a claw at x, there has to be an edge between two vertices $y, z \in N(x) \cap S$. By symmetry, we may assume that $y \in V(P_v)$ and $z \in V(P_w)$. By the minimality of $|S|$, the only other possible additional edges in $\langle S \rangle$ are the edges $\{uv, uw, vw\}$.

Furthermore, note that there are no edges between $S \setminus \{u, v, w\}$ and $V(C) \setminus \{u, v, w\}$. Otherwise, either $|S|$ is not minimal, or G, being claw-free, forces a situation like in Fact 6, guaranteeing $N_{2,2,1}$. This observation, together with the fact that for any two vertices $a, b \in V(C)$ with $ab \in E$ we have $N(a) \cap N(b) \cap V(H) = \emptyset$ (Fact 6), implies that $\langle N(u) \cap V(C) \rangle$, $\langle N(v) \cap V(C) \rangle$ and $\langle N(w) \cap V(C) \rangle$ are complete graphs.

Let $P_x = P_u$, $P_y = P_v - x$ and $P_z = P_w - x$. By symmetry we may assume that $|V(P_z)| \le |V(P_y)| \le |V(P_x)|$, and that u, w and v appear on C in this order. By Fact 6, $|V(P_y)| \ge 2$.

Case 1. $|V(P_z)| = 1$*, i.e.,* $z = w$*.*

Suppose first that $vw \in E$. Thus, $\langle v^-, v, v^+, w^-, w, w^+ \rangle$ is complete as $\langle N(v) \cap V(C) \rangle$ and $\langle N(w) \cap V(C) \rangle$ are complete. As $t \ge 5$, there is a vertex $a \in \{w^+, w^-, v^+, v^-\} - \{u, v, w\}$. If $|V(P_y)| \ge 4$, then $\langle \{w, a\} \cup V(P_x) \cup V(P_y) \rangle$ contains an induced $N_{2,2,1}$. Thus, $|V(P_y)| \le 3$.

Consider the cycle $C' = wyP_yvC^-w^+v^+Cw$. We have $t \leq |V(C')| \leq t+1$. As $C_t \not\subseteq G$, we know that $|V(C')| = t+1$. But now the cycle obtained from C' by skipping u (this is always possible as $\langle N(u) \cap V(C)\rangle$ is complete) has length t, a contradiction. Therefore, $vw \notin E$.

If $|V(P_y)| \geq 4$, then $\langle \{w, w^+\} \cup V(P_x) \cup V(P_y)\rangle$ contains an induced $N_{2,2,1}$. Thus, $|V(P_y)| \leq 3$.

Now suppose that $wv^- \in E$. Then $w^-v^- \in E$ as $\langle N(w) \cap V(C)\rangle$ is complete. Consider the cycle $C' = wyP_yvCw^-v^-C^-w$. Then $t \leq |V(C')| \leq t+1$ and, since $C_t \not\subseteq G$, we have $|V(C')| = t+1$. But now the cycle obtained from C' by skipping u has length t, a contradiction. Therefore, $wv^- \notin E$.

Let b be the first vertex on wCv with $wb \notin E$. If $vb \in E$, then the cycle $C' = vbCv^-v^+Cw^-w^+Cb^-wyP_yv$ has length t or $t+1$. We can then skip u if needed to create a cycle of length t, a contradiction. Thus, $vb \notin E$ and, by an analogous argument, $vb^- \notin E$. If $|V(P_x)| \geq 4$, then $\langle \{w, b^-, b\} \cup V(P_x) \cup V(P_y)\rangle$ contains an induced $N_{2,2,1}$. Thus, $|V(P_x)| \leq 3$.

If $ub \in E$, then the cycle $C' = ubCu^-u^+Cw^-w^+Cb^-wxP_xu$ has length t or $t+1$. Then omitting v if necessary, one can find a cycle of length t in G, a contradiction. Thus, $ub \notin E$ and, by a similar argument $ub^- \notin E$.

Observe that $\langle \{w, b^-, b\} \cup V(P_x) \cup V(P_y)\rangle$ contains an induced $N_{2,2,1}$, unless $|V(P_x)| = |V(P_y)| = 2$. But then since $C_t \not\subseteq G$, we see that $\langle x, y, w, u, u^+, v, v^+, w^+\rangle$ is an induced copy of $N_{2,2,1}$.

Case 2. $|V(P_z)| = 2$.

If $|V(P_y)| \geq 4$, then $\langle \{z, w\} \cup V(P_x) \cup V(P_y)\rangle$ contains an induced $N_{2,2,1}$. Thus, $|V(P_y)| \leq 3$.

Suppose that $v^+w^+ \in E$. Let $C' = wzyP_yvC^-w^+v^+Cu^-u^+Cw$. Then $t \leq |V(C')| \leq t+1$, so, as $C_t \not\subseteq G$, $|V(C')| = t+1$. Since $C_t \not\subseteq G$, C' contains no hops. Hence, no vertex of $V(C) \setminus \{u, u^-, u^+, v, v^+, w, w^+\}$ has a neighbor in $V(G) \setminus V(C)$. Observe also that all neighbors of u, v and w on C are completely connected. Consequently, the chordless vertices of C' are contained in the set $\{z, u^-, u^+\} \cup V(P_y) \setminus \{v\}$. Thus, C' has at most five chordless vertices and one can use Lemma 5 to reduce it to a cycle of length t, which contradicts the assumption that $C_t \not\subseteq G$. Therefore, $v^+w^+ \notin E$. This also implies that $vw, vw^+ \notin E$.

