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TWO-DIMENSIONAL ASYMPTOTIC EXPANSIONS FOR LARGE DEVIATIONS OF SPHERICALLY DISTRIBUTED RANDOM VECTORS IF THE DOMINATING POINT DEGENERATES ASYMPTOTICALLY

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Abstract. A geometric approach to asymptotic expansions for large-deviation probabilities, developed for the Gaussian law by Breitung and Richter [*J. Multivariate Anal.*, **58**, 1–20 (1996)], will be extended in the present paper to the class of spherical measures by utilizing their common geometric properties. This approach consists of rewriting the probabilities under consideration as large parameter values of the Laplace transform of a suitably defined function, expanding this function in a power series, and then applying Watson's lemma. A geometric representation of the Laplace transform allows one to combine the global and local properties of both the underlying measure and the large-deviation domain. A special new type of difficulty is to be dealt with because the so-called dominating points of the large-deviation domain degenerate asymptotically. As is shown in Richter and Schumacher (in print), the typical statistical applications of large-deviation theory lead to such situations. In the present paper, consideration is restricted to a certain two-dimensional domain of large-deviations having asymptotically degenerating dominating points. The key assumption is a parametrized expansion for the inverse $\bar{g}^{-1}$ of the negative logarithm of the density-generating function of the two-dimensional spherical law under consideration.

Key words: asymptotic expansion, spherical distribution, geometric representation, large deviations, Watson's lemma.

1. INTRODUCTION

Asymptotic expansions for large deviations of one-dimensional random variables have been dealt with by many authors. Such expansions can be derived for explicitly known distributions by standard methods of asymptotic analysis, which were outlined, e.g., in the monographs of Erdélyi [7], Berg [1], Sirovich [17], Riekstinš [15], Bleistein and Handelsman [3], Fedoryuk [8], Wong [18] and Breitung [4]. The large-deviation probability integrals can be approximated through the values of a suitable, explicitly defined function at well-defined points, which will be called dominating points. In typical cases, the integrand attains its maximum over the large-deviation domain at the dominating points and the same points have the smallest distance from the distribution center among all points from the large-deviation domain. Asymptotic expansions for large deviations of certain not necessarily explicitly known probability distributions have been treated, e.g., in the monographs of Petrov [10], Saulis and Statulevičius [16], and the many papers cited therein. For some classes of known multivariate distributions, the existence of asymptotic expansions for large deviations has been discussed theoretically under sufficient smoothness conditions, e.g., in Bleistein and Handelsman [3], Fedoryuk [8], and Wong [18]. In Breitung and Richter [6], it was shown for the Gaussian law how it is possible to explicitly obtain the coefficients of the higher-order expansion by studying the behavior of certain surface integrals near the dominating points of the large-deviation domain, where the normal density is maximal. Note that this understanding of dominating points is not related to convexity as in Ney [9].

The general idea behind the approach in Breitung and Richter [6] is to combine considerations concerning the global and local geometric properties of both the Gaussian law and the large-deviation domain. The main steps of dealing with the tail probabilities consist of, first, rewriting these probabilities as large parameter values of the Laplace transform of a suitably defined function f_k, second, making a series expansion of this function, and, third, applying Watson's lemma. The function f_k is deduced by applying a geometric representation formula for

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the underlying multivariate distribution to the domain of the large deviations under consideration. Assuming a suitable expansion for the function describing the boundary of the large-deviation domain in a neighborhood of the above-mentioned dominating points, the authors derive a series expansion for the function f_k.

In Richter and Schumacher (in print), a large-deviation asymptotic expansion is derived for a specific domain of large deviations having an asymptotically degenerated dominating point. Although the large-deviation domain considered by the authors is a specific one, their paper can be understood as a typical example for statistical applications of large-deviation theory, because dealing with noncentral statistical distributions other than the there-considered noncentral chi-square distributions leads to similar large-deviation domains. With respect to the general way of deriving an asymptotic expansion, the authors follow the geometric approach of Breitung and Richter [6] and extend it to the case where the underlying multivariate distribution is a spherical one. This allows us to study the noncentral g-generalized chi-square distributions. The respective function f_k is deduced again by applying a geometric representation formula for the underlying multivariate spherical measure to the domain of large deviations. Let $\|\cdot\|$ denote the Euclidean norm in R^2. Assuming a suitably parametrized expansion for the inverse $\tilde{g}^{-1}$ of the negative logarithm $\tilde{g}$ of the density-generating function g of the multivariate spherical measure in a neighborhood of $\tilde{g}(\|x_0\|^2)$, x_0 belonging to the set of the above-mentioned dominating points, the authors then derive a series expansion for the function f_k. Both this expansion and the resulting final expansion for the large-deviation probabilities reflect the influence of the density-generating function g on the asymptotic behavior of the large-deviation probabilities under consideration.

Note that related results, however concerning only the leading term, are obtained in Richter and Steinebach [14], where the probability that a spherically distributed random vector falls into a half space having a large distance from the origin was considered.

