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A Geometric Approach to an Asymptotic Expansion for Large Deviation Probabilities of Gaussian Random Vectors

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For the probabilities of large deviations of Gaussian random vectors an asymptotic expansion is derived. Based upon a geometric measure representation for the Gaussian law the interactions between global and local geometric properties both of the distribution and of the large deviation domain are studied. The advantage of the result is that the expansion coefficients can be obtained by making a series expansion of a surface integral avoiding the calculation of higher order derivatives.

1. INTRODUCTION

In many problems of probability theory, mathematical statistics and their applications probabilities of large deviations are of interest. Note that there are both many papers where the large deviation probabilities are studied for themselves and many papers where the logarithms of these probabilities are considered. Results concerning the second type of large deviation asymptotics are sometimes called rough limit theorems whereas the results of the first type are called sharp limit theorems. In the present paper we shall derive a third type of results in the form of asymptotic expansions for large deviation probabilities. These expansions include coefficients describing the local geometric structure of the boundary of the large deviation domain near the points around which the measure under consideration gives its main influence onto the asymptotics. While the first

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coefficient is a function of the curvatures of the boundary at these points, the following coefficients in the expansion reflect certain higher order local geometric properties of the large deviation domain in the neighborhoods of the just mentioned points.

Here we will consider a Gaussian random vector $\mathbf{X} = (X_1, ..., X_n)$ having a standard normal distribution, i.e. its mean vector is the zero vector $\mathbf{0} = (0, ..., 0)$, its covariance matrix is the n-dimensional unity matrix and its density is given by

$$\varphi_n(\mathbf{x}) = (2\pi)^{-n/2} \exp(-\tfrac{1}{2} \|\mathbf{x}\|^2)$$

with $\|\mathbf{x}\|$ the euclidean norm of the vector $\mathbf{x}$. The probability content of a set $A \subset \mathbb{R}^n$ for this probability measure is denoted by $\Phi_n(A)$. In the following we will study the asymptotic behaviour of the large deviation probabilities given by

$$\Phi_n(\lambda A) = \int_{\lambda A} \varphi_n(\mathbf{x}) \, d\mathbf{x}$$

as $\lambda \to \infty$, where we assume that A is a closed set and A^c—the complement of A—is a neighborhood of the origin in $\mathbb{R}^n$. This means that A^c is an absorbing set and therefore $\Phi_n(\lambda A) = 1 - \Phi_n(\lambda A^c) \to 0$ as $\lambda \to \infty$. So here we have to determine how fast $\Phi_n(\lambda A)$ approaches zero.

We set

$$a(A) = \inf\{ \|\mathbf{z}\|^2; \mathbf{z} \in A\}$$

and have $a(A) > 0$, since the origin lies in the interior of A^c. Let be defined further

$$M(A) = A \cap \{\mathbf{x}; \|\mathbf{x}\| = \sqrt{a(A)}\}.$$

Because A is a closed set, $M(A)$ is the subset of A which consists of all points with minimal distance $\sqrt{a(A)}$ to the origin. In this paper we shall exploit a geometric representation formula for Φ_n given in [10]

$$\Phi_n(A) = \frac{\omega_n}{(2\pi)^{n/2}} \int_0^\infty F(A; u) \, u^{n-1} \exp(-u^2/2) \, du \tag{1}$$

with $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ the surface area of the n-dimensional unit sphere $S_n(1) = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| = 1\}$ and $F(A; v)$ is defined by

$$F(A; v) = U_n(v^{-1} A \cap S_n(1)). \tag{2}$$

Here $U_n(\cdot)$ denotes the uniform probability distribution on the n-dimensional unit sphere. From Eq. (2) follows that $\omega_n F(A; u)$ is the surface area of the central projection of $A \cap S_n(u)$ on the n-dimensional sphere $S_n(1)$. The asymptotic behavior of the integral in Eq. (1) was studied, e.g. in [3], [10] and [11]. For the sets λA we have

$$\Phi_n(\lambda A) = \frac{\omega_n}{(2\pi)^{n/2}} \int_0^\infty F(\lambda A; u)\, u^{n-1} \exp(-u^2/2)\, du.$$

Making the substitution $u = \lambda v$, we get

$$\Phi_n(\lambda A) = \frac{\omega_n}{(2\pi)^{n/2}} \lambda^n \int_0^\infty F(\lambda A; \lambda v)\, v^{n-1} \exp(-(\lambda v)^2/2)\, dv.$$

From its definition in Eq. (2) we get $F(\lambda A; \lambda v) = U_n([\lambda v]^{-1} \cdot \lambda A \cap S_n(1)) = U_n(v^{-1} A \cap S_n(1)) = F(A; v)$, this gives

$$\Phi_n(\lambda A) = \frac{\omega_n}{(2\pi)^{n/2}} \lambda^n \int_0^\infty F(A; v)\, v^{n-1} \exp(-(\lambda v)^2/2)\, dv. \tag{3}$$

Note that on one side Eq. (3) reflects the global geometric property of the measure Φ_n to be invariant with respect to orthogonal transformations. On the other side it enables one to relate the local geometric properties of the measure Φ_n in a fixed neighborhood of the set $M(A)$ with the asymptotic behaviour of $\Phi_n(\lambda A)$ as $\lambda \to \infty$. Generally, this approach does not depend on whether the boundary ∂A of the large deviation region A is smooth in a neighborhood of $M(A)$ or not. But if one assumes smoothness of ∂A around $M(A)$ then it will be natural to reflect the local geometric properties of $\Phi_n(\lambda A)$ with the help of the respective differential geometric tools. In this respect we follow the approach in [5], [7] and [8] where the large deviation behavior of $\Phi_n(\lambda A)$ as $\lambda \to \infty$ is described, roughly spoken, by using the main curvatures of ∂A at the points in $M(A)$.

The expansions derived in the following are asymptotic expansions of Poincaré type (see [4], chapter 1.4). Let be given an finite or infinite so called asymptotic sequence $\{g_n(x)\}$, $n = 1, ..., N$ of functions g_n as $x \to \infty$, i.e. a sequence of continuous functions with $g_{n+1}(x) = o(g_n(x))$ as $x \to \infty$ for every n with $1 \leqslant n \leqslant N-1$. Following the notation of Bleistein and Handelsman, the relation

$$f(x) \sim \sum_{i=1}^{N} g_i(x), \qquad x \to \infty$$

means that for all n with $1 \leqslant n \leqslant N-1$ always

$$f(x) = \sum_{i=1}^{n} g_i(x) + O(g_{n+1}(x)), \qquad x \to \infty$$

and if N is finite that

$$f(x) = \sum_{i=1}^{N} g_i(x) + o(g_N(x)), \qquad x \to \infty.$$

Such a relation is called an asymptotic expansion of the function $f(x)$ as $x \to \infty$ with respect to the asymptotic sequence $\{g_n(x)\}$, $n = 1, ..., N$. Such expansions need not to be convergent.