A similar argument shows that $uw, uw^+ \notin E$ if $|V(P_x)| \leq 3$. If $|V(P_y)| = 3$, this implies that $\langle \{z, w, w^+\} \cup V(P_x) \cup V(P_y)\rangle$ contains an induced $N_{2,2,1}$. Thus, $|V(P_y)| = 2$.

We have already seen that $v^+w^+ \notin E$, so there are no edges between $\{w, w^+\}$ and $\{v, v^+\}$. Similarly, there are no edges between u and

$\{v, v^+, w, w^+\}$ if $|V(P_x)| = 2$. But now $\langle \{z, y, w, w^+, v, v^+\} \cup V(P_x)\rangle$ contains an induced $N_{2,2,1}$.

Case 3. $|V(P_z)| \geq 3$.

If $|V(P_x)| \geq 4$, then $\langle V(P_z) \cup V(P_x) \cup V(P_y)\rangle$ contains an induced $N_{2,2,1}$. Thus, $|V(P_z)| = |V(P_x)| = |V(P_y)| = 3$. Furthermore, we know that $uv, uw, vw \in E$ for the same reason. This implies that the graph $\langle (N(u) \cup N(v) \cup N(w)) \cap V(C)\rangle$ is complete. Since $|V(C)| = t - 1 \geq 5$, we know that $|(N(u) \cup N(v) \cup N(w)) \cap V(C)| \geq 5$, and so $\langle (N(u) \cup N(v) \cup N(w)) \cap V(C) \cup S\rangle$ is a pancyclic graph on at least eleven vertices. Thus $t \geq 12$.

Let us assume that uCw is the longest among the paths uCw, wCv, and vCu. Since $t \geq 12$, $|V(uCw)| \geq 4$. In fact, since none of the cycles of the type

$$wP_zz[x]yP_yvC^-w^+v^+Cu^-[u][u^+][w^-]w$$

has length t, we have $|V(uCw)| \geq 8$.

We call a chord ab peripheral, if $V(aCb) \subseteq V(u^+Cw^-)$, $a^{++} \neq b$, and each other chord cd such that $c, d \in V(aCb)$, is a hop, i.e., c and d lie at distance two on C. Note that since $u^+w^- \in E$, there exists at least one peripheral chord. Consider the cycle

$$C' = uP_xxzP_zwCv^-v^+Cu^-w^-C^-u$$

of length $t+2$. If the path u^+Cw^- contains two hops a^-a^+ and b^-b^+ such that a and b are non-consecutive vertices of C (and C'), then clearly we can omit a and b in C' obtaining a cycle of length t, contradicting the fact that $C_t \not\subseteq G$. Hence, we may assume that there are at most two hops on u^+Cw^-, say a^-a^+ and aa^{++}. Let bc be a peripheral chord of C. Assume first that $|V(b^+Cc^-)| \geq 4$ and consider the cycle $C'' = uP_xxyzP_zwCu^-w^-C^-u$ of length $t+4$. Note that all vertices from $V(b^+Cc^-)$, except at most four contained in the set $X = \{a^-, a, a^+, a^{++}\}$, are ends of chords of C (and C'') with one end outside $V(bCc)$. Thus, one can mimic the argument from the proof of Lemma 5 to show that all except four vertices of b^+Cc^- can be incorporated to $bC''cb$ to transform it into a cycle of length t. If $|V(b^+Cc^-)| = 2$, then $uP_xxzP_zwCv^-v^+Cu^-w^-C^-cbC^-u$ is a cycle of length t. If $|V(b^+Cc^-)| = 3$, then $uP_xxzP_zwCu^-w^-C^-cbC^-u$ is a cycle of length t. This contradiction with the assumption that $C_t \not\subseteq G$ completes the proof of Lemma 7. □

Theorem 8. *Every 3-connected $\{K_{1,3}, N_{2,2,1}\}$-free graph G on $n \geq 6$ vertices contains cycles of each length t, for $6 \leq t \leq n$.*

Proof. By Lemma 7, it is enough to show that G contains a copy of either C_5 or C_6. Suppose that this is not the case. Since G is claw-free and 3-connected, it contains a triangle xyz. Let $u \in V(G) \setminus \{x, y, z\}$. As G is 3-connected, there are three vertex-disjoint paths from u to $\{x, y, z\}$. Since G is a $N_{2,2,1}$-free graph without C_5 and C_6, there is a vertex w on one of these paths such that $\langle x, y, z, w\rangle$ is either K_4, or K_4^-, the graph with four vertices and five edges.

