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Approximating large quantiles

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ABSTRACT. Asymptotic expansions for probabilities of large deviations are used to construct an iteration procedure for approximating quantiles in the far tails of a distribution. Expansions for analytically known distributions are based on Laplace's method while quantile approximation for the arithmetical mean relies on large deviation results of the Linnik type. A comparison of quantile approximations based upon the δ-method or upon Cornish-Fisher type expansions with those based upon our large deviation approach show both formal similarities and substantial differences.

Key words: Poincaré type expansions, large deviation iteration procedure, large deviations in Linnik's zones, skewness-kurtosis adjusted quantiles, adjustments for the arithmetical mean, refined δ-method quantile approximation.

1 Introduction

Quantiles of statistical distributions are needed in many theoretical and practical situations. From a certain mathematical point of view one can distinguish between four different general situations of quantile approximation. On the one hand side we shall speak about central quantile approximation (CQA) or tail quantile approximation (TQA) in dependence of whether we are dealing with quantiles from the central part of a distribution or from its far tails, respectively. On the other hand side we shall speak simply about quantile approximation (QA) or about asymptotic quantile approximation (AQA) in dependence of whether we are keeping in mind only one fixed distribution or even a (weakly convergent) sequence of distributions, respectively. Thus we shall distinguish between the four situations CQA, TQA, ACQA and ATQA.

To recall shortly the idea of CQA let F denote a continuous and strongly monotonous cumulative distribution function (c.d.f.) and x_q its q-th order quantile, i. e. the uniquely determined solution of the quantile equation

$$F(x_q) = q, \quad q \in (0,1) . \tag{1}$$

The Newton iteration procedure

$$x_q(m+1) = x_q(m) - \frac{\Psi(x_q(m))}{\Psi'(x_q(m))} \quad , m = 0, 1, 2, \ldots$$

with

$$\Psi(x) = F(x) - q, x \in R$$

and a suitably chosen initial value $x_q(0)$ will converge to x_q under very general assumptions as well as other standard numerical algorithms will yield satisfactory numerical approximations for q from the central part of the distribution.

If, however, q is very small or very large then $\Psi'(x_q(m))$ will be approximately zero for sufficiently large m and the described algorithms fail more or less. In such situations one can try to use TQA. Roughly spoken, this method is based upon first approximating tail probabilities of a distribution by deriving a large deviation asymptotic representation or even expansion of Poincaré type and second inverting it in some suitably defined asymptotic sense. This will be outlined in Section 2 in more details.

If on the other hand side instead of a fixed c.d.f. F a weakly convergent sequence $(F_n)_{n=1,2,\ldots}$ of c.d.f.'s has to be considered then the problem of solving the quantile equation (1) can be replaced by finding a sequence $(x_{q,n})_{n=1,2,\ldots}$ satisfying the asymptotic quantile relation

$$F_n(x_{q,n}) \longrightarrow q, n \to \infty. \tag{2}$$

This is the situation of ACQA which in certain cases can be dealt with the so called δ-method or with the Cornish-Fisher expansion as only to recall two related standard methods. To be a little more specific assume for describing the first of the just mentioned methods that for a sequence $(Z_n)_{n=1,2,\ldots}$ of random variables (r.v.) over a common probability space the normed sequence $(Z_n/\sqrt{n})_{n=1,2,\ldots}$ converges in probability to zero,

$$Z_n/\sqrt{n} \xrightarrow{P} 0, n \to \infty$$

and the corresponding sequence of distributions $(P^{Z_n})_{n=1,2,\ldots}$ converges weakly to the standard Gaussian probability distribution $\Phi^*_{0,1}$,

$$P^{Z_n} \Rightarrow \Phi^*_{0,1}, n \to \infty.$$

Let f be a function which is strongly monotonous and sufficiently often differentiable in a neighbourhood of zero. Denote by $\xi_{q,n}$ a q–th order quantile of the distribution of Z_n and put

$$Y_n = \frac{\sqrt{n}}{f'(0)}(f(Z_n/\sqrt{n}) - f(0))$$

and

$$C_{q,n} = \frac{\sqrt{n}}{f'(0)}(f(\xi_{q,n}/\sqrt{n}) - f(0)).$$

Then

$$P^{Y_n} \Longrightarrow \Phi^*_{0,1}, \qquad n \to \infty$$

as well as

$$C_{q,n} \longrightarrow z_q, \qquad n \to \infty\ , \tag{3}$$

where z_q is the q-th order quantile of $\Phi^*_{0,1}$, i. e., for the c.d.f. Φ corresponding to the measure $\Phi^*_{0,1}$ holds

$$\Phi(z_q) = q.$$

Replacing the asymptotic relation (3) by the equation

$$\frac{\sqrt{n}}{f'(0)}\left(f(\hat{\xi}_{q,n}/\sqrt{n}) - f(0)\right) = z_q \tag{4}$$

and inverting (4) yields the so called δ-method quantile approximation $\hat{\xi}_{q,n}$ for $\xi_{q,n}$ which satisfies the quantile approximation equation

$$\hat{\xi}_{q,n} = \sqrt{n} f^{-1}\left(f(0) + \frac{f'(0)z_q}{\sqrt{n}}\right) . \tag{5}$$

Expanding f^{-1} at the point $f(0)$ and suppressing higher order terms results in the quantile approximation formula

$$\hat{\xi}_{q,n} \approx \hat{\xi}^{*}_{q,n}$$

where

$$\hat{\xi}^{*}_{q,n} = z_q - \frac{z_q^2 f''(0)}{2\sqrt{n} f'(0)} . \tag{6}$$

A more precise δ-method quantile approximation formula will be given in Appendix A, formula (36). The question of how to chose the function f has been discussed for two special cases in Richter and Gundlach (1990) and in Davids and Richter (1990).