In the case that $N = 1$, the relation

$$f(x) \sim g_1(x), \qquad x \to \infty$$

means that

$$f(x) = g(x) + o(g(x)), \qquad x \to \infty.$$

or, equivalently,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

In the case that A is defined by a twice differentiable function $g\colon \mathbb{R}^n \to \mathbb{R}$ by $A = \{\mathbf{x};\, g(\mathbf{x}) \leqslant 0\}$, A^c is an absorbing set and there is only one point $\mathbf{x}_0$ in $M(A)$, i.e. $M(A) = \{\mathbf{x}_0\}$, then we have that the following asymptotic equation holds

$$\Phi_n(\lambda A) \sim \Phi(-\lambda \cdot \sqrt{a(A)}) \prod_{i=1}^{n-1} (1 - \sqrt{a(A)} \cdot \kappa_i)^{-1/2}, \qquad \lambda \to \infty, \tag{4}$$

if the minimum of the function $\|\mathbf{x}\|^2$ with respect to $\partial A = \{\mathbf{x};\, g(\mathbf{x}) = 0\}$ is regular at $\mathbf{x}_0$ (see Appendix). Here $\Phi(x)$ denotes the one-dimensional standard normal integral and the κ_i's are the main curvatures of the surface ∂A at $\mathbf{x}_0$. From the fact that at $\mathbf{x}_0$ the function $\|\mathbf{x}\|$ has a minimum with respect to the surface ∂A follows that $\kappa_i \leqslant 1$ for $i = 1, ..., n-1$ and the minimum is regular, if $\kappa_i < 1$ for $i = 1, ..., n-1$. This result is proved in [5] and [8]. In the lecture note [7] main results for approximations of multivariate integrals by the Laplace method are collected.

In the case that $M(A)$ consists of a finite number of points $\mathbf{x}_1, ..., \mathbf{x}_k$ such that at all these points the minimum is regular, we get an analogous result

$$\Phi_n(\lambda A) \sim \Phi(-\lambda \cdot \sqrt{a(A)}) \sum_{j=1}^{k} \left[\prod_{i=1}^{n-1} (1 - \sqrt{a(A)} \cdot \kappa_{j,i})^{-1/2} \right], \qquad \lambda \to \infty$$

with the $\kappa_{j,1}, ..., \kappa_{j,n-1}$'s the main curvatures of ∂A at $\mathbf{x}_j$.

In this paper we will show the connection between the function $F(A; v)$ and the local structure of ∂A near $M(A)$ in the case that $M(A)$ consists of only one point. This result is generalized easily to the case of a finite number of points in $M(A)$.

If one assumes an expansion for $F(A; v)$ to hold as $v \to \sqrt{a(A)}$, then by Eq. (3) and Watson's lemma, given below, an asymptotic expansion for the large deviation probabilities $\Phi_n(\lambda A)$ as $\lambda \to \infty$ can be derived.

LEMMA 1 (Watson's Lemma). *Let $f\colon (0, \infty) \to \mathbb{R}$ be a locally integrable function, which is bounded on finite intervals. Further assume to hold*

$$f(t) = O(e^{at}), \qquad t \to \infty$$

for a real number a. For $t \to 0+$ let the function $f(t)$ have the following expansion

$$f(t) \sim \sum_{m=0}^{\infty} c_m t^{a_m},$$

where $\mathrm{Re}(a_m)$ *increases monotonically to* $+\infty$ *as* $m \to \infty$ *and* $\mathrm{Re}(a_0) > -1$. *Then the Laplace transform of $f(t)$ has the following asymptotic expansion*

$$\int_0^{\infty} e^{-\lambda t} f(t)\, dt \sim \sum_{m=0}^{\infty} \frac{c_m \Gamma(a_m + 1)}{\lambda^{a_m + 1}}, \qquad \lambda \to \infty. \tag{5}$$

A proof is given for example in [4], p. 103/4.

The possibility of asymptotic expansions of multivariate integrals with boundary maxima is discussed in [4], chap. 8, [9], p. 82, Theorem 4.5 and [14], chap. IX.5, but no methods for obtaining higher order terms are outlined. The following results can be used for this purpose. They show in which way higher order local geometric properties of the range of integration near the so called dominating point can be used for determining respective coefficients $c_0, c_1, \ldots$.

2. LARGE DEVIATIONS IN TWO DIMENSIONS

In this paragraph we treat the case of sets $A \subset \mathbb{R}^2$ and show here how the representation of the probabilities $\Phi_2(\lambda A)$ given in Eq. (3) and the asymptotic approximation given in Eq. (4) are related. A short review about basic theory of curves and surfaces is given in the Appendix.

We assume first that A is given in the form $A = \{\mathbf{x}; g(\mathbf{x}) \leqslant 0\}$ with g a continuous function; the boundary ∂A is then given by $\partial A = \{\mathbf{x}; g(\mathbf{x}) = 0\}$. Further it is assumed that the following conditions 1–4 are fulfilled:

1. $a(A) = 1$.

2. $M(A) = \{\mathbf{x}_0\} = \{(0, 1)^T\}$.

3. In an open neighborhood V of $\mathbf{x}_0$ the function g is twice continuously differentiable and the minimum of $\|\mathbf{x}\|^2$ at $\mathbf{x}_0$ with respect to ∂A is regular.

4. There is an open neighborhood U of $\mathbf{x}_0$ such that the curve part $U \cap \partial A = \{(x, y) \in U;\ g(x, y) = 0\}$ can be represented in the form

$$y = 1 + \sum_{r=2}^{\infty} \frac{k_r}{r!} x^r \tag{6}$$

by an absolutely convergent power series, i.e. there is an $\varepsilon > 0$ such that $\sum_{r=2}^{\infty} |k_r| \, \varepsilon^r / r! < \infty$.