Let $v \in V(G)\setminus\{x, y, z, w\}$. Consider three vertex-disjoint paths from v to $\{x, y, z, w\}$. If $\langle x, y, z, w\rangle = K_4$, the above argument guarantees a vertex w' on one of the paths with $|N(w') \cap \{x, y, z, w\}| \geq 2$, and C_5 can be found. If $\langle x, y, z, w\rangle = K_4^-$, say $xw \notin E$, then one of the three paths ends in y or z, say in y. Let w' be the predecessor of y on this path. One of the edges $w'w$ and $w'x$ has to be there to avoid the claw $\langle y, w, x, w'\rangle$, but this implies that $C_5 \subseteq G$, contradicting the choice of G. □

4. Forbidding P_7, $N_{4,0,0}$, and $N_{3,1,0}$

In this section we deal with 3-connected claw-free graphs which contain no induced copy of one of the graphs P_7, $N_{4,0,0}$ and $N_{3,1,0}$. We start with the following simple consequence of Lemma 5.

Lemma 9. *Let G be a 3-connected claw-free graph on n vertices which, for some $5 \leq t \leq n-1$, contains a cycle of length t with at least one chord but contains no cycles of length $t-1$. Then G contains an induced copy of each of the graphs P_7, $N_{4,0,0}$ and $N_{3,1,0}$.*

Proof. Let G be a 3-connected claw-free graph, C be a cycle of length $t \geq 5$ in G which contains at least one chord, and let us assume that G contains no cycles of length $t-1$. Let X be the set of chordless vertices on C. Choose a chord xy in C for which $|V(xCy) \cap X|$ is minimal, and for no other chord $x'y'$ such that $x' \in V(x^+Cy^-)$, $y' \in V(y^+Cx^-)$, and $|V(xCy) \cap X| = |V(x'Cy') \cap X|$, we have $|V(x'Cy')| < |V(xCy)|$. Since $C_{t-1} \not\subseteq G$, C contains no hops. Hence, by Lemma 5, $|V(xCy) \cap X| \geq 3$.

We first show that a chord xy can be chosen in such a way that $|V(xCy)| \geq 6$. Suppose that this is not the case and let xy be a chord which minimizes $|V(xCy) \cap X|$ and $V(x^+Cy^-) = \{x^+, x^{++}, y^-\} \subseteq X$. Let uw be a chord in yCx that minimizes $|X \cap V(uCw)|$, and assume that $|V(uCw)|$ is minimal under this restriction. Then, again, $V(u^+Cw^-) = \{u^+, u^{++}, w^-\} \subseteq X$. If the set $\{u^+, u^{++}, w^-\}$ has more than one neighbor outside of C, we can extend $yCxy$ through two of these neighbors and obtain a cycle of length $t-1$. Thus, there is only one vertex z in $N(\{u^+, u^{++}, w^-\})\setminus V(C)$, and since $\{u^+, u^{++}, w^-\} \subset X$, we have $zu^+, zu^{++}, zy^- \in E$. But G is 3-connected, so there has to be a path in $G - \{u, w\}$ from $\{u^+, u^{++}, w^-\}$ to x^+. Therefore, z has another neighbor $z' \notin N(\{u^+, u^{++}, w^-\})$; this however leads to the claw $\langle z, z', u^+, w^- \rangle$. Thus, we may assume that $|V(xCy)| \geq 6$.

Note that, by the choice of $|V(xCy)|$, $xy^-, yx^+ \notin E$. To avoid the claws $\langle x, x^+, x^-, y \rangle$ and $\langle y, y^+, y^-, x \rangle$ we have $xy^+, yx^- \in E$. If $x^+y^+ \in E$, then the cycle $x^+Cyx^-C^-y^+x^+$ has length $t-1$, thus $x^+y^+ \notin E$. To avoid the claw $\langle x, x^+, x^-, y^+ \rangle$ we have $x^-y^+ \in E$. Moreover, since $C_{t-1} \not\subseteq G$, the pairs $x^{--}y$, $x^{--}y^-$, x^-y^-, $x^{--}y^{--}$, x^-y^{--} are not edges of G and the choice of $|V(xCy)|$ guarantees that $x^{--}y^{3-}$, x^-y^{3-}, $x^{--}y^{4-}$, $x^-y^{4-} \notin E$. Now $\langle x^{--}, x^-, y, y^-, y^{--}, y^{3-}, y^{4-} \rangle$ is a copy of P_7, $\langle y^+, x^-, y, y^-, y^{--}, y^{3-}, y^{4-} \rangle$ is $N_{4,0,0}$, and $\langle y, x, x^-, x^+, x^{++}, x^{3+}, x^{--} \rangle$ is an induced copy of $N_{3,1,0}$. □

The following result has been shown by Łuczak and Pfender [3].

Theorem 10. *Every 3-connected $\{K_{1,3}, P_{11}\}$-free graph G is hamiltonian.* □

As an immediate consequence of Lemma 9 and Theorem 10 we get the following theorem.