In the present paper, we consider another type of large-deviation asymptotics, namely, when the dominating point degenerates asymptotically. In the multidimensional case, there is a great variety of ways for the dominating point to degenerate. It turns out from what follows that the large-deviation asymptotics will depend quite strongly on how the dominating point degenerates asymptotically. That is why we restrict our present study to the much simpler two-dimensional case to give a first impression of what happens.

Assume that the random vector X follows a two-dimensional spherically symmetric distribution with density $p(x; g) = C(g)g(\|x\|^2)$, $x \in R^2$, where

$$0 < \frac{1}{2\pi C(g)} = \int_0^\infty rg(r^2)\,dr < \infty.$$

Denote the cumulative distribution function corresponding to the density p by $\Phi(A; g)$, $A \in B^2$, where B^2 stands for the Borel σ-field in R^2. Let $0 < \sigma_2(\lambda) < \sigma_1 < \infty$ for all $\lambda > 0$ and put

$$A(\lambda) = \left\{(x_1, x_2) \in R^2\colon\ \sigma_1^2 x_1^2 + \sigma_2^2(\lambda) x_2^2 \geqslant 1\right\}.$$

We consider the large-deviation probabilities

$$\Phi\big(\lambda A(\lambda); g\big) \quad \text{as} \quad \lambda \to \infty, \tag{1}$$

allowing that

$$\sigma_2(\lambda) \longrightarrow \sigma_1 \quad \text{as} \quad \lambda \to \infty. \tag{2}$$

The well-determined set of dominating points, i.e., of points from the large-deviation domain where the normal density is maximal, is $\{(1/\sigma_1, 0)^T,\ (-1/\sigma_1, 0)^T\}$. In the case where (2) actually holds, both dominating points degenerate asymptotically as $\lambda \to \infty$. In Section 3, we shall derive an asymptotic expansion for probabilities (1) under the more specific assumption (5) concerning the possible asymptotic degeneracy of the dominating points. In Section 4, we shall compare the leading term of our expansion with the results of other authors and give a certain reformulation of the main result in this case. Section 2, which follows, deals with a geometric measure representation formula for the large-deviation probabilities (1), which will be the starting point for our asymptotic considerations. Concerning the density-generating function g, we shall assume that the conditions (D1) and (D2,m) in Richter and Schumacher (in print) are satisfied, i.e., g admits the representation

$$g(r) = \mathrm{e}^{-\tilde{g}(r)}, \qquad r > 0, \tag{D1}$$

with $\tilde{g}$ being a function first-order continuously differentiable and being invertible for large $r (r \geqslant r_0^2)$. It is also assumed that an "artificial" parameter $\rho = \rho(\tilde{g}, \lambda)$ can be chosen in such a way that $\tilde{g}^{-1}$ allows a power-series expansion in the form

$$\frac{\tilde{g}^{-1}(\rho z + \tilde{g}(\lambda^2/\sigma_1^2))}{\lambda^2} = \sum_{j=0}^{m} c_j z^j + \mathrm{O}(z^{m+1}), \qquad z \to 0, \tag{D2,m}$$

where m is a natural number, the coefficients $c_j = c_j(\rho, \lambda)$ approach certain constants c_j^* as λ tends to infinity, $c_j = c_j(\rho, \lambda) \to c_j^*$, $\lambda \to \infty$, and $c_1^* > 0$. Note that from (D2,m) it follows for $m = 0, 1, 2, 3$ that

$$c_0 = \frac{1}{\sigma_1^2}, \qquad c_1 = \frac{\rho}{\lambda^2 \tilde{g}'(\lambda^2/\sigma_1^2)}, \qquad c_2 = -\frac{\rho}{2} c_1 \frac{\tilde{g}''(\lambda^2/\sigma_1^2)}{[\tilde{g}'(\lambda^2/\sigma_1^2)]^2},$$

and

$$c_3 = -\frac{\rho^2}{12} c_1 \left[\frac{\tilde{g}'''(\lambda^2/\sigma_1^2)}{[\tilde{g}'(\lambda^2/\sigma_1^2)]^3} - 3 \frac{[\tilde{g}''(\lambda^2/\sigma_1^2)]^2}{[\tilde{g}'(\lambda^2/\sigma_1^2)]^4} \right],$$

respectively.

Example 1. Let $\gamma > 0, \beta > 0$, and $g(r) = r^{N-1-\beta r^\gamma} = r^{N-1} \mathrm{e}^{-\beta r^\gamma \ln r}$. Then $\tilde{g}(r) = \beta r^\gamma \ln r - (N-1) \ln r$ and

$$\begin{aligned}
\tilde{g}'(r) &= \beta \gamma r^{\gamma-1} \ln r + \beta r^{\gamma-1} - \frac{N-1}{r}, \\
\tilde{g}''(r) &= \beta \gamma (\gamma - 1) r^{\gamma-2} \ln r + \beta r^{\gamma-2}(2\gamma - 1) + \frac{N-1}{r^2}, \\
\tilde{g}'''(r) &= \beta \gamma (\gamma - 1)(\gamma - 2) r^{\gamma-3} \ln r + \beta r^{\gamma-3}(3\gamma^2 - 6\gamma + 2) - \frac{2(N-1)}{r^3}.
\end{aligned}$$