The first condition is only a standardization to simplify the derivation of the next two lemmas. The final result will be given for arbitrary $a(A) > 0$. Due to the rotational symmetry of the standard normal distribution the second condition says only that $M(A)$ consists of one point only; by a suitable rotation we can achieve always that, if $a(A) = 1$ and $M(A)$ is a one-point set, this point is $(0, 1)^T$. The third condition says that the curvature κ of the curve ∂A at $\mathbf{x}_0$ is less than unity. The last condition states that there is a power series expansion of the curve near $\mathbf{x}_0$. Since at $\mathbf{x}_0$ the function $\|\mathbf{x}\|^2$ has a minimum with respect to the curve $U \cap \partial A$, due to the Lagrange multiplier theorem the first coefficient k_1 in the representation $y = 1 + \sum_{r=1}^{\infty} (k_r / r!) \, x^r$ must be zero. We have further that $k_2 = -\kappa$ with κ the curvature of ∂A at $\mathbf{x}_0$, see Eq. (26) in the Appendix. As $x \to 0$ this gives

$$y = 1 + \frac{k_2}{2} x^2 + o(x^2).$$

If we use the representation for $\Phi_n(\lambda A)$ given in Eq. (3), we get with $n = 2$ that

$$\Phi_2(\lambda A) = \lambda^2 \int_0^{\infty} F(A; u) \, u \exp(-(\lambda u)^2 / 2) \, du.$$

Since $a(A) = 1$, we have $F(A; u) = 0$ for all u with $0 \leqslant u < 1$. Therefore we can write

$$\Phi_2(\lambda A) = \lambda^2 \int_1^{\infty} F(A; u) \, u \exp(-(\lambda u)^2 / 2) \, du. \tag{7}$$

Making the substitution $v = u - 1$ we can write the integral in the form

$$\Phi_2(\lambda A) = \lambda^2 \int_0^\infty F(A; 1+v)(1+v) \exp(-(\lambda(1+v))^2/2)\, dv$$

$$= \lambda^2 \exp(-\lambda^2/2) \int_0^\infty F(A; 1+v)(1+v) \exp(-\lambda^2(v + v^2/2))\, dv.$$

To get an asymptotic expansion for the probabilities $\Phi_2(\lambda A)$ as $\lambda \to \infty$, we will show in the following that these probabilities can be obtained as the Laplace transform of $F(A; (1+v)^{1/2})$.

For small v the function $F(A; 1+v)$ has under the conditions made above a simple geometric interpretation. Since only at $\mathbf{x}_0$ the domain A has minimal distance to the origin and g is continuous, for small v the function $F(A; 1+v)$ is determined by the local structure of A and its boundary ∂A near $\mathbf{x}_0$. For v small enough the set $A \cap S_2(1+v)$ is a simply connected arc on $S_2(1+v)$. Since $F(A; 1+\delta) = U_2((1+\delta)^{-1} A \cap S_2(1))$ in this case $F(A; 1+\delta)$ is the arc length on the circle with radius $1+\delta$ around the origin from the intersection point (x_-, y_-) of this circle and the curve ∂A to the intersection point (x_+, y_+) divided by $2\pi(1+\delta)$ or the arc length on the unit circle around the origin from $(x_-(1+\delta)^{-1}, y_-(1+\delta)^{-1})$ to $(x_+(1+\delta)^{-1}, y_+(1+\delta)^{-1})$ divided by 2π. Due to the conditions 1–4 there exist for $\delta \to 0$ exactly two intersection points, which move towards $(0, 1)$ as $\delta \to 0+$. To expand the function $F(A; 1+\delta)$ we therefore need the positions of these points.

The arc length can be computed using the first coordinate x as curve parameter for a part of the unit circle around the origin near $\mathbf{x}_0$. The arc length is then

$$F(A, 1+\delta) = (2\pi)^{-1} \int_{x_-(1+\delta)^{-1}}^{x_+(1+\delta)^{-1}} [1 + y'^2(x)]^{1/2}\, dx.$$

Here the function $y'(x)$ is given by

$$y'(x) = \frac{d(1-x^2)^{1/2}}{dx} = -x(1-x^2)^{-1/2}.$$

Since $(1 + y'^2(x))^{1/2} = (1-x^2)^{-1/2}$, we get

$$F(A; 1+\delta) = (2\pi)^{-1} \int_{x_-(1+\delta)^{-1}}^{x_+(1+\delta)^{-1}} (1-x^2)^{-1/2}\, dx$$

$$= (2\pi)^{-1} [\arcsin(x_+(1+\delta)^{-1}) + \arcsin(|x_-|(1+\delta)^{-1})]. \quad (8)$$

LEMMA 2. *Under the conditions* 1–4 *the function* x_+ (*resp.* $|x_-|$) *of* δ *has for* δ *small enough a convergent series expansion in terms of* $z^{1/2} = (2\delta + \delta^2)^{1/2}$ *in the form*

$$x_+ = \sum_{m=1}^{\infty} \beta_m^+ z^{m/2}, \qquad \textit{resp. } |x_-| = \sum_{m=1}^{\infty} \beta_m^- z^{m/2}$$

with

$$\beta_1^+ = \beta_1^- = (1-\kappa)^{-1/2} \tag{9}$$

$$\beta_2^+ = -\beta_2^- = \frac{-k_3}{6(1-\kappa)^2} \tag{10}$$

$$\beta_3^+ = \beta_3^- = \frac{5k_3^2}{72(1-\kappa)^{7/2}} - \frac{3\kappa^2 + k_4}{24(1-\kappa)^{5/2}} \tag{11}$$

and as $\delta \to 0+$

$$x_+ = \left(\frac{2}{1-\kappa}\right)^{1/2} \cdot \delta^{1/2} + o(\delta^{1/2}), \qquad \textit{resp. } x_- = -\left(\frac{2}{1-\kappa}\right)^{1/2} \cdot \delta^{1/2} + o(\delta^{1/2}).$$

Proof. The point (x_+, y_+) is that intersection point of the circle around the origin with radius $1+\delta$ and the curve ∂A in the first quadrant, which has a small positive x-value. Here we assume that δ is so small that $A \cap S_2(1+\delta)$ is a simply connected set. Due to condition 4, near $\mathbf{x}_0$ the curve ∂A can be represented by the absolutely convergent power series expansion

$$y = 1 + \sum_{r=2}^{\infty} \frac{k_r}{r!} x^r. \tag{12}$$

The circle $S_2(1+\delta)$ is defined by the equation $x^2 + y^2 = (1+\delta)^2$. By squaring Eq. (12), we get again a convergent power series expansion with coefficients $\hat{k}_r$, i.e.