Theorem 11. *Let G be a 3-connected $\{K_{1,3}, P_7\}$-free graph on n vertices. Then G contains a cycle of length t, for each $7 \leq t \leq n$.*

Proof. Let G be a 3-connected $\{K_{1,3}, P_7\}$-free graph on n vertices. From Theorem 10 it follows that G is hamiltonian. Let C_t, $8 \leq t \leq n$, be a cycle of length t in G. Since G is P_7-free, C_t must have a chord. Hence, Lemma 9 implies that G contains a cycle of length $t-1$. □

The next result states that 3-connected $\{K_{1,3}, N_{4,0,0}\}$-free graphs contain cycles of all possible lengths, except, perhaps, four and five.

Theorem 12. *Every 3-connected $\{K_{1,3}, N_{4,0,0}\}$-free graph G on n vertices contains cycles of each length t, for $6 \leq t \leq n$.*

Proof. We show first that every 3-connected $\{K_{1,3}, N_{4,0,0}\}$-free graph is hamiltonian. Let G be a 3-connected claw-free graph G which is not hamiltonian. From Theorem 10 it follows that G contains an induced path $P = v_1 \ldots v_{11}$. Since G is 3-connected, v_6 has at least one neighbor w outside P. Furthermore, G is claw-free and P is induced, so w cannot have neighbors in both sets $\{v_1, v_2, v_3, v_4\}$ and $\{v_8, v_9, v_{10}, v_{11}\}$. Thus, suppose that w has no neighbors in $\{v_1, v_2, v_3, v_4\}$ and let i_0 denote the minimum i such that v_i is adjacent to w (i.e., i_0 is 5 or 6). Since G is claw-free, v_{i_0+1} is adjacent to w, and so the vertices $v_{i_0-4}, v_{i_0-3}, v_{i_0-2}, v_{i_0-1}v_{i_0}v_{i_0+1}w$ span an induced copy of $N_{4,0,0}$ in G.

Hence, each 3-connected $\{K_{1,3}, N_{4,0,0}\}$-free graph on n vertices contains a cycle of length n.

Thus, to show the assertion, it is enough to verify that if a 3-connected $\{K_{1,3}, N_{4,0,0}\}$-free graph G contains a cycle $C = v_1 \dots v_t v_1$ of length t, $7 \le t \le n$, then it also contains a cycle of length $t-1$. From Lemma 9 it follows that it is enough to consider the case in which C has no chords, i.e., each vertex of C has at least one neighbor outside C. Note that since G is claw-free, each $w \in N(C)$ must have at least two neighbors on C. But G is also $N_{4,0,0}$-free which implies that for each such vertex $|N(w) \cap V(C)| \ge 3$, and one can use the fact that G is $\{K_{1,3}, N_{4,0,0}\}$-free again to verify that each $w \in N(C)$ has precisely four neighbors on C: v_i, v_{i+1}, v_j and v_{j+1}. If $j \ge i+6$, then G contains an induced copy of $N_{4,0,0}$ on vertices v_j, v_{j+1}, w, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}. Moreover, if $j \le i+4$, then there is a cycle of length $t-1$ in G containing the vertex w. Thus, we may assume that $j - i = i - j = 5$, i.e., $t = 10$ and each $w \in N(C)$ is adjacent to vertices v_i, v_{i+1}, v_{i+5}, v_{i+6} for some $i = 1, \dots, 10$. Let w be adjacent to v_1, v_2, v_6, v_7, and let w' be a neighbor of v_4. Assume that $N(w') = \{v_3, v_4, v_8, v_9\}$. Then the vertices $v_1, v_2, w, v_6, v_5, v_4, w'$ span a copy of $N_{4,0,0}$; since G is $N_{4,0,0}$-free, this copy is not induced; consequently, w and w' must be adjacent. But this leads to a cycle $v_3 w' w v_7 v_8 \dots v_2 v_3$ of length $t - 1 = 9$ in G. □

We conclude this section with a result on 3-connected $\{K_{1,3}, N_{3,1,0}\}$-free graphs.

Theorem 13. *Every 3-connected claw-free graph G on n vertices which contains no induced copy of $N_{3,1,0}$ contains a cycle of length t for each $6 \le t \le n$.*

Proof. We show first that each $\{K_{1,3}, N_{3,1,0}\}$-free 3-connected graph is hamiltonian. Suppose that it is not the case and let G be a non-hamiltonian $\{K_{1,3}, N_{3,1,0}\}$-free 3-connected graph with the minimum number of vertices. From Theorem 10 it follows that G contains an induced path $P = v_1 v_2 \dots v_{11}$. Since G is claw-free and P is induced, every vertex $w \in V(G) \setminus V(P)$ adjacent to v_i, $i = 2, \dots, 10$, must be also adjacent to either v_{i-1}, or v_{i+1}. Note however, that since G contains no induced copy of $N_{3,1,0}$, we have $|N(w) \cap V(P)| \ge 3$, unless $N(w) \cap V(P)$ is either $\{v_1, v_2\}$, or $\{v_{10}, v_{11}\}$. Moreover, if $w \in V(G) \setminus V(P)$ is adjacent to three non-consecutive vertices in $\{v_2, v_3, \dots, v_{10}\}$, then the fact that G is claw-free implies that $|N(w) \cap V(P)| = 4$, which, as one can easily check by a direct examination of all cases, would lead to an induced copy of $N_{3,1,0}$. Hence, each vertex $w \in V(G) \setminus V(P)$ which is adjacent to one of the vertices $v_3, \dots, v_9$, has precisely three neighbors on P:

v_{i-1}, v_i, and v_{i+1} for some $i \in \{2, 3, \dots, 10\}$. Hence, for $i = 3, \dots, 9$, set

$$\begin{aligned} V_i &= \{v_i\} \cup \{w \in V(G) \setminus V(P) : N(w) \cap V(P) = \{v_{i-1}, v_i, v_{i+1}\}\} \\ &= N(V_{i-1}) \cap N(V_{i+1}). \end{aligned}$$

Claim 1.