Let us shortly review now the other above mentioned standard asymptotic quantile approximation method in the situation of ACQA. To this end assume that F_n is the c.d.f. of a sum of i.i.d. r.v.'s $X_1,\ldots,X_n$. Let X_1 satisfy all conditions such that F_n admits an Edgeworth type expansion

$$F_n(x) = \Phi(x) + \sum_{\nu=1}^{2} Q_\nu(x)/n^{\nu/2} + 0\left(\frac{1}{n\sqrt{n}}\right) . \tag{7}$$

uniformly for all x from any finite interval and with the two Q_ν's being

$$Q_1(x) = -g_1\, \varphi''(x)/6$$

and

$$Q_2(x) = \left(3g_2\, \varphi'''(x) + g_1^2\, \varphi^{(5)}(x)\right)/72.$$

Here, φ denotes the derivative of Φ while

$$g_1 = \mathbb{E}(X_1 - \mathbb{E}X_1)^3 \,/\, \left(\mathbb{E}(X_1 - \mathbb{E}X_1)^2\right)^{3/2}$$

and

$$g_2 = \mathbb{E}(X_1 - \mathbb{E}X_1)^4 \,/\, \left(\mathbb{E}(X_1 - \mathbb{E}X_1)^2\right)^2 - 3$$

denote skewness and kurtosis of the distribution of X_1. For respective details see, e.g., in Petrov (1975). Starting from the equations

$$z_q = \Phi^{-1}(q) = \Phi^{-1}\left(\Phi(z_{q,n}) + [F_n(z_{q,n}) - \Phi(z_{q,n})]\right),$$

expanding the inverse function Φ^{-1} at the point $\Phi(z_{q,n})$ and plugging in the expansion (7) leads to the asymptotic relation

$$
\begin{aligned}
z_q = z_{q,n} &- \frac{g_1}{6\sqrt{n}}(z_{q,n}^2 - 1) \\
&+ \frac{1}{72n}\left((8g_1^2 - 3g_2)z_q^3 - (14g_1^2 - 9g_2)z_q\right) \\
&+ 0\left(\frac{1}{n\sqrt{n}}\right), \quad n \longrightarrow \infty.
\end{aligned} \tag{8}
$$

Assuming the series expansion

$$
z_{q,n} = z_q + \sum_i \psi_i(z_q)/n^{i/2} \tag{9}
$$

to hold, replacing $z_{q,n}$ in (8) by its series representation in (9) and suppressing higher order terms one gets from a comparison of coefficients to $n^{-1/2}$ a.d n^{-1} the Cornish-Fisher expansion

$$
\begin{aligned}
z_{q,n} = z_q &+ \frac{g_1}{6\sqrt{n}}\left(z_q^2 - 1\right) \\
&+ \frac{1}{72n}\left((3g_2 - 4g_1^2)z_q^3 - (9g_2 - 10g_1^2)z_q\right) \\
&+ 0\left(\frac{1}{n\sqrt{n}}\right) \quad n \longrightarrow \infty
\end{aligned} \tag{10}
$$

which holds uniformly with respect to q from [c,d] for all $[c,d] \subset (0,1)$, see, e.g., in Fisher and Cornish (1960) and in Bolschev and Smirnov (1983).

As we have seen above when discussing how to solve the quantile equation (1) for a fixed distribution one cannot expect in general to get satisfactory numerical approximations for large quantiles by applying CQA-methods. Additional reasons lead to the circumstance that one can also not expect that the quantiles $z_{q,n}$ can be represented in terms of the quantiles z_q in a satisfactory way by applying ACQA-methods. To understand this let us recall that the Edgeworth type expansion (7) plays an essential role in the derivation of the Cornish-Fisher type expansion (10). If x is very large or very small and n is a fixed integer then it may happen that

$$
\mid F_n(x) - \Phi(x) - \sum_{\nu=1}^{2} Q_\nu(x)/n^{\nu/2} \mid
$$

is extremly smaller than $\frac{C}{n\sqrt{n}}$ for a certain $C \in (0, \infty)$. An application of the asymptotic relation (7) in the fixed situation that x is very large and n is relatively small fails as well as it fails if x increases too fast when n approaches infinity. As a consequence, ACQA fails if $x \geq x(n)$ for a certain function $x(n)$ satisfying

$$
x(n) \longrightarrow \infty \qquad as \qquad n \longrightarrow \infty.
$$

If one exploits a suitable large deviation theorem instead of the Edgeworth type expansion (7) then one can overcome the described difficulties. To be more specific we shall replace (7) by a large deviation theorem for Linnik type zones and including two terms of Cramér's series. This will be outlined in Section 4 in more details. Note that theoretical and numerical comparisons of normal and large deviation approximations for tail probabilities have been made in Field and Ronchetti (1990), Fu, Len and Peng (1990) and Jensen (1995).

2 Large deviation quantile approximation for analytically known distribution functions

Methods for approximating large quantiles make usually more or less explicitely use of one or another type of approximating tail probabilities of the respective distributions. These probabilities, however, are small. Techniques for approximating tail probabilities which control relative approximation errors instead of absolute errors are therefore favoured in dealing with the problem. Such techniques are available from large deviation theory. Asymptotic expansions for probabilities of large deviations are the most precise results in this field and will be exploited in this section for deriving quantile approximation formulas of various orders. This approach will be demonstrated in the present section for a certain general situation and will be illustrated by some examples in Section 3.

Let R be an open interval on the real line and assume that $x_o \in R$. A sequence of functions $(\varphi_n)_{n=0,1,2,\ldots}$ will be called an asymptotic sequence as $x \to x_o$ in R if, for every n, φ_n is defined and continuous in R and

$$\varphi_{n+1}(x) = o(\varphi_n(x)) \qquad as \qquad x \to x_o.$$

Let f be a continuous function on R and $(\varphi_n)_{n=0,1,2,\ldots}$ an asymptotic sequence as $x \to x_o$ in R. Then the formal series

$$\sum_{k=o}^{\infty} a_k\, \varphi_k(x)\ ,\ x \in R$$

will be said to be an infinite asymptotic expansion of f as $x \to x_o$ and with respect to $(\varphi_n)_{n=0,1,2,\ldots}$ if for $n = 0, 1, 2, \ldots$ holds

$$f(x) = \sum_{k=o}^{n} a_k \varphi_k(x) + 0(\varphi_{n+1}(x)) \quad as \quad x \to x_o.$$

Following Bleistein and Handelsman (1975) we write symbolically in this case

$$f(x) \sim \sum_{k=o}^{\infty} a_k \varphi_k(x), \quad x \to x_o.$$

Such kind of asymptotic expansion is said to be of Poincaré type.