$$y^2 = 1 + \sum_{r=2}^{\infty} \frac{\hat{k}_r}{r!} x^r$$

with $\hat{k}_2 = 2k_2 = -2\kappa$. To find the point (x_+, y_+) we use the two equations for y_+^2, which are

$$y_+^2 = 1 + \sum_{r=2}^{\infty} \frac{\hat{k}_r}{r!} x_+^r$$

$$y_+^2 = (1+\delta)^2 - x_+^2.$$

This gives then

$$(1+\delta)^2 - x_+^2 = 1 + \sum_{r=2}^{\infty} \frac{\hat{k}_r}{r!} x_+^r$$

$$2\delta + \delta^2 - x_+^2 = \sum_{r=2}^{\infty} \frac{\hat{k}_r}{r!} x_+^r.$$

By defining $z = 2\delta + \delta^2$, we get further that

$$z = \sum_{r=2}^{\infty} \frac{\hat{k}_r}{r!} x_+^r x_+^2 = (1+k_2)\, x_+^2 + \sum_{r=3}^{\infty} \frac{\hat{k}_r}{r!} x_+^r$$

$$= (1-\kappa)\, x_+^2 + \sum_{r=3}^{\infty} \frac{\hat{k}_r}{r!} x_+^r.$$

Since $\kappa < 1$ due to assumption 3 the coefficient of x_+^2 is larger than zero. We rewrite the equation

$$z = (1-\kappa)\, x_+^2 \left(1 + \sum_{r=3}^{\infty} \tilde{k}_r x_+^{r-2}\right)$$

with

$$\tilde{k}_r = \frac{\hat{k}_r}{r!\,(1-\kappa)}, \qquad \text{for} \quad r = 3, 4, \ldots. \tag{13}$$

Taking the square root gives

$$z^{1/2} = (1-\kappa)^{1/2} \cdot x_+ \left(1 + \sum_{r=3}^{\infty} \tilde{k}_r x_+^{r-2}\right)^{1/2}.$$

Expanding the square root of the function in the parenthesis into a power series in terms of x_+ gives then

$$z^{1/2} = \sum_{r=1}^{\infty} \tilde{\alpha}_r x_+^r$$

with $\tilde{\alpha}_1 = (1-\kappa)^{1/2} > 0$. Now, since $\tilde{\alpha}_1 \neq 0$, we can invert this series to get an expansion of x_+ in terms of $z^{1/2}$, i.e.

$$x_+ = \sum_{i=1}^{\infty} \beta_i^+ z^{i/2}.$$

The result for $|x_-|$ can be derived in the same way.

To find the first three coefficients, we note first that

$$y_+ = 1 + \frac{k_2}{2!} x_+^2 + \frac{k_3}{3!} x_+^3 + \frac{k_4}{4!} x_+^4 + o(x_+^4)$$

$$y_+^2 = 1 + \frac{\hat{k}_2}{2!} x_+^2 + \frac{\hat{k}_3}{3!} x_+^3 + \frac{\hat{k}_4}{4!} x_+^4 + o(x_+^4)$$

with $\hat{k}_2 = 2k_2$, $\hat{k}_3 = 2k_3$ and $\hat{k}_4 = 6k_2^2 + 2k_4$.

From this follows then as above

$$z = (1 - \kappa)\, x_+^2 (1 + \tilde{k}_3 x_+ + \tilde{k}_4 x_+^2 + o(x_+^2)),$$

or

$$\frac{z}{1-\kappa} = x_+^2 (1 + \tilde{k}_3 x_+ + \tilde{k}_4 x_+^2 + o(x_+^2)).$$

Using Eq. (3.6.18) in [1] for the expansion of the square root of a series gives then

$$\left(\frac{z}{1-\kappa}\right)^{1/2} = x_+ + \underbrace{\frac{\tilde{k}_3}{2}}_{=b} x_+^2 + \underbrace{\left(\frac{\tilde{k}_4}{2} - \frac{\tilde{k}_3^2}{8}\right)}_{=c} x_+^3 + o(x_+^3).$$

Now using Eq. (3.6.25) in [1] for the inversion of a series, we can invert this expansion, yielding

$$x_+ = \left(\frac{z}{1-\kappa}\right)^{1/2} - b \cdot \frac{z}{1-\kappa} + (2b^2 - c) \cdot \left(\frac{z}{1-\kappa}\right)^{3/2} + o(z^{3/2}).$$

Inserting the values for b and c gives then

$$\begin{aligned} x_+ &= \left(\frac{z}{1-\kappa}\right)^{1/2} - \frac{\tilde{k}_3}{2} \cdot \frac{z}{1-\kappa} + \left[\frac{\tilde{k}_3^2}{2} - \left(\frac{\tilde{k}_4}{2} - \frac{\tilde{k}_3^2}{8}\right)\right] \frac{z^{3/2}}{(1-\kappa)^{3/2}} + o(z^{3/2}) \\ &= \left(\frac{z}{1-\kappa}\right)^{1/2} - \frac{\tilde{k}_3}{2} \cdot \frac{z}{1-\kappa} + \left[\frac{5\tilde{k}_3^2}{8} - \frac{\tilde{k}_4}{2}\right] \frac{z^{3/2}}{(1-\kappa)^{3/2}} + o(z^{3/2}). \end{aligned}$$

Inserting further the values for $\tilde{k}_3$ and $\tilde{k}_4$ from Eq. (13) gives finally

$$\begin{aligned} x_+ = &\left(\frac{z}{1-\kappa}\right)^{1/2} - \frac{k_3}{6(1-\kappa)^2} z \\ &+ \left(\frac{5k_3^2}{72(1-\kappa)^{7/2}} - \frac{3\kappa^2 + k_4}{24(1-\kappa)^{5/2}}\right) z^{3/2} + o(z^{3/2}). \end{aligned}$$

The result for $|x_-|$ can be derived in the same way. ∎

Using this result, we can find an expansion of the function $F(A; 1+\delta)$ as $\delta \to 0+$.

LEMMA 3. *Under the conditions* 1–4 *the function* $F(A; 1+\delta)$ *has an absolutely convergent series expansion in terms of* $z^{1/2} = (2\delta + \delta^2)^{1/2}$ *in the form*

$$F(A; 1+\delta) = F(A; (1+z)^{1/2}) = \sum_{m=1}^{\infty} \gamma_m z^{m/2} \tag{14}$$

for sufficiently small positive δ, *where*

$$\gamma_1 = \pi^{-1}(1-\kappa)^{-1/2} \tag{15}$$

$$\gamma_2 = 0 \tag{16}$$

$$\gamma_3 = \pi^{-1}(\beta_3^+ - \beta_1^+/2 + (\beta_1^+)^3/6). \tag{17}$$

Here β_1^+, β_2^+ *and* β_3^+ *are defined in* (9), (10) *and* (11).

Proof. The function $F(A; 1+\delta)$ is given in Eq. (8). We split it up into two parts

$$\begin{aligned} &F(A; 1+\delta) \\ &\quad = (2\pi)^{-1}\left[\int_{x_-(1+\delta)^{-1}}^{0} (1-x^2)^{-1/2}\,dx + \int_0^{x_+(1+\delta)^{-1}} (1-x^2)^{-1/2}\,dx\right] \\ &\quad = (2\pi)^{-1}\,[\arcsin(x_+(1+\delta)^{-1}) + \arcsin(|x_-|(1+\delta)^{-1})] \\ &\quad = (2\pi)^{-1}\,[\arcsin(x_+(1+z)^{-1/2}) + \arcsin(|x_-|(1+z)^{-1/2})]. \end{aligned} \tag{18}$$

Here we used that $(1+\delta)^{-1} = (1+z)^{-1/2}$. Since x_- and $|x_+|$ have expansions in terms of $z^{1/2}$ we get by making first a series expansion of $x_+(1+z)^{-1/2}$ and $|x_-|(1+z)^{-1/2}$ in terms of $z^{1/2}$ and then inserting these into the power series expansion of arc sine at zero the result above.