(i) *The path $v_1 \dots v_{i-1} v_i' v_{i+1} \dots v_{11}$ is induced for every $i = 3, \dots, 9$ and $v_i' \in V_i$.*
(ii) *Every two vertices of V_i, $i = 3, \dots, 9$, are adjacent.*
(iii) *All vertices of V_i and V_{i+1}, $i = 3, \dots, 8$, are adjacent.*
(iv) *$N(V_i) = V_{i-1} \cup V_{i+1}$ for $i = 4, 5, \dots, 8$.*

Proof. Each $v_i' \in V_i \setminus \{v_i\}$ has only three neighbors v_{i-1}, v_i, v_{i+1} on P, so (i) follows. Let $v_i', v_i'' \in V_i$. Consider the claw $\langle v_{i+1}, v_i', v_i'', v_{i+2} \rangle$. From (i) it follows that v_{i+2} is adjacent to neither v_i', nor v_i'', so $v_i' v_i'' \in E(G)$, showing (ii).

Now let $v_i' \in V_i$, $v_j' \in V_j \setminus \{v_j\}$, for $3 \le i < j \le 9$. Since the path $v_1 \dots v_{i-1} v_i' v_{i+1} \dots v_{11}$ is induced, v_j' must have on it precisely three consecutive neighbors. Hence, from the definition of V_j we infer that v_i' and v_j' are adjacent if $j = i + 1$, and non-adjacent otherwise. Finally, note that if $v_i' \in V_i$, $i = 4, \dots, 8$, has a neighbor $w \in V(G) \setminus V(P)$, then, because of the claw $\langle v_i', w, v_{i-1}, v_{i+1} \rangle$, w must have a neighbor on P, and thus $w \in V_{i-1} \cup V_i \cup V_{i+1}$. ◇

Let G' denote the graph obtained from G by deleting all vertices from V_6, and connecting all vertices of V_5 with all vertices of V_7. Then G' is 3-connected, claw-free, and contains no induced copy of $N_{3,1,0}$ (note that no induced copy of $N_{3,1,0}$ in G' contains vertices of both V_3 and V_9). Thus, since G is a smallest 3-connected $\{K_{1,3}, N_{3,1,0}\}$-free non-hamiltonian graph, G' is hamiltonian. But each hamiltonian cycle in G' can be easily modified to get a hamiltonian cycle in G, contradicting the choice of G. Hence, each 3-connected $\{K_{1,3}, N_{3,1,0}\}$-free graph is hamiltonian.

Now let us assume that a 3-connected $\{K_{1,3}, N_{3,1,0}\}$-free graph G contains a cycle $C = v_1 v_2 \dots v_t v_1$ of length t, $7 \le t \le n$. We shall show that it must also contain a cycle of length $t - 1$. If C contains at least one chord, the existence of such a cycle follows from Lemma 9, so assume that C contains no chords. If a vertex $w \in V(G) \setminus V(C)$ has a neighbor v on C, then, since G is claw-free, one of the vertices v^-, v^+, must be adjacent to w as well. Furthermore, since G is $N_{3,1,0}$-free, w cannot have only two neighbors on P. On the other hand, using the fact that G is claw-free once again, we infer that if v has three non-consecutive neighbors on P, then it must have precisely four of them. Furthermore, each choice of four neighbors on P leads either to an induced copy of $N_{3,1,0}$, or to a cycle of length $t-1$. Thus, we may assume that each vertex $w \in V(G) \setminus V(C)$ adjacent to at least one vertex from C is, in fact, adjacent to precisely three vertices v_i, v_{i+1}, and v_{i+2}, for $i = 1, \ldots, t$, where, of course, the addition is taken modulo t. Let us define

$$\begin{aligned} V_i &= \{v_i\} \cup \{w \in V(G) \setminus V(P) : N(w) \cap V(P) = \{v_{i-1}, v_i, v_{i+1}\}\} \\ &= N(V_{i-1}) \cap N(V_{i+1}), \end{aligned}$$

for $i = 1, 2, \ldots, t$. Then one can use an argument identical with the one used in the proof of Claim 1 to show that $V(G) = V_1 \cup \cdots \cup V_t$ is a partition of the set of the vertices of G into complete graphs, each vertex from V_i is adjacent to each vertex from V_{i+1}, and $N(V_i) = V_{i-1} \cup V_{i+1}$, for $i = 1, \ldots, t$. Note that if $|V_i| = |V_j| = 1$ for some $i \neq j$, then $|j-i| = 1$ since otherwise the set $V_i \cup V_j = \{v_i, v_j\}$ would be a vertex-cut, while G is 3-connected. Hence, for some i, in the sequence $V_i, V_{i+1}, \ldots, V_{i-1}$, each V_j, $i+1 \leq j \leq i-2$, has at least two elements. Clearly, it implies that G contains cycles of all lengths t, $3 \leq t \leq n$; in particular a cycle of length $t-1$. □

