Complete subgraphs in multipartite graphs

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Abstract

Turán's Theorem states that every graph of a certain edge density contains a complete graph K^k and describes the unique extremal graphs. We give a similar Theorem for ℓ-partite graphs. For large ℓ, we find the minimal edge density d_ℓ^k, such that every ℓ-partite graph whose parts have pairwise edge density greater than d_ℓ^k contains a K^k. It turns out that $d_\ell^k = \frac{k-2}{k-1}$ for large enough ℓ. We also describe the structure of the extremal graphs.

1 Introduction

All graphs in this note are simple, and we follow the notation of [3]. Let G be an ℓ-partite graph on finite sets $V_1, V_2, \ldots V_\ell$. For a vertex $x \in V(G)$, let $d(x) := |N(x)|$. The density between two parts is defined as

$$d_{ij} := d(V_i, V_j) := \frac{\|G[V_i \cup V_j]\|}{|V_i||V_j|}.$$

For a graph H with $|H| \geq \ell$, let $d_\ell(H)$ be the minimum number such that every ℓ-partite graph with $\min_{i<j} d_{ij} > d_\ell(H)$ contains a copy of H. Clearly, $d_\ell(H)$ is monotone decreasing in ℓ. In [2], Bondy at al. study the quantity $d_\ell(H)$, and in particular $d_\ell^3 := d_\ell(K^3)$, i.e. the values for the complete graph on three vertices, the triangle. Their main results about triangles can be written as follows.

Theorem 1. *[2]*

1. $d_3^3 = \tau \approx 0.618$*, the golden ration, and*

2. d_ω^3 *exists and* $d_\omega^3 = \frac{1}{2}$.

They go on and show that $d_4^3 \geq 0.51$ and speculate that $d_\ell^3 > \frac{1}{2}$ for all finite ℓ. We will show that this is false. In fact, $d_\ell^3 = \frac{1}{2}$ for $\ell \geq 13$ as we will proof in Section 3. In Section 4, we will extend the proof ideas to show that $d_\ell^k := d_\ell(K^k) = \frac{k-2}{k-1}$ for large enough ℓ.

In order to state our results, we need to define classes $\mathcal{G}_\ell^k$ of extremal graphs. We will do this properly in Section 2. Our main result is the following theorem.

Theorem 2. *Let ℓ be large enough and let $G = (V_1 \cup V_2 \cup \ldots \cup V_\ell, E)$ be an ℓ-partite graph, such that the pairwise edge densities*

$$d(V_i, V_j) := \frac{\|G[V_i \cup V_j]\|}{|V_i||V_j|} \geq \frac{k-2}{k-1} \text{ for } i \neq j.$$

Then G contains a K^k or G is isomorphic to a graph in $\mathcal{G}_\ell^k$.

Corollary 3. *For ℓ large enough, $d_\ell^k = \frac{k-2}{k-1}$.*

The bound on ℓ one may get out of the proof is fairly large, and we think that the true bound is much smaller. For triangles ($k = 3$), we can give a reasonable bound on ℓ. We think that this bound is not sharp, either. In fact we would not be surprised if $\ell \geq 5$ turns out to be sufficient.

Theorem 4. *Let $\ell \geq 13$ and let $G = (V_1 \cup V_2 \cup \ldots \cup V_\ell, E)$ be an ℓ-partite graph, such that the pairwise edge densities*

$$d(V_i, V_j) := \frac{\|G[V_i \cup V_j]\|}{|V_i||V_j|} \geq \frac{1}{2} \text{ for } i \neq j.$$

Then G contains a triangle or G is isomorphic to a graph in $\mathcal{G}_\ell^3$.

Corollary 5. $d_{13}^3 = \frac{1}{2}$.