Let F be a continuous cumulative distribution function with unbounded support and denote its q-th order quantile by $x_q, q \in (0,1)$. For the purpose of approximating large quantiles of F we put

$$f(x) = 1 - F(x)$$

and assume that we are given a Poincaré type asymptotic expansion

$$f(x) \sim \sum_{k=o}^{\infty} a_k \varphi_k(x), \quad x \to \infty$$

with $a_o\, \varphi_o(x) \neq 0$. In view of this expansion and because

$$x_{1-\alpha} \longrightarrow \infty \qquad \text{iff} \qquad \alpha \longrightarrow +0$$

we are motivated to replace the quantile equation (1) by the N-th order approximative quantile equation

$$\alpha = \sum_{k=o}^{N} a_k\, \varphi_k(x_{1-\alpha,N}) \tag{11}$$

for a suitably chosen integer N. Here, $x_{1-\alpha,N}$ will be called the N-th order approximative quantile. Note that in typical examples F is strongly monotonous for sufficiently large arguments and even

$$\sum_{k=o}^{N} a_k \varphi_k$$

has this property and is continuous. In this situation $x_{1-\alpha,N}$ is uniquely defined by equation (11) if α is sufficiently small.

Because (11) is a nonlinear equation we cannot expect to solve it explicitely, in general. Let us rewrite therefore equation (11) as

$$\alpha = a_o\, \varphi_o(x_{1-\alpha,N})[1 + f_N(x_{1-\alpha,N})].$$

Here,

$$f_N(x) = \sum_{k=1}^{N} \frac{a_k \varphi_k(x)}{a_o \varphi_o(x)}$$

satisfies the asymptotic relation

$$f_N(x) = o(1) \quad \text{as} \quad x \to \infty.$$

Consider now the reduced quantile approximation equation

$$\alpha = a_o \varphi_o(C_o) \tag{12}$$

as well as the iteration procedure which generates C_{n+1} from C_n by

$$\alpha = a_o \varphi_o(C_{n+1})[1 + f_N(C_n)], \tag{13}$$

$n = 0, 1, 2, \ldots$. Because the background of the derivation of the iteration procedure (13) comes from the theory of probabilities of large deviations we shall call (13) a Large Deviation Iteration Procedure having as initial value the solution C_o of the reduced equation (12) or a suitable approximation for it.

Define

$$G_N(x, \alpha) = \alpha - a_o \varphi_o(x)[1 + f_N(x)],\ N = 1, 2, \ldots$$

and put

$$G_o(x, \alpha) = \alpha - a_o \varphi_o(x).$$

The N-the order approximative quantile $x_{1-\alpha,N}$ is a solution of the equation

$$G_N(x, \alpha) = 0,\ N = 0, 1, 2, \ldots . \tag{14}$$

The function $x = x(\alpha)$ will be called an asymptotic solution of the equation (14) if

$$G_N(x(\alpha), \alpha) = o(\Psi(\alpha)),\ \alpha \to +0$$

holds for a function Ψ satisfying

$$\Psi(\alpha) \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow +0\,.$$

In typical cases the functions $C_n = C_n(\alpha)$ build a sequence of asymptotic solutions of the equation (14) having the additional property

$$G_N(C_{n+1}(\alpha), \alpha) = o(1) G_N(C_n(\alpha), \alpha),\ \alpha \to 0.$$

If for all sufficiently small $\alpha > 0$ holds

$$C_n(\alpha) \longrightarrow x_{1-\alpha,N},\ n \to \infty \tag{15}$$

then we are well motivated to call $x_{1-\alpha,N}$ an asymptotic solution of the so called asymptotic equation

$$G_N(x_{1-\alpha,N}, \alpha) = 0 \quad \text{as} \quad \alpha \to +0. \tag{16}$$

Dealing with $x_{1-\alpha,N}$ as the asymptotic solution of the asymptotic equation means then practically to approximate $x_{1-\alpha,N}$ by C_n for a suitably chosen large n. The Large Deviation Iteration Procedure could be stoped, e.g., if

$$| C_{n+1} - C_n | / C_{n+1} < \varepsilon \tag{17}$$

for a given small $\varepsilon > 0$.

Let us turn now to the question when the assumption (15) will be satisfied. To this end we shall assume that the function φ_o is invertable and continuously differentiable for sufficiently large arguments and rewrite then the equation (13) as

$$C_{n+1} = \varphi_o^{-1} \left(\alpha / (a_o [1 + f_N(C_n)]) \right).$$

From the fixed point theory it is well known that the iteration procedure (13) converges to $x_{1-\alpha,N}$ if

$$| \varphi'(x_{1-\alpha,N}) | < 1 \tag{18}$$

holds for the iteration function

$$\varphi(x) = \varphi_o^{-1}(\alpha / (a_o[1 + f_N(x)])).$$

In typical cases we have even

$$\varphi'(x) \longrightarrow 0 \quad \text{as} \quad x \to \infty \tag{19}$$

so that condition (18) is fulfilled in such situations. The N-th order approximative quantile $x_{1-\alpha,N}$ is then an attracting fixed point satisfying the fixed point equation for the iteration function φ, i.e.

$$x = \varphi(x). \tag{20}$$

We shall not be concerned here with the problem of characterizing the class of distribution functions F for which all assumptions are satisfied making our algorithm of quantile approximation successful but we shall give in the next section some examples which demonstrate the usefulness of this new method.

3 Examples

Example 1: Gaussian distribution

Let

$$F(x) = \Phi(x), \qquad x \in R$$

be the standard Gaussian distribution function. It is well known , see e. g. in Bleistein and Handelsman (1975), that $1-\Phi$ admits the following Poincaré type asymptotic expansion as $x \to \infty$

$$1-\Phi(x) \sim \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!!}{x^{2k}}\right).$$

As an illustration of more general properties of the expansion terms note that for, e. g., $x=2$ the first six expansion terms are $1/(e^2\sqrt{2\pi})$ times

$$\frac{1}{2}=0,5, -\frac{1}{2^3}=-0,125, \frac{3}{2^5}=0,094, -\frac{15}{2^7}=-0,117, \frac{105}{512}=0,205 \text{ and } -\frac{945}{2048}=-0,461,$$

respectively.