5. Proof of Theorem 3

In this section we conclude the proof of Theorem 3, showing that if a 3-connected claw-free graph G does not contain an induced copy of one of the graphs P_7, $N_{4,0,0}$, $N_{3,1,0}$, $N_{2,2,0}$, $N_{2,1,1}$, then it contains a cycle of length t, for $t = 4, 5, 6$.

Lemma 14. *Let G be a 3-connected claw-free graph which contains a cycle of length seven but no cycles of length six. Then G contains an induced copy of P_7.*

Proof. Let G be a 3-connected claw-free graph without copies of C_6 and let $C = v_1 v_2 \ldots v_7 v_1$ be a cycle of length seven in G. Since $C_6 \nsubseteq G$, C contains no hops. Applying Lemma 5, we infer that C contains no chords.

Let $x \in N(v_1) \setminus V(C)$. Then xv_2 or xv_7 is an edge to avoid a claw $\langle v_1, x, v_2, v_7 \rangle$. By symmetry, we may assume that $xv_2 \in E$. To avoid the P_7 $\langle x, v_2, v_3, \ldots, v_7 \rangle$, x must have another neighbor on C. Since $C_6 \nsubseteq G$, the only possible candidates for neighbors of x are v_3 and v_7. Without loss of generality, we may assume that $xv_3 \in E$. Let $P = (v_2 =) y_0 y_1 \ldots y_k (= v_4)$ be the shortest path from v_2 to v_4 in $G - \{v_1, v_3\}$. As $v_4 v_1 \notin E$, this path contains a vertex which is not adjacent to both v_1 and v_3; let y_ℓ denote the first such vertex on P.

To avoid the claw $\langle y_{\ell-1}, y_\ell, v_1, v_3\rangle$, either v_1y_ℓ or v_3y_ℓ is an edge, say $v_3y_\ell \in E$. As $\langle y_\ell, v_3, v_4 \dots v_1\rangle$ is not P_7, $y_\ell v_4 \in E$. But now, if $\ell \geq 2$, then $v_1v_2v_3v_4y_\ell y_{\ell-1}v_1$ is a cycle of length six, and if $\ell = 1$, then such a cycle is spanned by the vertices $v_1, v_2, y_1, v_4, v_3, x$, contradicting the fact that $C_6 \not\subseteq G$. □

Lemma 15. *If a 3-connected claw-free graph G contains a cycle of length six but no cycles of length five, then G contains an induced copy of each of the graphs P_7, $N_{4,0,0}$, $N_{3,1,0}$, $N_{2,2,1}$.*

Proof. Let G be a 3-connected claw-free graph and let $C = v_1v_2 \dots v_6v_1$ be a cycle of length six contained in C. We split the proof into several simple steps.

Claim 1. *G contains no induced copy of K_4^-, i.e., the graph with four vertices and five edges.*

Proof. Let $X = \{v_1, v_2, v_3, v_4\} \subseteq V(G)$ be such that all pairs of vertices from X, except for $\{v_1, v_2\}$, are edges of G. Since G is 3-connected, one of the vertices $\{v_3, v_4\}$, say, v_3, must have a neighbor $w \notin X$. Because G is claw-free, w must be adjacent to one of the vertices v_1, v_2, say, to v_1. But this leads to a cycle $v_1wv_3v_2v_4v_1$. ◇

Claim 2. *C has no chords. Moreover, no two non-consecutive vertices v_i, v_j of C are connected by a path of either of the types v_iwv_j, $v_iww'v_j$, where $w, w' \notin V(C)$.*

Proof. Since $C_5 \not\subseteq G$, C contains no hops. Applying Lemma 5, we infer that C contains no chords.

Furthermore, each path of type v_iwv_j leads to either C_5 or K_4^-, so we can eliminate them using Claim 1. Finally, the only paths of type $v_iww'v_j$ which do not immediately yield C_5 are of type $v_iww'v_i^{+++}$, but then $\langle v_i, v_i^-, v_i^+, w\rangle$ is a claw, and any edge between vertices v_i^-, v_i^+, w leads to a cycle of length five. ◇

Claim 3. *G contains a vertex x which lies at distance two from C.*

Proof. Suppose that all vertices of G are within distance one from C. Then the fact that G is 3-connected implies that there exist two non-consecutive vertices v_i, $v_j \in V(C)$ which are joined by a path of length at most three, which contradicts Claim 2. ◇

Let x be a vertex which lies at distance two from C, and let w denote a neighbor of x which lies within distance one from C. Claim 2 and the fact that G is claw-free imply that w has two consecutive neighbors on C, say, v_1 and v_2. From Claim 2 we infer that the graph H induced by the vertices $V(C) \cup \{x, w\}$ has only nine edges: the six edges of C and three incident to w. Note that H contains induced copies of both P_7 and $N_{3,1,0}$.