The function $\tilde{g}^{-1}$ satisfies condition (D2,3) with

$$\rho = 2\lambda^{2\gamma} \ln \lambda, \qquad c_1^* = \frac{\sigma_1^{2\gamma-2}}{\beta\gamma}, \qquad c_2^* = -\frac{(\gamma-1)\sigma_1^{2\gamma}}{2\beta\gamma} c_1^*,$$

and

$$c_3^* = -\frac{2(2\gamma-1)\sigma_1^{2\gamma}}{\beta\gamma} c_2^*.$$

To prove this, note that, for example,

$$\begin{aligned}
c_1 &= \frac{\rho}{\lambda^2} \left(\beta\gamma \left(\frac{\lambda^2}{\sigma_1^2}\right)^{\gamma-1} \ln \left(\frac{\lambda^2}{\sigma_1^2}\right) + \beta \left(\frac{\lambda^2}{\sigma_1^2}\right)^{\gamma-1} - \frac{(N-1)\sigma_1^2}{\lambda^2} \right)^{-1} \\
&= \frac{\rho}{2\lambda^{2\gamma} \ln \lambda} \left(\frac{\beta\gamma}{\sigma_1^{2\gamma-2}} \right)^{-1} \left(1 - \frac{\ln \sigma_1}{\ln \lambda} + \frac{1}{2\gamma \ln \lambda} - \frac{(N-1)\sigma_1^{2\gamma}}{2\beta\gamma\lambda^{2\gamma} \ln \lambda} \right)^{-1},
\end{aligned}$$

i.e.,

$$c_1 = \frac{\varrho}{2\lambda^{2\gamma} \ln \lambda} c_1^* \left(1 + \mathrm{O}\left(\frac{1}{\ln \lambda} \right) \right), \qquad \lambda \to \infty.$$

One can check similarly that

$$c_2 = -\left(\frac{\varrho}{2\lambda^{2\gamma}\ln\lambda}\right)^2 \frac{(\gamma-1)\sigma_1^{4\gamma-2}}{2(\beta\gamma)^2}\left(1+\mathrm{O}\left(\frac{1}{\ln\lambda}\right)\right), \qquad \lambda\to\infty,$$

and

$$c_3 = \left(\frac{\varrho}{2\lambda^{2\gamma}\ln\lambda}\right)^3 \frac{\sigma_1^{6\gamma-2}}{\beta^3\gamma^3}\left[(\gamma-1)(\gamma-2)\left(1+\mathrm{O}\left(\frac{1}{\ln\lambda}\right)\right)\right.$$
$$\left.-3(\gamma-1)^2\left(1+\mathrm{O}\left(\frac{1}{\ln\lambda}\right)\right)\right], \qquad \lambda\to\infty.$$

2. THE GEOMETRIC LAPLACE INTEGRAL REPRESENTATION FORMULA FOR THE LARGE-DEVIATION PROBABILITIES

Using the general representation formula for spherical distributions in [12], we have from assumption (D1)

$$\Phi(\lambda A(\lambda); g) = \frac{1}{I_{\tilde g}}\int_0^\infty F(\lambda A(\lambda), r) r \mathrm{e}^{-\tilde g(r^2)}\,\mathrm{d}r,$$

with the constant

$$I_{\tilde g} = \int_0^\infty r\mathrm{e}^{-\tilde g(r^2)}\,\mathrm{d}r$$

and the intersection-percentage function of a Borel set $A\subset R^2$ $F(A,r) = \omega((r^{-1}A)\cap S_2(1))$, $r>0$, where ω denotes the uniform probability distribution on the unit circle $S_2(1) = \{(x_1,x_1)^T \in R^2\colon x_1^2+x_2^2 = 1\}$ and $S_2(r) = \{(rx_1, rx_2)^T : (x_1,x_2)^T\in S_2(1)\}$. The substitution $r=\lambda v$ leads to

$$\Phi(\lambda A(\lambda); g) = \frac{\lambda^2}{I_{\tilde g}}\int_0^\infty F(\lambda A(\lambda), \lambda v) v\mathrm{e}^{-\tilde g(\lambda^2 v^2)}\,\mathrm{d}v.$$

Since $F(\lambda A(\lambda),\lambda v)\equiv F(A(\lambda), v)$, $\forall\lambda>0$, and because of $F(A(\lambda), v)\equiv 0$ for $v\in[0, 1/\sigma_1)$, it follows that

$$\Phi(\lambda A(\lambda); g) = \frac{\lambda^2}{I_{\tilde g}}\int_0^\infty F(A(\lambda), v) v\mathrm{e}^{-\tilde g(\lambda^2 v^2)}\,\mathrm{d}v.$$