Let $N \in \{0,1,2,\dots\}$. The N-th order approximative quantile equation is

$$\alpha = \frac{e^{-x^2_{1-\alpha,N}/2}}{\sqrt{2\pi}x_{1-\alpha,N}} \left[1 + f_N(x_{1-\alpha,N})\right],$$

with

$$f_o(x) \equiv 0, f_N(x) = \sum_{k=1}^{N} a_k \varphi_k(x), N = 1,2,\dots$$

and

$$a_k\varphi_k(x) = (-1)^k \frac{(2k-1)!}{2^{k-1}(k-1)!x^{2k}} \quad ,$$

and whereby $x_{1-\alpha,N}$ denotes the N-th order approximation for $z_{1-\alpha}$. The reduced quantile equation

$$\alpha = \frac{1}{\sqrt{2\pi}} e^{-C_o^2/2}$$

leads to an explicit formula for the initial value C_o of the Large Deviation Iteration Procedure,

$$C_o = \sqrt{-2\ln\alpha - \ln(2\pi)}.$$

The iteration function

$$\varphi(x) = \sqrt{-2\ln\alpha - \ln(2\pi) - 2\ln x + 2\ln\left[1 + f_N(x)\right]}$$

has the derivative

$$\varphi'(x) = \frac{-\frac{2}{x} + \frac{2f_N'(x)}{1+f_N(x)}}{2\varphi(x)}$$

so that because of $x_{1-\alpha,N} = \varphi(x_{1-\alpha,N})$ holds

$$\varphi'(x_{1-\alpha,N}) = -x_{1-\alpha,N}^{-2} - \sum_{k=1}^{N}(-1)^k(2k)(2k-1)!!x_{1-\alpha,N}^{-2k-1} \Big/ \Big[x_{1-\alpha,N} + \sum_{k=1}^{N}(-1)^k(2k-1)!!x_{1-\alpha,N}^{-2k+1} \Big].$$

Since

$$\varphi'(x_{1-\alpha,N}) \longrightarrow -0 \ , \ \alpha \to +0$$

the attracting fixed point $x_{1-\alpha,N}$ is an oszillating fixed point. Note that, e. g. for $N = 0$, the subsequence $(C_{2k+1})_{k=0,1,2,\ldots}$ monotonously increases. For proving this assume that $C_{2k+1} > C_{2k-1}$. Then

$$\begin{aligned} C_{2k+3} &= (-2\ln\alpha - \ln(2\pi) - 2\ln C_{2k+2})^{1/2} \\ &= (-2\ln\alpha - \ln(2\pi) - 2\ln(-2\ln\alpha - \ln(2\pi) - \ln C_{2k+1})^{1/2})^{1/2} \\ &> (-2\ln\alpha - \ln(2\pi) - 2\ln(-2\ln\alpha - \ln(2\pi) - \ln C_{2k-1})^{1/2})^{1/2} \\ &= (-2\ln\alpha - \ln(2\pi) - 2\ln C_{2k}) = C_{2k+1}. \end{aligned}$$

It can be shown analogously, that the sequence $(C_{2k})_{k=1,2,\ldots}$ monotonously decreases. Besides using the rule (17) in terms of quantiles for stoping the Large Deviation Iteration Procedure it is also interesting to control the relative error in terms of quantile orders

$$r_N(\alpha) = \frac{[1 - \Phi(C_{n+1})] - \alpha}{\alpha}$$

in dependence of N and α. The following Table 1 gives some values of $r_N(\alpha)$ while Table 2 gives the related values of the absolute error

$$a_n(\alpha) = |C_{n+1} - z_{1-\alpha}|.$$

Table 1*) Relative error $r_N(\alpha)$

α	N=0	N=1	N=2	N=3
10^{-2}	0,1233	0,0737	0,0470	0,0654
10^{-3}	0,0810	0,0255	0,0110	0,0077
10^{-5}	0,0470	0,0083	0,0014	0,0013
10^{-9}	0,0256	0,0022	0,0002	0,0002

Table 2*)

Absolute error $a_N(\alpha)$

α	N=0	N=1	N=2	N=3
10^{-2}	0,0490	0,0268	0,0180	0,0239
10^{-3}	0,0151	0,0174	0,0067	0,0122
10^{-5}	0,0009	0,0117	0,0096	0,0102
10^{-9}	0,0058	0,0103	0,0100	0,0100

*)The author thanks J. Schumacher for making the calculations for this table.

Note that for starting the iteration procedure one can use several asymptotically more precise initial values than C_o. Consider

$$\varphi(C_o) = \sqrt{-2\ln\alpha - \ln(2\pi) - 2\ln C_o + 2\ln[1 + f_N(C_o)]}$$

and deduce from here by plugging in C_o and suppressing higher order terms

$$C_0^*(\alpha) = \sqrt{-2\ln\alpha - \ln(4\pi) - \ln(-\ln\alpha)}$$

as a new initial value.

Lemma 2.1 If $\alpha \longrightarrow 0$ then

$$z_{1-\alpha} = C_o^*(\alpha)[1 + O(\ln(-\ln\alpha)/(\ln\alpha)^2)].$$

Proof: It follows from the above considerations that for $\alpha \to 0$ holds

$$1 - \Phi(z_{1-\alpha}) = (2\pi)^{-1/2} z_{1-\alpha}^{-1} \quad \exp(-z_{1-\alpha}^2/2)(1 + 0(z_{1-\alpha}^{-2})).$$

The solution $x_{1-\alpha,0}$ of the zero-order approximative quantile equation

$$\alpha = (2\pi)^{-1/2}\, x_{1-\alpha,0}^{-1} \quad \exp(-x_{1-\alpha,0}^2\,/2) \tag{21}$$

satisfies therefore the asymptotic relation

$$z_{1-\alpha} = (1 + o(z_{1-\alpha}^{-2}))x_{1-\alpha,0}\ ,\ \alpha \to 0.$$

It has been shown in Richter (1987) that the solution $h(y) = h$ of the equation

$$h\exp(h^2/(2C^2)) = y, y \geq y_o > 0 \tag{22}$$

satisfies the asymptotic relation

$$h(y) = C[\ln y^2 - \ln(2C^2) - \ln\ln y]^{1/2}[1 + O(\ln\ln y/(\ln y)^2)]$$

as $y \to \infty$. Notice that (22) is equivalent to the zero-order approximative quantile equation (21) if

$$C = 1, y = 1/(\sqrt{2\pi}\,\alpha) \text{ and } h = x_{1-\alpha,0}.$$

Consequently,

$$x_{1-\alpha,0} = [-\ln(2\pi\,\alpha^2) - \ln 2 - \ln(-\ln[\sqrt{2\pi}\alpha])]^{1/2}$$
$$\cdot [1 + 0(\ln(-\ln\alpha)/(\ln\alpha)^2)], \quad \alpha \to 0$$

■

Example 2: Inverse Gaussian distribution (IGD)

Let $F(x) = F_v(x), x \in R$ denote the c.d.f. of the standard IGD, i.e.,

$$F_v(x) = I(x > 0)\left[\Phi((x - 1/v^2)/\sqrt{x}) + e^{2/v^2}\Phi(-(x + 1/v^2)/\sqrt{x})\right].$$

This distribution has been obtained by Schrödinger 1915 when studying first passage times of the Brownian motion. In Tweedie (1957) essential properties of this distribution have been discussed. In Wasan and Roy (1969) are given tables of quantiles of this distribution and in Folks and Chhikara (1978) main knowledge about the IGD is reviewed. Lehmann, Thiele and Tiedge (1989) discuss technical applications of the IGD.