Now let $w' \notin V(H)$ be a neighbor of v_3. Note that because $C_5 \nsubseteq G$, w' is adjacent neither to x nor to w. From Claim 2 and the fact that G is claw-free it follows that the only neighbor of w' in $V(H)$, except v_3, is in the set $\{v_2, v_4\}$. If $w'v_4 \in E$, then the vertices $x, w, v_1, v_2, v_3, w', v_6, v_5$ span an induced copy of $N_{2,2,1}$, and $\langle w, v_2, v_1, v_6, v_5, v_4, w' \rangle$ is $N_{4,0,0}$. Hence, assume that $w'v_2 \in E$. Now let x' be a neighbor of w' outside $V(H)$ which is not adjacent to both v_2 and v_3 (the fact that G is 3-connected and Claim 2 guarantee that such a vertex always exists). Then, since G is claw-free and $C_5 \nsubseteq G$, x' is adjacent to none of the vertices of $V(H)$. But now the vertices $x, w, v_1, v_2, w', x', v_6, v_5$ span an induced copy of $N_{2,2,1}$ in G.

Finally, let $w'' \in N(v_5) \setminus V(C)$. Then, either $v_4w'' \in E$, or $v_6w'' \in E$. If $v_4w'' \in E$, then $\langle w'', v_4, v_5, v_6, v_1, v_2, w' \rangle$ is $N_{4,0,0}$, if $v_6w'' \in E$, then $\langle w'', v_6, v_5, v_4, v_3, v_2, w \rangle$ is $N_{4,0,0}$, as $ww'', w'w'' \notin E$ by Claim 2. □

For our argument we also need the following simple observation on G_1 defined in the Introduction (see Figure 2).

Fact 16. *Let G be a 3-connected claw-free graph which contains no cycles of length four. Let $\tilde{G}_1$ be a copy of G_1 in G. Then*

(i) *The copy $\tilde{G}_1$ is induced. In particular, G contains induced copies of each of the graphs P_7, Ł, $N_{4,0,0}$, $N_{3,1,0}$, $N_{2,2,0}$, $N_{2,1,1}$.*
(ii) *If $G \neq \tilde{G}_1$, then G contains an induced copy of $N_{2,2,1}$.*

Proof. It is easy to check that if we add any edge to G_1, then either we create a copy of C_4, or we get $K_{1,3}$ which in turn, since G is claw-free, forces a cycle of length four. Thus, (i) follows. In order to show (ii) note that, since $\tilde{G}_1$ is induced, any vertex $x \in V(G) \setminus V(\tilde{G}_1)$ with a neighbor in $\tilde{G}_1$ must be adjacent to precisely two vertices of $\tilde{G}_1$, which are connected by an edge which belongs to none of the four triangles contained in $\tilde{G}_1$. Now it is easy to check that a subgraph spanned in G by $\{x\} \cup V(\tilde{G}_1)$ contains an induced copy of $N_{2,2,1}$ in which x has degree one and is adjacent to a vertex of degree three. □

Lemma 17. *Let G be a 3-connected claw-free graph which contains a cycle of length five but no cycles of length four. Then G contains an induced copy of each of the graphs P_7, $N_{4,0,0}$, $N_{3,1,0}$, $N_{2,2,0}$, $N_{2,1,1}$. Furthermore, if $G \neq G_1$, then G contains an induced copy of $N_{2,2,1}$.*

Proof. Let $C = v_1v_2v_3v_4v_5v_1$ be a cycle of length five in a 3-connected claw-free graph G which contains no cycles of length four. Clearly,

C contains no chords. Let $S = N(V(C))$. Since $C_4 \nsubseteq G$ and G is claw-free, each vertex $w \in S$ is adjacent to precisely two consecutive vertices of C, for each two vertices $w', w'' \in S$ we have $N(w') \cap V(C) \neq N(w'') \cap V(C)$, and S is independent. A vertex w from S we call w_i, if w is adjacent to v_i and v_{i+1}. Observe also that, since S is independent and G is claw-free, any vertex $x \notin V(C) \cup S$ has in S at most two neighbors; consequently, G must contain an edge with both ends in $V(G) \setminus (V(C) \cup S)$.

Now let us assume that there exists an edge xy, such that $x, y \notin V(C) \cup S$ and each of the vertices x and y has two neighbors in S, denoted x_1, x_2 and y_1, y_2 respectively. Because of the claw $\langle x, x_1, x_2, y \rangle$, we may assume that $x_1 = y_1 = w_1$. Furthermore, to avoid C_4, x and y must be adjacent to different vertices from the set $\{w_3, w_4\}$. But now the graph H induced in G by the set $V(C) \cup \{x, y, w_1, w_3, w_4\}$ contains a copy of the graph G_1 and the assertion follows from Fact 16.