2 Extremal graphs

For $\ell \geq (k-1)!$, a graph G is in $\bar{\mathcal{G}}_\ell^k$, if it can be constructed as follows.

$$\begin{aligned} V(G) &= \{(i,s,t) : 1 \leq i \leq \ell,\ 1 \leq s \leq k-1,\ 1 \leq t \leq n_i^s\}, \\ E(G) &= \{(i,s,t)(i',s',t') : i \neq i',\ s \neq s'\}, \end{aligned}$$

where for $\{\pi_1, \pi_2, \ldots, \pi_{(k-1)!}\}$ being the set of all permutations of the set $\{1, \ldots, k-1\}$,

$$\begin{aligned} n_i^{\pi_i(1)} \geq n_i^{\pi_i(2)} \geq \ldots \geq n_i^{\pi_i(k-1)} \qquad & \text{for } 1 \leq i \leq (k-1)!, \\ n_i^1 = n_i^2 = \ldots = n_i^{k-1} \qquad & \text{for } (k-1)! < i \leq \ell, \text{ and} \\ \sum n_i^s > 0 \qquad & \text{for } 1 \leq i \leq \ell. \end{aligned}$$

Let $\mathcal{G}_\ell^k$ be the class of graphs which can be obtained from graphs in $\bar{\mathcal{G}}_\ell$ by deletion of some edges in $\{(i, s, k)(i', s', k') : s \neq s' \wedge 1 \leq i < i' \leq (k-1)!\}$.

It is easy to see that all graphs in $\mathcal{G}_\ell^k$ are ℓ-partite and that $\mathcal{G}_\ell^k$ contains graphs with $d_{ij} \geq \frac{k-2}{k-1}$ (and $d_{ij} = \frac{k-2}{k-1}$ for $j > (k-1)!$ for all graphs in $\mathcal{G}_\ell^k$).

For $k = 3$, the density condition is fulfilled for all graphs in $\bar{\mathcal{G}}_\ell^3$, and for all graphs in $\mathcal{G}_\ell^3$ which have $d_{12} \geq \frac{1}{2}$.

For $k > 3$, this description is not a full characterization of the extremal graphs in the problem. We would need some extra conditions on the n_i^s to make sure that all graphs in $\bar{\mathcal{G}}_\ell^k$ fulfill the density conditions.

3 Theorem 4—triangles

In this section we prove Theorem 4. We will start with a few useful lemmas. An important lemma for the study of d_ω^3 is the following.

Lemma 6. *[2] Let $G = (V_1 \cup V_2 \cup V_3 \cup V_4, E)$ be a 4-partite graph with $|V_1| = 1$, such that the pairwise edge densities $d(V_i, V_j) > \frac{1}{2}$ for $i \neq j$. Then G contains a triangle.*

With the same proof one gets a slightly stronger result which we will use in our proof. In most cases occurring later, X will be the neighborhood of a vertex, and the Lemma will be used to bound the degree of the vertex.

Lemma 7. *Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a 3-partite graph and X an independent set, such that the pairwise edge densities $d(V_i, V_j) \geq \frac{1}{2}$ for $i \neq j$ and $|X \cap V_i| \geq \frac{1}{2}|V_i|$ for $1 \leq i \leq 3$, with a strict inequality for at least two of the six inequalities. Then G contains a triangle.*

In order to prove the first part of Theorem 1, the authors show a stronger result.

Theorem 8. *[2] Let $G = (V_1 \cup V_2 \cup V_3, E)$ be a 3-partite graph with edge densities $d_{ij} := d(V_i, V_j)$, and $d_{ij}d_{ik} + d_{jk} > 1$ for $\{i, j, k\} = \{1, 2, 3\}$. Then G contains a triangle.*

As a corollary from Lemma 7 and Theorem 8 we get

Corollary 9. *Let $G = (V_1 \cup V_2 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph on $n\ell$ vertices with edge densities $d_{ij} \geq \frac{1}{2}$, which does not contain a triangle. Then for every independent set $X \subseteq V(G)$, $|X| \leq \frac{(\ell+1)n}{2}$.*

Proof of Theorem 4. Suppose that G contains no triangle. Without loss of generality we may assume that each of the $\ell \geq 13$ parts of G contains exactly n vertices, where n is a sufficiently large even integer. Otherwise, blow up each part by an appropriate factor, which has no effect on the densities or the membership in $\mathcal{G}_\ell$, and creates no triangles.