The following considerations have been startet in a slightly other form in Richter and Nicol (1990). Our large deviation assumption will be formulated as

$$(x - 1/v^2)/\sqrt{x} \longrightarrow \infty. \tag{23}$$

Using the results from Example 1 we can replace the quantile equation

$$F_v(\overline{Q}_\beta) = \beta$$

by the N-th order approximative quantile equation

$$1 - ß = (2\pi)^{-1/2}\exp\left(-y_1^2/2\right)(f_N(y_1) - f_N(y_2))$$

where

$$y_1 = \left(\overline{Q}_\beta - 1/v^2\right)/\sqrt{\overline{Q}_\beta},\ y_2 = \left(\overline{Q}_\beta + 1/v^2\right)/\sqrt{\overline{Q}_\beta},$$

$$f_N(x) = \frac{1}{x} + \sum_{k=1}^{N}\frac{(-1)^k(2k-1)!}{2^{k-1}(k-1)!x^{2k+1}}$$

and in the case $N = 0$ we put $f_N(x) = 1/x$.
Because of the condition (23) we can assume that

$$\overline{Q}_\beta = 1/v^2 + \delta \text{ for some } \delta = \delta(\beta) > 0.$$

We rewrite now the N-th order approximative quantile equation as

$$1 - \beta = (2\pi)^{-1/2}\exp\left(-y_1^2/2\right) f_N(y_1)K_v^N(\delta)$$

with

$$K_v^o(\delta) = 1 - \frac{\delta v^2}{2 + \delta v^2},$$

$$K_v^1(\delta) = 1 - \frac{\delta^3\left((2/v^2 + \delta)^2 - (1/v^2 + \delta)\right)}{(2/v^2 + \delta)^3(\delta^2 - (1/v^2 + \delta))},$$

$$K_v^2(\delta) = 1 - \frac{\delta^5\left((2/v^2 + \delta)^4 - (2/v^2 + \delta)^2(1/v^2 + \delta) + 3(1/v^2 + \delta)^2\right)}{(2/v^2 + \delta)^5(\delta^4 - \delta^2(1/v^2 + \delta) + 3(1/v^2 + \delta)^2)}.$$

The reduced quantile equation is as in the preceding example

$$1 - \beta = (2\pi)^{-1/2} e^{-y_{1,0}^2/2}$$

and leads here to the formula

$$y_{1,0} = \sqrt{-2\ln(1-\beta) - \ln(2\pi)}.$$

Using this formula or its refinement

$$y_{1,0}^{*} = \sqrt{-2\ln(1-\beta) - \ln(4\pi) - \ln(-\ln(1-\beta))}$$

we get the initial value C_o (or C_o^*) for approximating $\overline{Q}_\beta$ with the help of the Large Deviation Iteration Procedure by solving the equation

$$y_{1,0} = \left(C_o - 1/v^2\right)/\sqrt{C_o}.$$

The corresponding initial value δ_o for approximating δ is then

$$\delta_o = C_o - 1/v^2.$$

The iteration function for approximating y_1 is

$$\varphi(x) = \sqrt{-2\ln(1-\beta) - \ln(2\pi) + 2\ln f_N(x) + 2\ln K_v^N(x - 1/v^2)}$$

and has similar properties as the function φ from the preceding example. Convergence of the iteration procedure

$$y_{1,n+1} = \varphi(y_{1,n}), n = 0, 1, 2, \ldots$$

for approximating y_1, however, is equivalent to convergence of the sequence $(C_n)_{n=1,2,\ldots}$ for approximating $\overline{Q}_\beta$, where C_n is defined by

$$y_{1,n} = \left(C_n - 1/v^2\right)/\sqrt{C_n}.$$

As we have seen in Examle 1, the value y_1 corresponding to the quantile $\overline{Q}_\beta$ can be approximated with a small relative approximation error if the considerations concern quantiles from the far tails of the distribution. If we are dealing, however, with quantiles from the more central part of the distribution, then the Large Deviation Iteration Procedure may still yield initial values for standard numerical iteration procedures.

Example 3: Generalized gamma distribution (g.g.d)

Let $F(x) = F_{\alpha,b,p}(x)$, $x \in R$ denote the c.d.f. of the g.g.d. with parameters $\alpha > 0, b > 0$ and $p > 0$, i.e.

$$F_{\alpha,b,p}(x) = I(x > 0)\frac{\alpha b^{p/\alpha}}{\Gamma(p/\alpha)} \int_o^x y^{p-1} e^{-by^\alpha} dy, \ x \in R.$$

The following Poincaré type asymptotic expansion for large deviations has been shown in Richter and Schumacher (1990) based upon a much more general result in Fedoryuk (1977):

$$1 - F_{\alpha,b,p}(x) \sim \frac{\alpha\, b^{p/\alpha}}{\Gamma(p/\alpha)}\, x^p e^{-bx^\alpha} \sum_{k=o}^{\infty} c_k / x^{\alpha(k+1)},\ x \to \infty$$

with

$$c_o = (\alpha b)^{-1},$$

$$c_k = (\alpha b)^{-1-k} \prod_{i=1}^{k} (p - i\alpha), k = 1, 2 \ldots$$

where in the case that p/α is an integer the infinite sum becomes finite and includes only terms up to the summation index $k = p/\alpha - 1$. Under the assumtion that $N \leq p/\alpha - 1$ if p/α is an integer the N-th order approximative quantile equation is

$$1 - \beta = \frac{\alpha b^{p/\alpha}}{\Gamma(p/\alpha)}\, \overline{Q}_\beta^p e^{-b\overline{Q}_\beta^\alpha} \sum_{k=0}^{N} c_k / \overline{Q}_\beta^{\alpha(k+1)},$$

where $\overline{Q}_\beta$ denotes the N-th order approximation for the 100β-percentage point $Q_\beta^{\alpha,b,p}$ of $F_{\alpha,b,p}$. The reduced quantile equation

$$1 - \beta = \frac{b^{p/\alpha - 1}}{\Gamma(p/\alpha)}\, e^{-bC_o^\alpha}$$

leads to the explicit formula for the initial value C_o of the Large Deviation Iteration Procedure

$$C_o = \frac{1}{b^{1/\alpha}} \left(-\ln(1-\beta) - \ln\Gamma\left(\frac{p}{\alpha}\right) + \left(\frac{p}{\alpha} - 1\right)\ln b\right)^{\frac{1}{\alpha}}.$$