Thus, we may assume that each edge contained in $V(G) \setminus (V(C) \cup S)$ has at least one end which is adjacent to at most one vertex from S. Note also that if a vertex $x \in V(G) \setminus (V(C) \cup S)$ has just one neighbor in S, then it must have at least two neighbors x', x'' in $V(G) \setminus (V(C) \cup S)$, and all three vertices x, x', x'' cannot share the same neighbor in S because $C_4 \nsubseteq G$. Consequently, as G is claw-free, we may assume that G contains vertices x and y such that x is adjacent to y, y is adjacent to w_1, x has at most one neighbor in S, and it is different than w_1, and y has at most one more neighbor in S (then it must be either w_3 or w_4). Let F be the graph spanned in G by $V(C) \cup \{x, y, w_1\}$. It contains precisely nine edges: five edges of C, three edges incident to w_1, and xy.

Clearly, $xyw_1v_2v_3v_4v_5$ is an induced copy of P_7 in $F \subseteq G$. In order to find in G induced copies of $N_{4,0,0}$ and $N_{3,1,0}$ consider the neighbor of v_4 in S: without loss of generality we may assume that it is w_3. If w_3 is not adjacent to y, then G contains an induced copy of $N_{4,0,0}$ (on the vertices $y, w_1, v_1, v_5, v_4, v_3, w_3$) as well as an induced copy of $N_{3,1,0}$ (with the vertex set $\{y, w, v_2, v_3, w_3, v_4, v_5\}$). Thus, assume that w_3 is the only neighbor other than w_1 of y in S. Because of the claw $\langle y, x, w_1, w_3 \rangle$, w_3 is also the only neighbor of x in S. But then the vertices $v_2, v_1, v_5, v_4, w_3, x, y$ span in G an induced copy of $N_{4,0,0}$, while the vertices $w_1, v_1, v_5, v_4, v_3, w_3, x$ span an induced copy of $N_{3,1,0}$.

Finally, we shall show that G contains an induced copy of $N_{2,2,1}$. Thus, let x, y be defined as above and let w_3 be a neighbor of v_4. Consider now two possible choices for a neighbor of v_5. Assume first, that there is a vertex w_4 adjacent to both v_4 and v_5. Then vertices y, w_1, v_1, v_2, v_3, w_3, v_5 and w_4 span a copy of $N_{2,2,1}$. It is induced unless y

is adjacent to one of the vertices w_3, w_4, say w_3. Then, because of the claw $\langle y, x, w_1, w_3\rangle$, x is also adjacent to w_3, and none of the vertices x, y, is adjacent to w_4. But then the vertices x, y, w_1, v_1, v_2, v_3, v_5 and w_4 span an induced copy of $N_{2,2,1}$.

Thus, suppose that G contains a vertex w_5, adjacent to both v_5 and v_1. Note that the vertices x, y, w_1, v_1, v_2, v_3, v_4, and w_5 span an induced copy of $N_{2,2,1}$, unless $w_5x \in E$. But if $w_5x \in E$, then w_3 is adjacent to neither x nor y, and so there is an induced copy of $N_{2,2,1}$ on the vertices y, x, w_5, v_1, v_2, v_5, v_4, w_3. □

As an immediate consequence of Theorem 8 and Lemmas 15 and 17 we get the following result.

Theorem 18. *Each* 3*-connected* $\{K_{1,3}, N_{2,2,1}\}$*-free graph is either isomorphic to* G_1*, or pancyclic.* □

Finally we can complete the proof of the main result of the paper.

Proof of Theorem 3. We have already seen that (i) implies (ii). Since the graphs $N_{2,2,0}$ and $N_{2,1,1}$ are induced subgraphs of $N_{2,2,1}$, the fact that (i) follows from (ii) is an immediate consequence of Theorems 4, 11, 12, and 13, Lemmas 14, 15, 17, and Theorem 18. □

We conclude the paper with a remark that for Theorem 3, the graphs G_0 and G_1 we introduced at the beginning of the paper are, in a way, extremal. It follows that the smallest 3-connected claw-free graph G which is not pancyclic has ten vertices. Indeed, by Theorem 3, we may assume that G contains an induced path P on seven vertices. The minimal degree of G is at least three, so there are at least nine edges incident to $V(P)$ which do not belong to P. But G is claw-free, so no vertex from $V(G)\setminus V(P)$ is adjacent to more than four vertices from P. Consequently, $|V(G)\setminus V(P)| \geq 3$. In fact, one can examine the proof of Lemma 17 to verify that the graph G_1 is the only 3-connected claw-free graph G on ten vertices which is not pancyclic. In a similar manner one can also deduce from Theorem 10 and the proof of Lemma 15 that the graph G_0 (Figure 2) is the unique smallest 3-connected claw-free graph on at least five vertices which does not contain a cycle of length five.

References

[1] B. Bollobás, "*Modern Graph Theory*", Springer Verlag, New York, 1998.

[2] R.J. Faudree, R.J. Gould, *Characterizing forbidden pairs for hamiltonian properties*, Discrete Math. **173** (1997), pp. 45–60.

[3] T. Łuczak, F. Pfender, *Claw-free* 3*-connected* P_{11}*-free graphs are hamiltonian*, submitted.

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