For a vertex x let $d_i(x) = |N(x) \cap V_i|$. For each edge $xy \in E(G)$, choose i and j such that $x \in V_i$ and $y \in V_j$, and let

$$s(xy) := d(x) - d_j(x) + d(y) - d_i(y).$$

We have

$$\sum_{xy \in E(G)} s(xy) = \frac{1}{2} \sum_{\substack{x \in V(G) \\ y \in N(x)}} s(xy) = \sum_{x \in V(G)} \left(d(x)^2 - \sum_{j=1}^{\ell} d_j(x)^2 \right).$$

The set $N(x)$ is independent, so by Lemma 7, at most two of the $d_j(x)$ may be larger than $\frac{n}{2}$, and by Lemma 8, $d_j(x)d_k(x) \leq \frac{1}{2}n^2$ for every vertex $x \in V_i$ and $j \neq k$.

Therefore, for fixed $d(x) \geq n$, the last sum is minimized if $d_j(x) = n$ for one j, $d_j(x) = \frac{n}{2}$ or $d_j(x) = 0$ for all but one of the other j, and $0 \leq d_j(x) \leq \frac{n}{2}$ for the last remaining j. For $d(x) < n$, the last sum is non negative. Thus,

$$\begin{aligned}
\frac{1}{\|G\|} \sum_{xy \in E(G)} s(xy) &\geq \frac{2}{\sum d(x)} \sum_{x \in V(G)} \left(d(x)^2 - n^2 - (d(x) - n)\tfrac{n}{2}\right) \\
&= \frac{2 \sum d(x)^2}{\sum d(x)} - n - \frac{\ell n^3}{\sum d(x)} \\
&\geq \frac{2}{\ell n} \sum d(x) - n - \frac{\ell n^3}{\sum d(x)} \\
&\geq (\ell - 2)n - \frac{2n}{\ell - 1}.
\end{aligned}$$

Therefore, there is an edge $xy \in E(G)$ with $s(xy) \geq (\ell - 2)n - \frac{2n}{\ell-1}$. By symmetry, we may assume that $x \in V_1$, $y \in V_2$ and $d(x) - d_2(x) \geq d(y) - d_1(y)$. Let

$$N'(x) := N(x) \setminus V_2, \; N'(y) := N(y) \setminus V_1, \text{ and } W' := \bigcup_{i=3}^{\ell} V_i \setminus (N(x) \cup N(y)).$$

Let $G' := G[\bigcup_{i=3}^{\ell} V_i]$. Since $N'(x)$ and $N'(y)$ are independent sets, and $|W'| \leq \frac{2n}{\ell-1} \leq \frac{n}{6}$, and by Lemma 7 and Theorem 8, for fixed $|W'|$ $G'[N'(x) \cup N'(y)]$ has at most as many edges as in the graph we would get if $|N(x) \cap V_3| = |N(y) \cap V_4| = n$ and $|(W' \cup N'(Y)) \cap V_5| = |N(x) \cap V_5| = |N(x) \cap V_i| = |N(y) \cap V_i| = \frac{n}{2}$ for $6 \leq i \leq \ell$, and all possible edges (i.e., all edges not inside $N(x)$, $N(y)$ or one of the V_i) are there. So,

$$\|G'[N'(x) \cup N'(y)]\| \leq \binom{\ell-2}{2}\frac{n^2}{2} + \frac{n^2}{2} - |W'|\frac{\ell-3}{2}n.$$

Further, by Corollary 9, no vertex in G' can have degree larger than $\frac{\ell-2}{2}n$, so

$$\|G'\| \leq \binom{\ell-2}{2}\frac{n^2}{2} + \frac{n^2}{2} + |W'|\frac{n}{2} \leq \binom{\ell-2}{2}\frac{n^2}{2} + \frac{7}{12}n^2.$$

On the other hand, by the density condition,

$$\|G'\| \geq \binom{\ell-2}{2}\frac{n^2}{2},$$

so at most $\frac{7}{12}n^2$ of the possible edges between $N'(x)$ and $N'(y)$ are missing. In particular, no vertex z can have large neighborhoods in both $N'(x)$ and $N'(y)$, i.e.