The iteration function is

$$\varphi(x) = \frac{1}{b^{1/\alpha}} (-\ln(1-\beta) - \ln\Gamma(\frac{p}{\alpha}) + (\frac{p}{\alpha} - 1)\ln b + p\ln x + \ln \sum_{k=o}^{N} c_k / x^{\alpha(k+1)})^{1/\alpha}.$$

From

$$\varphi'(x) = \frac{1}{b\alpha\varphi(x)^{\alpha-1}} \left[\frac{p}{x} - \frac{\alpha}{x} \sum_{k=o}^{N} c_k(k+1)x^{-\alpha(k+1)} / \sum_{k=o}^{N} c_k x^{-\alpha(k+1)}\right]$$

it follows that

$$\varphi'(x) \longrightarrow 0 \qquad \text{as} \qquad x \longrightarrow \infty.$$

Hence,

$$| \, \varphi'(Q_\beta^{\alpha,b,p}) \, | < 1$$

for sufficiently large quantile order $\beta < 1$. Consequently, $Q_\beta^{\alpha,b,p}$ is an attracting fixed point for sufficiently large $\beta < 1$.

Further examples of large deviation quantile approximation are discussed in Ittrich, Krause and Richter (in preparation) based upon an asymptotic expansion for large deviation probabilities derived in Richter and Schumacher (in preparation). Note that these papers concern large deviations for noncentral distributions which lead to the basic difficulty that the so called dominating points from large deviation theory degenerate asymptotically.

4 Large deviation quantile approximation for the standardized arithmetical mean

Assume that *a* r.v. X has expectation μ, dispersion σ^2 and satisfies the Linnik condition

$$\mathbb{E} \exp \left(| \, X \, |^{4\gamma/(2\gamma+1)} \right) < \infty \tag{24}$$

for some $\gamma \in (0, 1/2)$. For a discussion of condition (24) see, e.g., in Petrov (1975). Let $X_{1,\ldots,}X_n$ be independent r.v.'s which are distributed as X and denote by F_n the c.d.f. of the standardized arithmetical mean

$$T_n = \sqrt{n} \, \frac{\overline{X}_n - \mu}{\sigma},$$

where

$$\overline{X}_n = \frac{1}{n} \, (X_1 + \ldots + X_n).$$

Define the functions

$$f_{n,s}(x) = \frac{1}{\sqrt{2\pi x}} \, exp \left(-\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \sum_{k=o}^{s-1} a_k \left(\frac{x}{\sqrt{n}} \right)^k \right),$$

where s is the uniquely determined integer satisfying

$$\frac{s}{2(s+2)} < \gamma \leq \frac{s+1}{2(s+3)},$$

$\sum_{k=o}^{-1} \equiv 0$, and a_k is the k-th order coefficient of Cramér's powers series. From Petrov (1975) we have, e.g.,

$$a_o = g_1/6 \quad \text{and} \quad a_1 = (g_2 - 3g_1^2)/24,$$

whereby

$$g_1 = \mathbb{E}\,(X-\mu)^3/\sigma^3 \quad \text{and} \quad g_2 = \mathbb{E}\,(X-\mu)^4/\sigma^4 - 3$$

denote skewness and kurtosis of (the distribution of) X, respectively. In Petrov (1975) it has been proved that if

$$x \longrightarrow \infty,\ x = o(n^\gamma) \quad \text{as} \quad n \longrightarrow \infty \tag{25}$$

then Linnik's asymptotic relation

$$1 - F_n(x) \sim f_{n,s}(x),\ n \longrightarrow \infty \tag{26}$$

holds.

Lemma 4.1 If condition (24) is fulfilled for some

$$\gamma \in (1/6, 1/4] \text{ and } \rho(x) = \frac{g_1 x^2}{6\sqrt{n}} + O\left(\frac{x^3}{n}\right)$$

then the asymptotic quantile correction formula

$$1 - F_n(x + \rho(x)) \sim f_{n,o}(x),\ n \longrightarrow \infty \tag{27}$$

holds for x satisfying assumption (25).

Proof: Under the assumption (25) holds

$$\rho(x) = o(x) \qquad \text{as} \qquad n \to \infty.$$

From (26) follows therefore

$$1 - F_n(x + \rho(x)) \sim f_{n,1}(x + \rho(x))$$

and

$$1 - F_n(x+\rho(x)) \sim \frac{1}{\sqrt{2\pi}x} \exp\left\{-\frac{(x+\rho(x))^2}{2} + \frac{g_1}{6\sqrt{n}}(x+\rho(x))^3\right\}$$
$$= \frac{1}{\sqrt{2\pi}x} \exp\left\{-\frac{x^2}{2} - x\rho(x) + \frac{g_1 x^3}{6\sqrt{n}} + O(\frac{x^4}{n})\right\} \quad \blacksquare$$

Lemma 4.2. If X satisfies assumtion (24) for some

$$\gamma \epsilon (1/4, 3/10] \quad \text{and} \quad \rho(x) = \frac{g_1 x^2}{6\sqrt{n}} + \frac{(3g_2 - 4g_1^2)\, x^3}{72n} + O\left(\frac{x^4}{n^{3/2}}\right)$$

then formula (27) holds for all x satisfying assumtion (25).

Proof: Analogously to the proof of Lemma 1,

$$1 - F_n(x + \rho(x)) \sim \frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{(x+\rho(x))^2}{2} + \frac{g_1}{6\sqrt{n}}(x+\rho(x))^3 + \frac{g_2 - 3g_1^2}{24n}(x+\rho(x))^4\right\}$$

$$= \frac{1}{\sqrt{2\pi x}} \exp\left\{-\frac{x^2}{2} - \frac{g_1^2 x^4}{72n} - \frac{(3g_2 - 4g_1^2)x^4}{72n} + \frac{3g_1^2 x^4}{36n} + \frac{g_2 - 3g_1^2}{24n} x^4 + O\left(\frac{x^5}{n^{3/2}}\right)\right\} \quad \blacksquare$$

Definition 4.3 Any quantity $z_{1-\alpha(n)}(s)$ admitting the representation

$$z_{1-\alpha(n)}(s) = z_{1-\alpha(n)} + \frac{z_{1-\alpha(n)}^2}{\sqrt{n}} \sum_{k=-1}^{s-2} b_{k+1} \left(\frac{z_{1-\alpha(n)}}{\sqrt{n}}\right)^{k+1} + O\left(\frac{{z_{1-\alpha(n)}}^{s+2}}{n^{(s+1)/2}}\right) \qquad (28)$$

as $n \to \infty$ with constants

$$b_o = g_1/6 \quad \text{and} \quad b_1 = \left(3g_2 - 4g_1^2\right)/72$$

will be called a skewness or a skewness-kurtosis adjusted $(1-\alpha(n))$-quantile of the distribution of T_n in dependence of whether $s = 1$ or $s = 2$.