If $v\geqslant 1/\sigma_1$ and $\lambda\geqslant r_0/\sqrt{c_0}$, then the relation

$$\tilde g(\lambda^2 v^2) = \rho y \tag{3}$$

is invertible for all "artificial" parameters $\rho>0$ and substitution (3) yields

$$\Phi(\lambda A(\lambda); g) = \frac{\rho}{2I_{\tilde g}}\int_{\tilde g(\lambda^2/\sigma_1^2)/\rho}^\infty F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde g^{-1}(\rho y)}\right)\tilde g^{-1\prime}(\rho y)\mathrm{e}^{-\rho y}\,\mathrm{d}y.$$

Thus, if the density-generating function satisfies assumption (D1), then for $\lambda\geqslant r_0/\sqrt{c_0}$ and all "artificial" parameters $\rho>0$, after the change of variables $z = y-\tilde g(\lambda^2/\sigma_1^2)/\rho$ we get the following geometric representation formula for the large-deviation probabilities under consideration:

$$\Phi(\lambda A(\lambda); g) = \frac{\rho}{2I_{\tilde{g}}} e^{-\tilde{g}(\lambda^2/\sigma_1^2)} \int_0^\infty F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(\rho z + \tilde{g}(\lambda^2/\sigma_1^2))}\right) \tilde{g}^{-1'}(\rho z + \tilde{g}(\lambda^2/\sigma_1^2)) e^{-\rho z}\, dz. \tag{4}$$

This formula, as well as that in the following Lemma 2.1, can also be interpreted as a Laplace integral representation for the large-deviation probabilities under consideration. Our final representation formula follows from the just-proved equation (4) by the change of variables $w = z/[\sigma_1^2 - \sigma_2^2(\lambda)]$.

LEMMA 2.1. *If assumption* (D1) *is satisfied, then for all* $\rho > 0$ *and* $\lambda \geqslant r_0/\sqrt{c_0}$ *the formula*

$$\Phi(\lambda A(\lambda); g) = \frac{[\sigma_1^2 - \sigma_2^2(\lambda)]\rho e^{-\tilde{g}(\lambda^2/\sigma_1^2)}}{2I_{\tilde{g}}} \int_0^\infty f(\lambda, \rho, w) e^{-[\sigma_1^2 - \sigma_2^2(\lambda)]\rho w}\, dw$$

holds; here

$$f(\lambda, \rho, w) = F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))}\right) \tilde{g}^{-1'}(u(\lambda, \rho, w))$$

with $u(\lambda, \rho, w) = [\sigma_1^2 - \sigma_2^2(\lambda)]\rho w + \tilde{g}(\lambda^2/\sigma_1^2)$.

Before defining the "artificial" parameter $\rho = \rho(\tilde{g}, \lambda)$ more precisely, let us assume that

$$[\sigma_1^2 - \sigma_2^2(\lambda)]\rho(\tilde{g}, \lambda) \longrightarrow \infty \quad \text{as} \quad \lambda \to \infty. \tag{5}$$

It turns out from the asymptotic theory of the Laplace integrals that under certain conditions an asymptotic expansion for $f(\lambda, \rho, w)$ as $w \to +0$ should yield an asymptotic expansion for the large-deviation probabilities $\Phi(\lambda A^*(\lambda); g)$ as $\lambda \to \infty$. Therefore, for our purposes it suffices to know an asymptotic expansion for the intersection-percentage function $F(A(\lambda), v)$ as $v \to 1/\sigma_1 + 0$ instead of the whole function F. In Lemma 2.2 we consider $F(A(\lambda), v)$ for v not much bigger than or equal to $1/\sigma_1$.

LEMMA 2.2. *If* $w \in [0, \frac{\tilde{g}(\lambda^2/\sigma_2^2(\lambda)) - \tilde{g}(\lambda^2/\sigma_1^2)}{\rho[\sigma_1^2 - \sigma_2^2(\lambda)]}]$, *then*

$$F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))}\right) = \frac{2}{\pi} \arcsin\sqrt{\frac{\tilde{g}^{-1}(u(\lambda, \rho, w)) - \lambda^2/\sigma_1^2}{(1 - \sigma_2^2(\lambda)/\sigma_1^2)\tilde{g}^{-1}(u(\lambda, \rho, w))}}.$$

Proof. Note that w belongs to the interval given in the assumption of the lemma if and only if

$$\frac{1}{\sigma_1^2} \leqslant \frac{\tilde{g}^{-1}(u(\lambda, \rho, w))}{\lambda^2} \leqslant \frac{1}{\sigma_2^2(\lambda)}.$$

In this case, the formula

$$F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))}\right) = \frac{4\alpha}{2\pi}$$

holds with

$$\alpha = \arcsin \frac{x_{20}}{\sqrt{x_{10}^2 + x_{20}^2}},$$

where (x_{10}, x_{20}) is to denote that the intersection point of the boundary of the set $A(\lambda)$ and the circle $S_2(\frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))})$, which belongs to the positive orthant, i.e., (x_{10}, x_{20}), satisfies the equations

$$\sigma_1^2 x_{10}^2 + \sigma_2^2(\lambda) x_{20}^2 = 1 \tag{6}$$

and

$$x_{10}^2 + x_{20}^2 = \frac{1}{\lambda^2}\tilde{g}^{-1}(u(\lambda, \rho, w)). \tag{7}$$