$$(|N(z) \cap N'(x)| - n)|N(z) \cap N'(y)| < \frac{7}{12}n^2.$$

Let

$$\begin{aligned} X' &:= \{v \in V(G') : |N(v) \cap N'(x)| > \tfrac{1}{2}|N'(x)|\}, \\ Y' &:= \{v \in V(G') : |N(v) \cap N'(y)| > \tfrac{1}{2}|N'(y)|\}, \text{ and} \\ Z' &:= V(G') \setminus (X' \cup Y'). \end{aligned}$$

If $z \in Z'$, then

$$d_{G'}(z) \leq \frac{|N'(x)|}{2} + \frac{7n^2}{6|N'(x)| - 12n} + |W'| \leq \frac{(\ell-1)n}{4} + \frac{7n}{3\ell - 15} + \frac{2n}{\ell - 1},$$

and for $z \in Z' \setminus W'$, at least $\frac{|N'(y)|}{2} - n \geq \frac{\ell-7}{4}n \geq \frac{3}{2}n$ of the missing possible edges edges between $N'(x)$ and $N'(y)$ are incident to z. Therefore, $|Z'| \leq |W'| + \frac{7}{9}n \leq \frac{17}{18}n$.

Again, since X' is an independent set, at most two of the sets $V_i \cap X'$ contain more than $\frac{n}{2}$ vertices. We may assume that these sets are contained in $V_3 \cup V_4$. Let $G'' = G' \setminus (V_3 \cup V_4)$, and X'', Y'' and Z'' the according subsets of X', Y' and Z'. Then by the density condition,

$$\|G''\| \geq \binom{\ell-4}{2}\frac{n^2}{2}.$$

On the other hand, $\|G''\| \leq |E(X'', Y'')| + |E(Z'', V(G''))|$, and $|E(X'', Y'')|$ is maximized for fixed $|Z''| < n$ if $|V_5 \cap Y''| = n$ and $|V_i \cap X''| = \frac{n}{2}$ for $6 \leq i \leq \ell$. Thus,

$$\begin{aligned}
\|G''\| &\leq \frac{n^2(\ell-5)}{2} + \frac{n(\ell-6)}{2}\left(\frac{n(\ell-5)}{2} - |Z''|\right) + |Z''|\left(\frac{\ell-1}{4} + \frac{7}{3\ell-15} + \frac{2}{\ell-1}\right)n \\
&= \binom{\ell-4}{2}\frac{n^2}{2} + |Z''|\left(\frac{\ell-1}{4} + \frac{7}{3\ell-15} + \frac{2}{\ell-1} - \frac{\ell-6}{2}\right)n \\
&\leq \binom{\ell-4}{2}\frac{n^2}{2}.
\end{aligned}$$

Equality is only attained for $Z'' = \emptyset$, in which case it is easy to show that G is isomorphic to a graph in $\mathcal{G}_\ell^3$. □

4 Theorem 2—complete subgraphs

Graphs which have almost enough edges to force a K^k either contain a K^k or have a structure very similar to the Turán graph. This is described by the following theorem from [1], where a more general version is credited to Erdös and Simonovits.

Theorem 10. *[1, Theorem VI.4.2] Let $k \geq 3$. Suppose a graph G contains no K^k and*

$$\|G\| = \left(1 - \frac{1}{k-1} + o(1)\right)\binom{|G|}{2}.$$

Then G contains a $(k-1)$-partite graph of minimal degree $(1-\frac{1}{k-1}+o(1))|G|$ as an induced subgraph.

Proof of Theorem 2. For the ease of reading and since we are not trying to minimize the needed ℓ, we will use some variables ℓ_i and $c_i > 0$. As ℓ is chosen larger, the ℓ_i grow without bound and the c_i approach 0.

Let G be an ℓ-partite graph with $V(G) = V_1 \cup V_2 \cup \ldots \cup V_\ell$ with densities $d_{ij} \geq \frac{k-2}{k-1}$, and suppose that G contains no K^k. Without loss of generality we may assume that each of the V_i contains exactly n vertices, where n is a sufficiently large integer divisible by $k-1$.