Remark 4.4 Assume that

$$\varliminf_{n\to\infty} z_{1-\alpha(n)}^3/\sqrt{n} \quad > 0.$$

The special quantile approximations for T_n,

$$z_{1-\alpha(n)}^*(1) = z_{1-\alpha(n)} + \frac{g_1}{6\sqrt{n}}\left(z_{1-\alpha(n)}^2 - 1\right)$$

and

$$z_{1-\alpha(n)}^*(2) = z_{1-\alpha(n)}^*(1) + \frac{1}{72n}\left((3g_2 - 4g_1^2)z_{1-\alpha(n)}^3 - (9g_2 - 10g_1^2)z_{1-\alpha(n)}\right),$$

corresponding both to the Cornish-Fisher expansions from (10) and to the δ-method quantile approximation formulas, satisfy Definition 4.3. If, e.g., a function $f(x,n)$ which is strongly monotonous and sufficiently often differentiable in a neighbourhood of $x = 0$ satisfies the assumption

$$\frac{f_{xx}(0,n)}{f_x(0,n)} = -\frac{g_1}{3} + O\left(\frac{z_{1-\alpha(n)}^3}{n}\right) \quad \text{as } n \to \infty \qquad (29)$$

then $\xi^*_{q,n}$ from (6) for this f satisfies Definition 4.3.

Definition 4.5 The zero-sequence $(\alpha(n))_{n=1,2,\ldots}$ will be said to fulfil the Osipov type condition of order γ, $\gamma\epsilon(0,1/2)$ concerning its speed of convergence to zero as n approaches infinity if

$$\alpha(n)n^{\gamma}\exp\left\{n^{2\gamma}/2\right\} \longrightarrow \infty \text{ as } n \rightarrow \infty. \tag{30}$$

For a discussion of condition (30) we refer to Richter (1990). We are now in a position to formulate the main result of this section dealing with precise asymptotic quantile approximation for (the distribution of) T_n.

Theorem 4.6 Assume that the random variable X satisfies the Linnik condition (24) for some $\gamma = \gamma(s)\,\epsilon\left(\frac{s}{2s+4}, \frac{s+1}{2s+6}\right)$ with $s\,\epsilon\{1,2\}$ and let the sequence $(\alpha(n))_{n=1,2,\ldots}$ be chosen such that the Osipov type condition (30) is fulfilled for this $\gamma = \gamma(s)$. Then

$$1 - F_n(z_{1-\alpha(n)}(s)) \sim \alpha(n),\ n \rightarrow \infty$$

for any skewness or skewness - kurtosis adjusted quantile $z_{1-\alpha(n)}(s)$ satisfying the representation (28).

Proof: Condition (30) means

$$[f_{n,o}(n^{\gamma})]^{-1}\left(1 - \Phi(z_{1-\alpha(n)})\right) \longrightarrow \infty\,,\ n \rightarrow \infty.$$

This relation is equivalent to

$$[f_{n,o}(n^{\gamma})]^{-1} f_{n,o}(z_{1-\alpha(n)}) \longrightarrow \infty\,,\ n \rightarrow \infty,$$

which is the same as

$$z_{1-\alpha(n)} = o(n^{\gamma})\,,\ n \rightarrow \infty.$$

This means that $z_{1-\alpha(n)}$ satisfies the assumtions (25). The theorem follows now from Lemmas 4.1 and 4.2 ■

5 Large deviation approach to skewness and kurtosis adjusted mean value statistics

Let

$$T_n^{(1)} = T_n - \frac{g_1 T_n^2}{6\sqrt{n}}$$

and

$$T_n^{(2)} = T_n^{(1)} + \frac{(8g_1^2 - 3g_2)T_n^3}{72n}$$

denote skewness and skewness - kurtosis adjusted mean statistics, respectively.

Theorem 5.1 If the assumptions of Theorem 4.6 are fulfilled then

$$P\left(T_n^{(s)} > z_{1-\alpha(n)}\right) \sim \alpha(n)\,,\; n \rightarrow \infty$$

for $s \epsilon \{1,2\}$.

Proof: Let us start from

$$P\left(T_n^{(s)} > z_{1-\alpha(n)}\right) \sim P\left(3\sqrt{n}/(g_1\,\rho(n)) \geq T_n^{(s)} > z_{1-\alpha(n)}\right)$$

where $\rho(n)$ denotes a function approaching infinity sufficiently slowly as $n \rightarrow \infty$. For sufficiently large n, the functions

$$T_n \longmapsto T_n^{(s)} = T_n^{(s)}(T_n)\,,\, s \,\epsilon \,\{1,2\}$$

are monotonously in the regions of arguments under consideration. The corresponding inverse functions can be represented as follows:

$$T_n^{(s)-1}(x) = x + \frac{x^2}{\sqrt{n}} \sum_{k=-1}^{s-2} b_{k+1} \left(\frac{x}{\sqrt{n}}\right)^{k+1} + O\left(\frac{x^{s+2}}{n^{(s+1)/2}}\right).$$

Hence, for some Θ with $\mid \Theta \mid < \infty$,

$$P\left(T_n^{(s)} > z_{1-\alpha(n)}\right) \sim P\left(T_n > T_n^{(s)-1}(z_{1-\alpha(n)})\right) = P\Big(T_n > z_{1-\alpha(n)}(s) + \Theta\, z_{1-\alpha(n)}^{s+2} \,/\, n^{(s+1)}$$

Because of

$$z_{1-\alpha(n)}^{s+2}\,/n^{(s+1)/2} = o\left(z_{1-\alpha(n)}(s)\right),\; n \rightarrow \infty$$