A combination of (6) and (7) gives

$$\frac{\sigma_1^2}{\lambda^2}\tilde{g}^{-1}(u(\lambda, \rho, w)) - 1 = x_{20}^2[\sigma_1^2 - \sigma_2^2(\lambda)].$$

From this equation and (7), it follows that

$$\alpha = \arcsin\sqrt{\frac{(\sigma_1^2/\lambda^2)\tilde{g}^{-1}(u(\lambda, \rho, w)) - 1}{((\sigma_1^2 - \sigma_2^2(\lambda))/\lambda^2)\tilde{g}^{-1}(u(\lambda, \rho, w))}}$$

3. MAIN RESULTS

In the spirit of Watson's lemma combined with Lemma 2.1, a main step in deriving an asymptotic expansion for the large-deviation probabilities $\Phi(\lambda A(\lambda) : g)$ as $\lambda \to \infty$ will be to derive an asymptotic expansion for the intersection-percentage function $F(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))})$ as $w \to +0$.

LEMMA 3.1. *If the density-generating function g satisfies assumption* (D2,m), *then under* (5)

$$F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))}\right) = \frac{2\sqrt{c_1}\sigma_1^2}{\pi}\left\langle w^{1/2} + \left(\frac{c_1\sigma_1^4}{6} + A_1[\sigma_1^2 - \sigma_2^2(\lambda)]\right)w^{3/2}\right.$$
$$\left. + \left(\frac{c_1\sigma_1^4 A_1}{2}[\sigma_1^2 - \sigma_2^2(\lambda)] + A_2[\sigma_1^2 - \sigma_2^2(\lambda)]^2 + \frac{3c_1^2\sigma_1^8}{40}\right)w^{5/2} + \cdots\right\rangle$$

as $w \to +0$, *where*

$$A_1 = \frac{1}{2}\left(\frac{c_2}{c_1} - c_1\sigma_1^2\right)$$

and

$$A_2 = \frac{1}{2}\left(\frac{c_3}{c_1} - \frac{c_2^2}{4c_1^2} - \frac{3c_2\sigma_1^2}{2} + \frac{3c_1^2\sigma_1^4}{4}\right).$$

Proof. Note that (5) ensures that the upper bound for w in the assumption of Lemma 2.2 tends to zero as $\lambda \to \infty$. We now rewrite the result of Lemma 2.2 with $z = w[\sigma_1^2 - \sigma_2^2(\lambda)]$ as

$$F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))}\right) = \frac{2}{\pi}\arcsin\sqrt{\frac{\tilde{g}^{-1}(\rho z + \tilde{g}(\lambda^2/\sigma_1^2))/\lambda^2 - 1/\sigma_1^2}{(1 - \sigma_2^2(\lambda)/\sigma_1^2)\tilde{g}^{-1}(\rho z + \tilde{g}(\lambda^2/\sigma_1^2))/\lambda^2}}$$

and get from assumption (D2,m) with $m \geqslant 1$ for $z \to 0$ that

$$F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))}\right) = \frac{2}{\pi}\arcsin\sqrt{\frac{\sum_{j=1}^m c_j z^j + O(z^{m+1})}{(1 - \sigma_2^2(\lambda)/\sigma_1^2)\left(\sum_{j=0}^m c_j z^j + O(z^{m+1})\right)}}$$
$$= \frac{2}{\pi}\arcsin\sqrt{\frac{c_1 z\left(1 + \sum_{j=2}^m \frac{c_j}{c_1}z^{j-1} + O(z^m)\right)}{(1 - \sigma_2^2(\lambda)/\sigma_1^2)\frac{1}{\sigma_1^2}\left(1 + \sum_{j=1}^m c_j z^j \sigma_1^2 + O(z^{m+1})\right)}}.$$

From the well-known expansions

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \mp \cdots, \qquad \frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} \pm \cdots,$$

it follows that

$$\begin{aligned}\left(\frac{1+\sum_{j=2}^{m}\frac{c_j}{c_1}z^{j-1}+\mathrm{O}(z^m)}{1+\sigma_1^2\sum_{j=1}^{m}c_j z^j+\mathrm{O}(z^{m+1})}\right)^{1/2} &= \left\langle 1+\frac{1}{2c_1}[c_2z+c_3z^2+\cdots]\right. \\ &\quad \left. -\frac{1}{8c_1^2}[c_2z+\cdots]^2+\cdots\right\rangle\left\langle 1-\frac{\sigma_1^2}{2}[c_1z+c_2z^2+\cdots]+\frac{3\sigma_1^4}{8}[c_1z+\cdots]^2+\cdots\right\rangle \\ &= \left\langle 1+\frac{c_2}{2c_1}z+\left(\frac{c_3}{2c_1}-\frac{c_2^2}{8c_1^2}\right)z^2+\cdots\right\rangle\left\langle 1-\frac{\sigma_1^2c_1}{2}z+\left(-\frac{\sigma_1^2c_2}{2}+\frac{3\sigma_1^4c_1^2}{8}\right)z^2+\cdots\right\rangle \\ &= 1+A_1z+A_2z^2+\cdots.\end{aligned}$$