We have

$$\begin{aligned} x_+(1+z)^{-1/2} &= (\beta_1^+ z^{1/2} + \beta_2^+ z + \beta_3^+ z^{3/2} + o(z^{3/2}))(1 - z/2 + o(z)) \\ &= \beta_1^+ z^{1/2} + \beta_2^+ z + (\beta_3^+ - \beta_1^+/2)\, z^{3/2} + o(z^{3/2}). \end{aligned} \tag{19}$$

Now, we have for the arc sine function the following expansion (see [1], Eq. 4.4.40)

$$\arcsin(y) = y + \frac{y^3}{6} + o(y^3), \qquad y \to 0.$$

Inserting in Eq. (19) the last equation gives the result

$$F(A;(1+z)^{1/2}) = \pi^{-1}[\beta_1^+ z^{1/2} + (\beta_3^+ - \beta_1^+/2 + (\beta_1^+)^3/6)\, z^{3/2}] + o(z^{3/2}). \quad \blacksquare$$

The last lemma gives an expansion of $F(A; 1+\delta) = F(A; (1+z)^{1/2})$ in terms of $z^{1/2} = (2\delta + \delta^2)^{1/2}$. To find now an asymptotic expansion of $\Phi_2(\lambda A)$ as $\lambda \to \infty$ we have to relate this expansion with the probabilities $\Phi_2(\lambda A)$ using Eq. (3).

We replace now conditions 1, 2 and 4 by:

1′. $a(A) > 0$.

2′. $M(A) = \{\mathbf{x}_0\} = \{(0, \sqrt{a(A)})^T\}$.

4′. There is an open neighborhood U of $\mathbf{x}_0$ such that the curve part $U \cap \partial A = \{(x, y) \in U;\ g(x, y) = 0\}$ can be represented in the form

$$y = \sqrt{a(A)} + \sum_{r=2}^{\infty} \frac{a(A)^{r/2}\, k_r}{r!}\, x^r \tag{20}$$

by an absolutely convergent power series, i.e. there is an $\varepsilon > 0$ such that $\sum_{r=2}^{\infty} |k_r|\, \varepsilon^r/r! < \infty$.

In the following theorem we derive the relation between the expansions of $\Phi_2(\lambda A)$ as $\lambda \to \infty$ and $F(A; (1+z)^{1/2})$ as $z \to 0+$.

THEOREM 4. *Under the assumptions* 1′, 2′, 3 *and* 4′ *the probabilities* $\Phi_2(\lambda A)$ *have the following asymptotic expansion*

$$\Phi_2(\lambda A) \sim \frac{\exp(-a(A)\, \lambda^2/2)}{\lambda \sqrt{a(A)\, 2\pi}} \sum_{i=0}^{\infty} a_i (\sqrt{a(A)} \cdot \lambda)^{-i}, \qquad \lambda \to \infty. \tag{21}$$

For even i the coefficients a_{i-1} *are zero and for odd i the coefficient* a_{i-1} *are given by*

$$a_{i-1} = \gamma_i \Gamma(i/2+1)\, \pi^{1/2} 2^{(i+1)/2}$$

with γ_i *from Eq.* (14) *and* $a_0 = (1-\kappa)^{-1/2}$ *and* $a_2 = 3(\beta_3^+ - \beta_1^+/2 + (\beta_1^+)^3/6)$. *Here* κ *is the curvature of* $\partial A/\sqrt{a(A)}$ *at* $\mathbf{x}_0/\sqrt{a(A)}$.

Recall that because of the asymptotic nature of the expansion, the formal series on the right hand side is not necessarily convergent. This is reflected by the fact that $a_i/(\lambda^i \gamma_{i+1})$ tends to infinity quite fast as $i \to \infty$.

Proof. We assume first that $a(A) = 1$. The fact that all coefficients a_i for odd i are zero follows from Theorem 4.5 in [9], where the existence of such an expansion under the conditions given above is proved. We have then

$$\Phi_2(\lambda A) = \lambda^2 \int_0^\infty F(A; 1+v)(1+v) \exp(-\lambda^2(1+v)^2/2)\, dv. \tag{22}$$

This can be written as

$$\Phi_2(\lambda A) = \lambda^2 \exp(-\lambda^2/2) \int_0^\infty F(A; 1+v)(1+v) \exp(-\lambda^2(v+v^2/2))\, dv.$$

Making the substitution $z = 2v + v^2$, we have $1 + v = (1+z)^{1/2}$ and $dv/dz = \frac{1}{2} \cdot (1+z)^{-1/2}$. This gives then for the integral

$$\Phi_2(\lambda A) = \frac{\lambda^2}{2} \exp(-\lambda^2/2) \int_0^\infty F(A; (1+z)^{1/2}) \exp\left(-\frac{\lambda^2}{2} \cdot z\right) dz.$$

Now this integral is the Laplace transform of the function $F(A; (1+z)^{1/2})$ at $\lambda^2/2$. From Lemma 3 we have for $F(A; (1+z)^{1/2})$ at $z = 0$ the convergent expansion

$$F(A; (1+z)^{1/2}) = \sum_{i=1}^\infty \gamma_i z^{i/2}.$$

From this we can derive an asymptotic expansion for the Laplace transform using Watson's lemma

$$\int_0^\infty F(A; (1+z)^{1/2}) \exp\left(-\frac{\lambda^2}{2} \cdot z\right) dz \sim \sum_{i=1}^\infty \frac{\gamma_i \Gamma(i/2+1)\, 2^{(i+2)/2}}{\lambda^{i+2}}, \qquad \lambda \to \infty.$$

Inserting this into Eq. (22) gives then

$$\Phi_2(\lambda A) \sim \frac{\exp(-\lambda^2/2)}{\lambda \sqrt{2\pi}} \cdot \sum_{i=0}^\infty a_i \lambda^{-i}, \qquad \lambda \to \infty.$$

This proves the result for $a(A) = 1$. The general case is derived in the same way by considering instead of A the standardized set $\tilde{A} = A/\sqrt{a(A)}$. ∎

3. THE MULTIVARIATE CASE

In this paragraph we consider the general case of a set A in the n-dimensional space. We assume that A is defined by a continuous function $g: \mathbb{R}^n \to \mathbb{R}$ by $A = \{\mathbf{x}; g(\mathbf{x}) \leqslant 0\}$. We assume further that the conditions 1–4 hold:

1. $a(A) > 0$.