We have

$$\|G\| \geq \left(1 - \frac{1}{k-1} - \frac{1}{\ell}\right)\binom{|G|}{2}.$$

Let H be the $(k-1)$-partite subgraph of G guaranteed by Theorem 10, with $V(H) = X_1 \cup X_2 \cup \ldots \cup X_{k-1}$ and $Z := V(G) \setminus V(H)$. There is a $c_1 > 0$ so that $|Z| \leq c_1|G|$ (and this c_1 becomes arbitrarily small if ℓ is chosen large enough). Let $X_{i,j} := V_i \cap X_j$ and $Z_i := V_i \setminus \bigcup_j X_{i,j}$. After renumbering the V_i and the X_j, we have $|Z_i| \leq 2c_1 n$ and $|X_{i,1}| \geq |X_{i,2}| \geq \ldots \geq |X_{i,k-1}|$ for $1 \leq i \leq \ell_1 \leq \ell$, where $\ell_1 \geq \frac{\ell}{2(k-1)!}$ is picked as large as possible. For some $c_2 > 0$ (with $c_2 \to 0$), there is at most one index $i \leq \ell_1$ with $|X_{i,1}| > \left(\frac{1}{k-1} + c_2\right) n$, as otherwise there is a pair $(V_i, V_{i'})$ with $d_{ii'} < \frac{k-2}{k-1}$. So we may assume that

$$\left(\frac{1}{k-1} - kc_2\right) n \leq |X_{i,j}| \leq \left(\frac{1}{k-1} + c_2\right) n$$

for $1 \leq i \leq \ell_1 - 1$ and $1 \leq j \leq k-1$. This implies that

$$\|G[X_{i,j}, X_{i',j'}]\| > |X_{i,j}||X_{i',j'}| - c_3 n^2$$

for $i \neq i'$, $j \neq j'$, $1 \leq i, i' \leq \ell_1 - 1$, $1 \leq j, j' \leq k-1$ and some $c_3 > 0$ with $c_3 \to 0$.

For every $v \in \bigcup_{i \leq \ell_1 - 1} V_i$, find a maximum set $\mathcal{P}_v$ of pairs (i_s, j_s) with $(1,1) \leq (i_s, j_s) \leq (\ell - 1, k-1)$, $i_s \neq i_{s'}$, $j_s \neq j_{s'}$, $|N(v) \cap X_{i_s,j_s})| > c_4 n$, where $c_4 := k\sqrt{c_3}$. If there is a vertex v with $|\mathcal{P}| = k-1$, then we have a K^k. So we may assume this is not the case. Assign $v \in Z$ to one set $Y_j \supseteq X_j \cap \bigcup_{i \leq \ell_1 - 1} V_i$, if there is no pair (i,j) in $\mathcal{P}_v$. If there is more than one available set, arbitrarily pick one.

Now we reorder the V_i and Y_j again to guarantee that $|Y_{i,1}| \geq \ldots \geq |Y_{i,k-1}|$ for $1 \leq i \leq \ell_2 < \ell_1$, with $\ell_2 \geq \frac{\ell_1 - 1}{(k-1)!}$ as large as possible. In the following, only consider indices $i \leq \ell_2$. Note that for

$v \in Y_{i,j}$, $|N(v) \cap Y_{i,j'}| < (c_4 + 2c_1)n$ for all but at most $k-2$ different j', as $Y_{i,j'} \setminus X_{i,j'} \subseteq Z_{j'}$.

Let $Y_i' \subseteq Y_i$ the set of all vertices $v \in Y_i$ with $|N(v) \cap Y_j)| < \frac{1}{2}(\frac{1}{k-1} + c_5)\ell_2 n$ for some $j \neq i$, $c_5 := c_2 + c_4$. Note that the sets $Y_i \setminus Y_i'$ are independent, as the intersection of the neighborhoods of every two vertices in this set contain a K^{k-2}. Every vertex in $v \in Y_i' \cap V_j$ may have up to $((c_4+2c_1)(\ell_2 - k + 1) + k - 2)n$ neighbors in Y_i. But, at the same time, v has at least $|Y_{i'}| - \frac{1}{2}(\frac{1}{k-1} + c_5)\ell_2 n - n > \frac{1}{3k}\ell_2 n$ non-neighbors in some $Y_{i'} \setminus V_j$, $i' \neq i$. Then