On combining the resulting representation for the intersection-percentage function

$$\begin{aligned}&F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda,\rho,w))}\right) \\ &= \frac{2}{\pi}\arcsin\left\langle\sqrt{\frac{c_1\sigma_1^4 z}{\sigma_1^2-\sigma_2^2(\lambda)}}(1+A_1z+A_2z^2+\cdots)\right\rangle \\ &= \frac{2}{\pi}\arcsin\left\langle\sqrt{c_1}\sigma_1^2 w^{1/2}\left(1+A_1w[\sigma_1^2-\sigma_2^2(\lambda)]+A_2w^2[\sigma_1^2-\sigma_2^2(\lambda)]^2+\cdots\right)\right\rangle\end{aligned}$$

with the expansion

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots, \qquad |x| < 1,$$

we get

$$\begin{aligned}&F\left(A(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda,\rho,w))}\right) \\ &= \frac{2}{\pi}\left\langle\sqrt{c_1}\sigma_1^2 w^{1/2}\left(1+A_1w[\sigma_1^2-\sigma_2^2\lambda)]+A_2w^2[\sigma_1^2-\sigma_2^2(\lambda)]^2+\cdots\right)\right. \\ &\quad +\frac{1}{6}c_1^{3/2}\sigma_1^6 w^{3/2}\left(1+A_1w[\sigma_1^2-\sigma_2^2(\lambda)]+\cdots\right)^3 \\ &\quad \left.+\frac{3}{40}c_1^{5/2}\sigma_1^{10}w^{5/2}(1+\cdots)^5+\cdots\right\rangle \\ &= \frac{2}{\pi}\sqrt{c_1}\sigma_1^2\left\langle w^{1/2}+w^{3/2}\left(A_1[\sigma_1^2-\sigma_2^2(\lambda)]+\frac{c_1\sigma_1^4}{6}\right)\right. \\ &\quad \left.+w^{5/2}\left(A_2[\sigma_1^2-\sigma_2^2(\lambda)]^2+\frac{c_1\sigma_1^4}{2}A_1[\sigma_1^2-\sigma_2^2(\lambda)]+\frac{3}{40}c_1^2\sigma_1^8\right)+\cdots\right\rangle.\end{aligned}$$

We are now in a position to expand the function $f = F\,\tilde{g}^{-1\prime}$.

LEMMA 3.2. *If the density-generating function satisfies conditions* (D1) *and* (D2,m), *then under* (5)

$$f(\lambda,\varrho,w) = \frac{2\sqrt{c_1}\sigma_1^2\lambda^2}{\pi\rho}\left[b_1w^{1/2}+b_2w^{3/2}+b_3w^{5/2}+\cdots+b_mw^{\frac{2m-1}{2}}+\mathrm{O}\left(w^{\frac{2m+1}{2}}\right)\right],$$

where the first coefficients are

$$b_1 = c_1,$$
$$b_2 = \frac{1}{6}c_1^2\sigma_1^4 + (c_1A_1 + 2c_2)[\sigma_1^2 - \sigma_2^2(\lambda)],$$

and

$$b_3 = \frac{3}{40}\sigma_1^8 c_1^3 + \left(\frac{1}{2}\sigma_1^4 c_1^2 A_1 + \frac{1}{3}\sigma_1^4 c_1 c_2\right)[\sigma_1^2 - \sigma_2^2(\lambda)] + (c_1A_2 + 2c_2A_1 + 3c_3)[\sigma_1^2 - \sigma_2^2(\lambda)]^2.$$

Proof. We start with the definition of f,

$$f(\lambda, \rho, w) = F\left(A^*(\lambda), \frac{1}{\lambda}\sqrt{\tilde{g}^{-1}(u(\lambda, \rho, w))}\right)\tilde{g}^{-1\prime}(u(\lambda, \rho, w))$$

and plug in both the expansion for F from Lemma 3.1 and an expansion for $\tilde{g}^{-1\prime}$ which follows from (D2,m) using the continuous differentiability of $\tilde{g}^{-1}$:

$$\begin{aligned}
f(\lambda, \rho, w) &= \frac{2\sqrt{c_1}\sigma_1^2}{\pi}\Bigg\langle w^{1/2} + \left(\frac{1}{6}\sigma_1^4 c_1 + A_1[\sigma_1^2 - \sigma_2^2(\lambda)]\right)w^{3/2} \\
&\quad + \left(\frac{3}{40}\sigma_1^8 c_1^2 + \frac{1}{2}\sigma_1^4 c_1 A_1[\sigma_1^2 - \sigma_2^2(\lambda)] + A_2[\sigma_1^2 - \sigma_2^2(\lambda)]^2\right)w^{5/2} \\
&\quad + \cdots\Bigg\rangle\frac{\lambda^2}{\rho}\langle c_1 + 2c_2 w[\sigma_1^2 - \sigma_2^2(\lambda)] + 3c_3 w^2[\sigma_1^2 - \sigma_2^2(\lambda)]^2 + \cdots\rangle \\
&= \frac{2\sqrt{c_1}\sigma_1^2\lambda^2}{\pi\rho}\Bigg[c_1 w^{1/2} + \left(\frac{1}{6}c_1^2\sigma_1^4 + (c_1A_1 + 2c_2)[\sigma_1^2 - \sigma_2^2(\lambda)]\right)w^{3/2} \\
&\quad + \Bigg(\frac{3}{40}\sigma_1^8 c_1^3 + \left(\frac{1}{2}\sigma_1^4 c_1^2 A_1 + \frac{1}{3}\sigma_1^4 c_1 c_2\right)[\sigma_1^2 - \sigma_2^2(\lambda)] \\
&\quad + (c_1A_2 + 2c_2A_1 + 3c_3)[\sigma_1^2 - \sigma_2^2(\lambda)]^2\Bigg)w^{5/2} + \cdots\Bigg].
\end{aligned}$$

THEOREM 3.3. *If assumptions* (D1), (D2,m), *and* (5) *are satisfied, then for* $\lambda \to \infty$ *we have*

$$\begin{aligned}
\Phi(\lambda A(\lambda); g) &= \frac{c_1^{3/2}}{2\sqrt{\pi}}\frac{\sigma_1}{I_{\tilde{g}}}\frac{\lambda^2}{\rho^{3/2}[1 - \sigma_2^2(\lambda)/\sigma_1^2]^{1/2}}e^{-\tilde{g}(\lambda^2/\sigma_1^2)}\Bigg[1 + \frac{D_{1,1}}{\rho[\sigma_1^2 - \sigma_2^2(\lambda)]} \\
&\quad + \frac{D_{1,2}}{\rho} + \frac{D_{2,1}}{[\rho[\sigma_1^2 - \sigma_2^2(\lambda)]]^2} + \frac{D_{2,2}}{\rho^2[\sigma_1^2 - \sigma_2^2(\lambda)]} + \frac{D_{2,3}}{\rho^2} + \cdots\Bigg]
\end{aligned}$$

with

$$D_{1,1} = \frac{c_1\sigma_1^4}{4}, \qquad D_{1,2} = \frac{15}{4}\frac{c_2}{c_1} - \frac{3}{4}c_1\sigma_1^2,$$
$$D_{2,1} = \frac{9}{32}\sigma_1^8 c_1^2, \qquad D_{2,2} = \frac{5}{16}(7\sigma_1^4 c_2 - 3\sigma_1^4 c_1^2),$$

and

$$D_{2,3} = \frac{15}{8}\left(7\frac{c_3}{c_1} + \frac{7}{4}\frac{c_2^2}{c_1^2} - \frac{7}{2}c_2\sigma_1^2 + \frac{3}{4}c_1^2\sigma_1^4\right).$$

Proof. From Lemmas 2.1 and 3.2 we have

$$\Phi(\lambda A(\lambda); g) = \frac{(\sigma_1^2 - \sigma_2^2(\lambda))\rho \exp\{-\tilde{g}(\lambda^2/\sigma_1^2)\}}{2I_{\tilde{g}}} \times \int_0^\infty \frac{2\sqrt{c_1}\sigma_1^2\lambda^2}{\pi\rho}(b_1 w^{1/2} + b_2 w^{3/2} + b_3 w^{5/2} + \cdots) e^{[\sigma_1^2 - \sigma_2^2(\lambda)]\rho w}\, dw.$$