2. $M(A) = \{\mathbf{x}_0\} = \{(0, ..., 0, \sqrt{a(A)})^T\}$.

3. In an open neighborhood V of $\mathbf{x}_0$ the function g is twice continuously differentiable and the minimum of $\|\mathbf{x}\|^2$ at $\mathbf{x}_0$ with respect to the surface ∂A is regular.

4. In the neighborhood $V/\sqrt{a(A)}$ of $(0, ..., 0, 1)$ the surface $\partial A/\sqrt{a(A)}$ can be parametrized locally at $(0, ..., 0, 1)$ by using the first $n-1$ coordinates as parameters in the form

$$x_n = 1 + f(\mathbf{x}^*), \qquad \mathbf{x}^* = (x_1, ..., x_{n-1}) \in U(\mathbf{0}).$$

Here $U(\mathbf{0})$ is a neighborhood of the origin in $\mathbb{R}^{n-1}$ and the function f allows an absolute convergent power series representation such that

$$x_n = 1 + \sum_{r=2}^{\infty} \frac{k_r(\boldsymbol{\tau})}{r!} \rho^r,$$

where $\rho = \|\mathbf{x}^*\|$ is the norm of $\mathbf{x}^*$ in $\mathbb{R}^{n-1}$ and $\boldsymbol{\tau} = \rho^{-1}\mathbf{x}^*$ is the unit vector in the direction of $\mathbf{x}^*$. Here the $k_r(\boldsymbol{\tau})$ are continuous functions and there is an absolutely convergent power series with coefficients k_r^* such that $\max_{\|\boldsymbol{\tau}\|=1} |k_r(\boldsymbol{\tau})| \leqslant k_r^*$ and for all $0 < \varepsilon \leqslant \varepsilon^*$ with $\varepsilon^* > 0$, i.e.

$$\sum_{r=2}^{\infty} \frac{k_r^*}{r!} \varepsilon^r < \infty.$$

Here condition 3 means that the matrix

$$\mathbf{H} = (-(\delta_{ij} - |\nabla g(\mathbf{x}_0)|^{-1} g_{ij}(\mathbf{x}_0)))_{i,j=1,...,n}$$

is positive definite with respect to the constraint $(\nabla g(\mathbf{x}_0))^T \mathbf{x} = 0$, see the Appendix. As shown in the appendix of [5] this is equivalent to the condition that $1 - \kappa_i > 0$, $i = 1, ..., n-1$ with $\kappa_1, ..., \kappa_{n-1}$ being the main curvatures of ∂A at $\mathbf{x}_0$. Therefore the minimum is regular, i.e. the matrix positive definite under the constraint, if all main curvatures are less than unity.

In the following we will again relate the representation of $\Phi_n(\lambda A)$ given in Eq. (3) with the local surface structure of the surface ∂A. First we assume again that $a(A) = 1$. We have then

$$\Phi_n(\lambda A) = \frac{\omega_n}{(2\pi)^{n/2}} \lambda^n \int_0^{\infty} F(A; u)\, u^{n-1} \exp(-(\lambda u)^2/2)\, du.$$

Making again the substitution $u^2 \to z$ as in the last paragraph gives then

$$\Phi_n(\lambda A) = \frac{\omega_n}{(2\pi)^{n/2}} \frac{\lambda^n}{2} \exp(-\lambda^2/2) \int_0^\infty [F(A; (1+z)^{1/2})(1+z)^{(n-2)/2}] \times \exp\left(-\frac{\lambda^2}{2} \cdot z\right) dz.$$

To apply Watson's lemma, we have to determine an expansion of the function in the square brackets as $z \to 0+$. As in the two-dimensional case the behavior of $F(A; (1+z)^{1/2})$ as $z \to 0+$ determines the asymptotic behavior of $\Phi_n(\lambda A)$ as $\lambda \to \infty$.

THEOREM 5. *Under the conditions* 1–4 *the probabilities* $\Phi_n(\lambda A)$ *have an asymptotic expansion in the form*

$$\Phi_n(\lambda A) \sim \frac{\exp(-a(A)\,\lambda^2/2)}{\lambda\sqrt{a(A)\,2\pi}} \sum_{i=0}^{\infty} a_i(\sqrt{a(A)} \cdot \lambda)^{-i}, \qquad \lambda \to \infty \tag{23}$$

with

$$a_0 = \prod_{i=1}^{n-1} (1-\kappa_i)^{-1/2}.$$

The κ_i*'s are the main curvatures of the surface* $\partial A/\sqrt{a(A)}$ *at* $\mathbf{x}_0/\sqrt{a(A)}$ *and the coefficients* a_i *are determined by the expansion of* $F(A/\sqrt{a(A)}; (1+z)^{1/2})$ *at* $z=0$ *in the following way. If*

$$F(A/\sqrt{a(A)}; (1+z)^{1/2})(1+z)^{(n-2)/2} = z^{(n-2)/2} \sum_{i=1}^{\infty} \gamma_i z_i,$$

then for even i the coefficient a_{i-1} *is zero and for odd i the coefficient* a_{i-1} *is given by*

$$a_{i-1} = \gamma_i \frac{\Gamma((n+i)/2)}{\Gamma(n/2)} \pi^{1/2} 2^{(i+1)/2}.$$

Proof. The vanishing of the odd coefficients follows again from Theorem 4.5 in [9]. We assume first that $a(A)=1$. Then

$$\Phi_n(\lambda A) = \frac{\omega_n}{(2\pi)^{n/2}} \frac{\lambda^n}{2} \exp(-\lambda^2/2) \int_0^\infty [F(A; (1+z)^{1/2})(1+z)^{(n-2)/2}] \times \exp\left(-\frac{\lambda^2}{2} \cdot z\right) dz.$$

To get an asymptotic expansion we have to find first an expansion of the function in the brackets as $z \to 0+$ and then to use again Watson's lemma in the same way as in the last theorem.