$$
\begin{aligned}
\|G[V_1 \cup \ldots V_{\ell_2}]\| &\leq \sum_{\substack{i \neq i' \\ j < j'}} |Y_{i,j}||Y_{i',j'}| + \sum_i |Y_i'| \left(((c_4 + 2c_1)(\ell_2 - k + 1) + k - 2)n - \tfrac{1}{3k}\ell_2 n\right) \\
&\leq \sum_{\substack{i \neq i' \\ j < j'}} |Y_{i,j}||Y_{i',j'}| + \sum_i |Y_i'| \underbrace{(c_4 + 2c_1 + \tfrac{k}{\ell_2} - \tfrac{1}{3k})}_{<0 \text{ for large enough } \ell} \ell_2 n \\
&\leq \binom{\ell_2}{2} n^2 - \sum_{j < j'} |Y_{i,j}||Y_{i,j'}| \\
&\leq \binom{\ell_2}{2} \tfrac{k-2}{k-1}\, n^2,
\end{aligned}
$$

where equality only holds if $|Y_i'| = 0$ for all i, and $|Y_{i,j}| = \frac{n}{k-1}$ for $1 \leq j \leq k-1$ and all but at most one index i.

This completes the proof of $d_\ell^k = \frac{k-2}{k-1}$ for large enough ℓ. We are left to analyze the extremal graphs. After reordering, we have $|Y_{i,j}| = \frac{n}{k-1}$ and $d(Y_{i,j}, Y_{i',j'}) = 1$ for $1 \leq j, j' \leq k-1$ and $1 \leq i, i' \leq k$, if $i \neq i'$ and $j \neq j'$.

Let $v \in V_{i'}$ for some $i' > k$. Then $|N(v) \cap \bigcup_{i \leq k} V_i| \leq \frac{k(k-2)}{k-1} n$, as otherwise there is a K^{k-1} in $N(v)$. On the other hand, equality must hold for all vertices $v \in V_{i'}$ due to the density condition. Therefore, $N(v) \cap \bigcup_{i \leq k} V_i = V_i \setminus Y_j$ for some $1 \leq j \leq k-1$. Define $Y_{i',j}$ accordingly for all $i' > k$, and let $Y_j = \bigcup_i Y_{i,j}$. Then $V = \bigcup Y_j$. For every permutation π of the set $\{1, \ldots, k-1\}$, there can be at most one set V_i with $|Y_{i,\pi(1)}| \geq |Y_{i,\pi(2)}| \geq \ldots \geq |Y_{i,\pi(k-1)}|$ and $|Y_{i,\pi(1)}| > |Y_{i,\pi(k-1)}|$. Otherwise, this pair of sets would have density smaller than $\frac{k-2}{k-1}$. Thus, all but at most $(k-1)!$ of the V_i have $|Y_{i,j}| = \frac{n}{k-1}$ for $1 \leq j \leq k-1$. Therefore, all extremal graphs are in $\mathcal{G}_\ell^k$. □

5 Open problems

As mentioned above, the characterization of the extremal graphs is not complete for $k > 3$. We need to determine all parameters n_i^s so that the resulting graphs in $\bar{\mathcal{G}}_\ell^k$ fulfill the density conditions.

The other obvious question left open is a good bound on ℓ depending on k in Theorem 2, and the determination of the exact values of d_ℓ^k for smaller ℓ. In particular, is it true that $d_5^3 = \frac{1}{2}$?

Another interesting open topic is the behavior of $d_\ell(H)$ for non-complete H. Bondy et al. [2] show that

$$\lim_{\ell\to\infty} d_\ell(H) = \frac{\chi(H)-2}{\chi(H)-1},$$

but it should be possible to show with similar methods as in this note that $d_\ell(H) = \frac{\chi(H)-2}{\chi(H)-1}$ for large enough ℓ depending on H.

References

[1] B. Bollobás, Extremal Graph Theory, Academic Press London (1978).

[2] A. Bondy, J. Shen, S. Thomassé and C. Thomassen, Density conditions for triangles in multipartite graphs, Combinatorica **26** (2006), 121–131.

[3] R. Diestel, Graph Theory, Springer-Verlag New York (1997).