Linnik's formula (26) applies so that

$$P\left(T_n^{(s)} > z_{1-\alpha(n)}\right) \sim f_{n,s}\left(z_{1-\alpha(n)}(s) + \Theta\, z_{1-\alpha(n)}/n^{(s+1)/2}\right).$$

From

$$f_{n,s}\left(x + g_{n,s}(x)\right) \sim f_{n,s}(x),\; n \rightarrow \infty$$

for

$$g_{n,s}(x) = o\left(x^{-(s+1)}\right),\; n \rightarrow \infty,$$

it follows

$$P\left(T_n^{(s)} > z_{1-\alpha(n)}\right) \sim f_{n,s}\left(z_{1-\alpha(n)}(s)\right).$$

Because of

$$f_{n,s}(z_{1-\alpha(n)}(s)) \sim f_{n,o}(z_{1-\alpha(n)}),\ n \to \infty$$

and

$$f_{n,o}(z_{1-\alpha(n)}) \sim \alpha(n),\ n \to \infty$$

the proof is finished ■

Appendix

Refined δ-method quantile approximation

Using the notations and assumptions from Section 1 define

$$U_{n,1} = \mathbb{E}\, Z_n + \frac{f''(0)\mathbb{E}Z_n^2}{2f'(0)\sqrt{n}} + \frac{f'''(0)\mathbb{E}Z_n^3}{6f'(0)n}$$

and

$$\begin{aligned} U_{n,2} = V(Z_n) &+ \frac{f''(0)}{f'(0)\sqrt{n}}\left(\mathbb{E}Z_n^3 - \mathbb{E}Z_n\mathbb{E}Z_n^2\right) \\ &+ \frac{1}{n}\left[\left(\frac{f''(0)}{2f'(0)}\right)^2\left(\mathbb{E}Z_n^4 - (\mathbb{E}Z_n^2)^2\right)\right. \\ &+ \left.\frac{f'''(0)}{3f'(0)}\left(\mathbb{E}Z_n^4 - \mathbb{E}Z_n\mathbb{E}Z_n^3\right)\right]. \end{aligned}$$

Expanding f in the zero point we get

$$Y_n = Z_n + \frac{f''(0)}{2f'(0)}\frac{Z_n^2}{\sqrt{n}} + \ldots \tag{31}$$

Taking expecation yields

$$\mathbb{E}Y_n = U_{n,1} + O\left(\frac{1}{n\sqrt{n}}\right)\ n \to \infty. \tag{32}$$

From (31) follows

$$Y_n^2 = Z_n^2 + \frac{f''(0)}{f'(0)}\frac{Z_n^3}{\sqrt{n}} + \left[\left(\frac{f''(0)}{2f'(0)}\right)^2 + \frac{f'''(0)}{3f'(0)}\right]\frac{Z_n^4}{n} + \ldots$$

and from (32) follows

$$(\mathbb{E}Y_n)^2 = (\mathbb{E}Z_n)^2 + \frac{f''(0)\mathbb{E}Z_n\mathbb{E}Z_n^2}{f'(0)\sqrt{n}}$$
$$+ \frac{1}{n}\left[\left(\frac{f''(0)\mathbb{E}Z_n^2}{2f'(0)}\right)^2 + \frac{f'''(0)\mathbb{E}Z_n\mathbb{E}Z_n^3}{3f'(0)}\right] + O\left(\frac{1}{n\sqrt{n}}\right), \ n \to \infty.$$

Consequently,

$$V(Y_n) = U_{n,2} + 0\left(\frac{1}{n\sqrt{n}}\right), \ n \to \infty$$

and therefore we have

$$\frac{Y_n - U_{n,1}}{\sqrt{U_{n,2}}} = \frac{Y_n - \mathbb{E}Y_n + O(\frac{1}{n\sqrt{n}})}{\sqrt{V(Y_n) + O(\frac{1}{n\sqrt{n}})}}.$$

Because of $P^{Y_n} \Longrightarrow \Phi_{0,1}$ we get

$$P^{(Y_n - U_{n,1})/\sqrt{U_{n,2}}} \Longrightarrow \Phi_{0,1}.$$

Let F_n^* denote the c.d.f. of $(Y_n - U_{n,1})/\sqrt{U_{n,2}}$
and put

$$C_{q,n}^* = \left[\sqrt{n}\left(f(\xi_{q,n}/\sqrt{n}) - f(0)\right)/f'(0) - U_{n,1}\right]/\sqrt{U_{n,2}}.$$

Then

$$F_n^*(C_{q,n}^*) = P(Z_n < \xi_{q,n}) \equiv q \equiv \Phi_{0,1}(z_q).$$

Hence,

$$C_{q,n}^* \longrightarrow z_q, \ n \to \infty. \tag{33}$$

Replacing the asymptotic relation (33) by the equation

$$\left[\sqrt{n}\left(f(\widehat{\widehat{\xi}}_{q,n}/\sqrt{n}) - f(0)\right)/f'(0) - U_{n,1}\right]/\sqrt{U_{n,2}} = z_q \tag{34}$$

and inverting (34) yields the refined δ-method quantile approximation $\widehat{\widehat{\xi}}_{q,n}$ for $\xi_{q,n}$ which satisfies the quantile approximation equation

$$\widehat{\widehat{\xi}}_{q,n} = \sqrt{n}\, f^{-1}\left(f(0) + \frac{f'(0)}{\sqrt{n}}\left[U_{n,1} + z_q\sqrt{U_{n,2}}\right]\right). \tag{35}$$

For expanding f^{-1} recall that

$$f^{-1'}(y) = 1/f'(f^{-1}(y))$$

and

$$f^{-1''}(y) = -f''(f^{-1}(y))/f'^3(f^{-1}(y))\,.$$

Suppressing higher order expansion terms we get the following refined δ-method quantile approximation formula

$$\widehat{\widehat{\xi}}_{q,n} \approx U_{n,1} + z_q\sqrt{U_{n,2}} - \tfrac{f''(0)}{2f'(0)\sqrt{n}}\left[U_{n,1} + z_q\sqrt{U_{n,2}}\right]^2$$

or

$$\widehat{\widehat{\xi}}^{*}_{q,n} = z_q\left[\sqrt{U_{n,2}}\left(1 - \frac{f''(0)U_{n,1}}{f'(0)\sqrt{n}}\right)\right] + U_{n,1}$$
$$- \frac{f''(0)\left[U_{n,1}^2 + z_q^2 U_{n,2}\right]}{2f'(0)\sqrt{n}}. \tag{36}$$

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