As $z \to 0+$ the domain $(1+z)^{-1/2} A \cap S_n(1)$ is contracting towards the set $\{\mathbf{x}_0\}$. Since g is continuous, there exists a $z_V > 0$ such that for all z with $0 < z < z_V$ the set $(1+z)^{-1/2} A \cap S_n(1)$ is a simply connected set. We assume now that z is so small that $(1+z)^{-1/2} A \cap S_n(1)$ is a subset of the set V defined in condition 3. Then using the local parametrization by the first $n-1$ coordinates, we can calculate the surface area of the set $(1+z)^{-1/2} A \cap S_n(1)$, which is $\omega_n \cdot F(A; (1+z)^{1/2})$, by an $n-1$-dimensional integral over the set $U(\mathbf{0})$ defined in condition 4

$$\omega_n F(A; (1+z)^{1/2}) = \int_U \mathbf{1}_{\{(\mathbf{x}^*, h(\mathbf{x}^*)) \in (1+z)^{-1/2} A \cap S_n(1)\}} T(\mathbf{x}^*)\, d\mathbf{x}^*.$$

Here the surface under consideration is $\{(\mathbf{x}^*, h(\mathbf{x}^*), \ \mathbf{x}^* \in U\}$, where $\mathbf{x}^* = (x_1, ..., x_{n-1})$ and $h(\mathbf{x}^*) = (1 - \|\mathbf{x}^*\|^2)^{1/2}$. The function $T(\mathbf{x}^*) = (1 + \sum_{i=1}^{n-1} h_i^2(\mathbf{x}^*))^{1/2} = (1 - \|\mathbf{x}^*\|^2)^{1/2}$ denotes the determinant of the transformation $\mathbf{x} \to (\mathbf{x}^*, h(\mathbf{x}^*))$, where the h_i are the partial derivatives of h with respect to x_i, $i = 1, ..., n-1$.

To reduce this essentially to a two-dimensional problem, we introduce now spherical coordinates in the set $U(\mathbf{0}) \subset \mathbb{R}^{n-1}$. The transformation is $\mathbf{x}^* \to (\boldsymbol{\tau}, \rho)$ with $\boldsymbol{\tau} = \|\mathbf{x}^*\|^{-1} \mathbf{x}^*$, $\rho = \|\mathbf{x}^*\|$; the cartesian coordinates are obtained from $\boldsymbol{\tau}$ and ρ by $\mathbf{x}^* = \rho\boldsymbol{\tau}$. The respective transformation determinant is ρ^{n-2} multiplied by a product of powers of trigonometric functions. This trigonometric part will be summarized into an $n-2$-dimensional surface integration.

We get then $T(\mathbf{x}^*) = \tilde{T}(\rho) = (1 - \rho^2)^{-1/2}$ and

$$F(A; (1+z)^{1/2}) = \omega_n^{-1} \int_{S_{n-2}(1)} \underbrace{\left[\int_0^\infty \mathbf{1}_{\{(\rho\boldsymbol{\tau}, h(\rho\boldsymbol{\tau})) \in (1+z)^{-1/2} A \cap S_n(1)\}} \rho^{n-2} \tilde{T}(\rho)\, d\rho \right]}_{= I(\boldsymbol{\tau}, z)} ds_{n-2}(\boldsymbol{\tau}) \tag{24}$$

Here $ds_{n-2}(\boldsymbol{\tau})$ denotes surface integration over $S_{n-2}(1)$. For fixed $\boldsymbol{\tau}$, the integral $I(\boldsymbol{\tau}, z)$ in the square brackets is a function of z only. Since $(1+z)^{-1/2} A \cap S_n(1)$ is simple connected, this integral has always the form

$$I(\boldsymbol{\tau}, z) = \int_0^{x_+(\boldsymbol{\tau}, z)} \rho^{n-2} \tilde{T}(\rho)\, d\rho$$

with $x_+(\boldsymbol{\tau}, z) = \max\{\rho > 0; (\rho\boldsymbol{\tau}, h(\rho\boldsymbol{\tau})) \in (1+z)^{-1/2} A \cap S_n(1)\}$.

The function $x_+(\boldsymbol{\tau}, z)$ can be found as in the two dimensional problem of the last paragraph in the form of an expansion in terms of $z^{1/2}$, where now the coefficients $\beta_i^+(\boldsymbol{\tau})$ depend on $\boldsymbol{\tau}$; here the condition that the minimum at $\mathbf{x}_0$ is regular ensures that always $k_2(\boldsymbol{\tau}) > 1$ respectively $\kappa(\boldsymbol{\tau}) < 1$. This gives with, e.g. $\beta_1(\boldsymbol{\tau}) = (1 - \kappa(\boldsymbol{\tau}))^{-1/2}$, $\beta_2(\boldsymbol{\tau}) = -k_3(\boldsymbol{\tau})\, 6^{-1}(1 - \kappa(\boldsymbol{\tau}))^{-2}$, etc.

$$x_+(\boldsymbol{\tau}, z) = \sum_{i=1}^{\infty} \beta_i(\boldsymbol{\tau})\, z^{i/2}.$$

Similar as in the last paragraph we have to consider the integral

$$I(\boldsymbol{\tau}, z) = \int_0^{x_+(\boldsymbol{\tau}, z)} \frac{\rho^{n-2}}{(1-\rho^2)^{1/2}}\, d\rho.$$

By making a Taylor series expansion of the integrand, integrating term by term and then inserting the expansion of $x_+(\boldsymbol{\tau}, z)$ we finally get an expansion of $I(\boldsymbol{\tau}, z)$ in the form

$$I(\boldsymbol{\tau}, z) = z^{(n-2)/2} \sum_{i=1}^{\infty} \gamma_i(\boldsymbol{\tau})\, z^{i/2}. \tag{25}$$

Here, e.g., $\gamma_1(\boldsymbol{\tau}) = \beta_1(\boldsymbol{\tau})^{n-1}/(n-1)$. Inserting the last equation into Eq. (24) gives then

$$\begin{aligned} F(A; (1+z)^{1/2}) &= \omega_n^{-1} \int_{S_{n-2}(1)} \left[z^{(n-2)/2} \sum_{i=1}^{\infty} \gamma_i(\boldsymbol{\tau})\, z^{i/2} \right] ds_{n-2}(\boldsymbol{\tau}) \\ &= z^{(n-2)/2} \sum_{i=1}^{\infty} \bar{\gamma}_i z^{i/2} \end{aligned}$$

with $\bar{\gamma}_i = \omega_n^{-1} \int_{S_{n-2}(1)} \gamma_i(\boldsymbol{\tau})\, ds_{n-2}(\boldsymbol{\tau})$.

To obtain an expansion of $F(A; (1+z)^{1/2})(1+z)^{(n-2)/2}$, we make an expansion of $(1+z)^{(n-2)/2}$ and multiply it by the expansion of $F(A; (1+z)^{1/2})$ given above. Then we have finally

$$F(A; (1+z)^{1/2})(1+z)^{(n-2)/2} = z^{(n-2)/2} \sum_{i=1}^{\infty} \gamma_i z^{i/2} = \sum_{i=1}^{\infty} \gamma_i z^{(n+i)/2-1}.$$

Here the coefficients γ_i are obtained from the multiplication of the two series. Using again Watson's lemma gives the result of the theorem. The value of the first coefficient a_0 follows from the results in [5] and [8], Corollary 2.

The general case for arbitrary $a(A) > 0$ is derived by considering the standardized set $\tilde{A} = A/\sqrt{a(A)}$. ∎

4. SUMMARY

In this paper a method for deriving asymptotic expansions for large deviation probabilities is outlined. Theoretically the existence of such expansions under sufficient smoothness conditions is known from [9] and [14], but here it is shown, how it is possible to obtain higher order coefficients by studying the behaviour of surface integrals near the points where the normal density is maximal; this avoids the calculation of higher order derivatives. The main idea behind is to combine considerations concerning global and local geometric properties of Gaussian laws and large deviation domains.

Such results can be generalized in various directions. For example, it is possible to consider the case of more complicted shaped sets $M(A)$, i.e. manifolds. Related results for normal vectors are proved in [12]. Some heuristic approaches for generalizations of such type results to non-normal random vectors are given in [6].

APPENDIX: SOME BASIC FACTS ABOUT SURFACES

A curve C in $\mathbb{R}^2$ is defined by a function $g: \mathbb{R}^2 \to \mathbb{R}$ by $C = \{(x, y); g(x, y) = 0\}$. If g is twice continuously differentiable, at a point $\mathbf{x}_0 = (x, y) \in C$, the curve has two important characteristics. The normal $\mathbf{n}(\mathbf{x}_0)$ at $\mathbf{x}_0$ is defined by

$$\mathbf{n}(\mathbf{x}_0) = \|\nabla g(\mathbf{x}_0)\|^{-1} \begin{pmatrix} g_1(\mathbf{x}_0) \\ g_2(\mathbf{x}_0) \end{pmatrix},$$

where g_1, g_2 denote partial derivatives of g with respect to x and y respectively. The curvature $\kappa(\mathbf{x}_0)$ describes a certain deviation of the curve form a straight line. It is given as

$$\kappa(\mathbf{x}_0) = \frac{-g_1^2(\mathbf{x}_0)\, g_{22}(\mathbf{x}_0) + 2g_1(\mathbf{x}_0)\, g_2(\mathbf{x}_0)\, g_{12}(\mathbf{x}_0) - g_2^2(\mathbf{x}_0)\, g_{11}(\mathbf{x}_0)}{(g_1^2(\mathbf{x}_0) + g_2^2(\mathbf{x}_0))^{3/2}}. \tag{26}$$

An $n-1$ dimensional surface G in the n-dimensional space $\mathbb{R}^n$ is defined by a continuously differentiable function $g: \mathbb{R}^n \to \mathbb{R}$ by $G = \{\mathbf{x}; g(\mathbf{x}) = 0\}$; here it is assumed that $\nabla g(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in G$. Then for such a surface a normal vector field is defined by $\mathbf{n}(\mathbf{x}) = \|\nabla g(\mathbf{x})\|^{-1} \nabla g(\mathbf{x})$, $\mathbf{x} \in G$. The tangential space $T_{\mathbf{x}}$ of G at $\mathbf{x} \in G$ consists of all vectors $\mathbf{y} + \mathbf{x}$ with $\mathbf{y}^T \mathbf{n}(\mathbf{x}) = 0$.

The Weingarten map $L_{\mathbf{x}}: T_{\mathbf{x}} \to T_{\mathbf{x}}$, defined by $\mathbf{y} \mapsto L_{\mathbf{x}}(\mathbf{y}) = -\nabla_{\mathbf{y}} \mathbf{n}(\mathbf{x})$, $\mathbf{y} \in T_{\mathbf{x}}$, measures the turning of the normal as one moves through $\mathbf{x}$ on the surface G with the tangential vector $\mathbf{y}$ as speed vector. When $\|\mathbf{v}\| = 1$, then the number

$$\kappa(\mathbf{v}) = L_{\mathbf{x}}(\mathbf{v}) \cdot \mathbf{v}$$

is called the normal curvature of G at $\mathbf{x}$ in the direction $\mathbf{v}$. The $n-1$ eigenvalues of the Weingarten map are called the main curvatures of G at $\mathbf{x}$ and denoted by $\kappa_1, ..., \kappa_{n-1}$. Their corresponding unit eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_{n-1}$ are called the main curvature directions. The normal curvature at $\mathbf{x}$ in the direction $\mathbf{v}$ is given by

$$\kappa(\mathbf{v}) = \sum_{i=1}^{n-1} \kappa_i (\mathbf{v}^T \cdot \mathbf{v}_i)^2 = \sum_{i=1}^{n-1} \kappa_i \cos^2(\theta_i)$$

with $\theta_i = \arccos(\mathbf{v}^T \cdot \mathbf{v}_i)$ the angle between $\mathbf{v}$ and $\mathbf{v}_i$. Further details can be found in [13], chap. 12.

We give a short review of necessary and sufficient conditions for extrema on surfaces. Let G be a surface in $\mathbb{R}^n$ defined by a twice continuously differentiable function $g: \mathbb{R}^n \to \mathbb{R}$ by $G = \{\mathbf{x}; g(\mathbf{x}) = 0\}$. Further let be given another twice differentiable function $f: \mathbb{R}^n \to \mathbb{R}$. The Lagrange multiplier theorem (see [2], p. 25) states that, if at $\mathbf{x}_0 \in G$ the function f has a local extremum with respect to G, then there exists a $\lambda \neq 0$, the so-called Lagrange multiplier, such that

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0),$$

i.e. the both gradients are parallel at $\mathbf{x}_0$. This is a necessary condition.

A sufficient condition is given in [13], p. 98–100. A function f has a local maximum (minimum) with respect to the surface G at a point $\mathbf{x}_0$, if:

1. There is a $\lambda \in \mathbb{R}$ with $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$.
2. The matrix $\mathbf{H}(\mathbf{x}_0) = (f_{ij}(\mathbf{x}_0) - \lambda g_{ij}(\mathbf{x}_0))_{i,j=1,...,n}$ is negative (positive) definite under the constraint $(\nabla g(\mathbf{x}_0))^T \mathbf{x} = 0$, i.e. $\mathbf{x}^T \mathbf{H}(\mathbf{x}_0)\, \mathbf{x} < 0$ (resp. $\mathbf{x}^T \mathbf{H}(\mathbf{x}_0)\, \mathbf{x} > 0$) for all vectors $\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ and $(\nabla g(\mathbf{x}_0))^T \mathbf{x} = 0$.

In the second condition appears a modified Hessian of the function f to take into account the curvatures of G at $\mathbf{x}_0$. An extremum at point $\mathbf{x}_0$ of a function f with respect to the surface G is called regular, iff the matrix $\mathbf{H}(\mathbf{x}_0)$ is definite under the constraint $(\mathbf{n}(\mathbf{x}_0))^T \mathbf{x} = 0$.

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