Applying Lemma 3.4 we get

$$\begin{aligned}
\Phi(\lambda A(\lambda); g) &= \frac{\sqrt{c_1}\sigma_1^2\lambda^2[\sigma_1^2 - \sigma_2^2(\lambda)]}{\pi I_{\tilde{g}}} \exp\{-\tilde{g}(\lambda^2/\sigma_1^2)\} \\
&\quad \times \left[\frac{b_1\Gamma(3/2)}{([\sigma_1^2 - \sigma_2^2(\lambda)]\rho)^{3/2}} + \frac{b_2\Gamma(5/2)}{([\sigma_1^2 - \sigma_2^2(\lambda)]\rho)^{5/2}} + \frac{b_3\Gamma(7/2)}{([\sigma_1^2 - \sigma_2^2(\lambda)]\rho)^{7/2}} + \cdots \right] \\
&= \frac{\sqrt{c_1}\sigma_1^2\lambda^2[\sigma_1^2 - \sigma_2^2(\lambda)]}{\pi I_{\tilde{g}}} \exp\{-\tilde{g}(\lambda^2/\sigma_1^2)\} \\
&\quad \times \left[\frac{c_1\sqrt{\pi}}{2([\sigma_1^2 - \sigma_2^2(\lambda)]\rho)^{3/2}} + \frac{3\sqrt{\pi}\left(\frac{1}{6}c_1^2\sigma_1^4 + (c_1A_1 + 2c_2)[\sigma_1^2 - \sigma_2^2(\lambda)]\right)}{4([\sigma_1^2 - \sigma_2^2(\lambda)]\rho)^{5/2}} \right. \\
&\quad + \frac{15\sqrt{\pi}(\frac{3}{40}\sigma_1^8 c_1^3 + (\frac{1}{2}\sigma_1^4 c_1^2 A_1 + \frac{1}{3}\sigma_1^4 c_1 c_2)[\sigma_1^2 - \sigma_2^2(\lambda)])}{8([\sigma_1^2 - \sigma_2^2(\lambda)]\rho)^{7/2}} \\
&\quad \left. + \frac{15\sqrt{\pi}(c_1A_2 + 2c_2A_1 + 3c_3)[\sigma_1^2 - \sigma_2^2(\lambda)]^2}{8([\sigma_1^2 - \sigma_2^2(\lambda)]\rho)^{7/2}} + \cdots \right] \\
&= \frac{c_1^{3/2}\sigma_1^2\lambda^2}{2\sqrt{\pi} I_{\tilde{g}}\rho^{3/2}[\sigma_1^2 - \sigma_2^2(\lambda)]^{1/2}} \exp\{-\tilde{g}(\lambda^2/\sigma_1^2)\} \\
&\quad \times \left[1 + \frac{3\left(\frac{1}{6}c_1^2\sigma_1^4 + (c_1A_1 + 2c_2)[\sigma_1^2 - \sigma_2^2(\lambda)]\right)}{2c_1\rho[\sigma_1^2 - \sigma_2^2(\lambda)]} \right. \\
&\quad + \frac{15\left(\frac{3}{40}\sigma_1^8 c_1^3 + \left(\frac{1}{2}\sigma_1^4 c_1^2 A_1 + \frac{1}{3}\sigma_1^4 c_1 c_2\right)[\sigma_1^2 - \sigma_2^2(\lambda)]\right)}{4c_1\rho^2[\sigma_1^2 - \sigma_2^2(\lambda)]^2} \\
&\quad \left. + \frac{\left((c_1A_2 + 2c_2A_1 + 3c_3)[\sigma_1^2 - \sigma_2^2(\lambda)]^2\right)}{4c_1\rho^2[\sigma_1^2 - \sigma_2^2(\lambda)]^2} \right. \\
&= \frac{c_1^{3/2}\sigma_1^2\lambda^2}{2\sqrt{\pi} I_{\tilde{g}}\rho^{3/2}[\sigma_1^2 - \sigma_2^2(\lambda)]^{1/2}} \exp\{-\tilde{g}(\lambda^2/\sigma_1^2)\} \\
&\quad \times \left[1 + \frac{c_1\sigma_1^4}{4\rho[\sigma_1^2 - \sigma_2^2(\lambda)]} + \frac{3(c_1A_1 + 2c_2)}{2c_1\rho} + \frac{9}{32}\frac{\sigma_1^8 c_1^2}{\rho^2[\sigma_1^2 - \sigma_2^2(\lambda)]^2} \right. \\
&\quad \left. + \frac{5(3\sigma_1^4 c_1 A_1 + 2\sigma_1^4 c_2)}{8\varrho^2[\sigma_1^2 - \sigma_2^2(\lambda)]} + \frac{15(c_1A_2 + 2c_2A_1 + 3c_3)}{4c_1\rho^2} + \cdots \right].
\end{aligned}$$

The proof is completed by plugging in the expressions for A_1 and A_2 given in Lemma 3.1.

Remark. Note that the ordering of the expansion terms depends on the behavior of $\sigma_1^2 - \sigma_2^2(\lambda)$ as $\lambda \to \infty$. To give an example, we compare $D_{1,2}/\rho$ and $D_{2,1}/(\rho^2[\sigma_1^2 - \sigma_2^2(\lambda)]^2)$. We have

$$\frac{D_{2,1}}{\rho^2[\sigma_1^2 - \sigma_2^2(\lambda)]^2} = o\left(\frac{D_{1,2}}{\rho}\right), \qquad \lambda \to \infty,$$

$$\text{if } = \frac{1}{\rho[\sigma_1^2 - \sigma_2^2(\lambda)]} = \mathrm{o}(\sigma_1^2 - \sigma_2^2(\lambda)), \qquad \lambda \to \infty,$$

and

$$\frac{D_{2.1}}{\rho^2(\sigma_1^2 - \sigma_2^2(\lambda))^2} \asymp \frac{D_{1.2}}{\rho}, \qquad \lambda \to \infty,$$

$$\text{if } \frac{1}{\rho[\sigma_1^2 - \sigma_2^2(\lambda)]} \asymp (\sigma_1^2 - \sigma_2^2(\lambda)), \qquad \lambda \to \infty,$$

as well as

$$\frac{D_{1.2}}{\rho} = \mathrm{o}\left(\frac{D_{2.1}}{\rho^2(\sigma_1^2 - \sigma_2^2(\lambda))^2}\right), \qquad \lambda \to \infty,$$

$$\text{if } \sigma_1^2 - \sigma_2^2(\lambda) = \mathrm{o}\left(\frac{1}{\rho(\sigma_1^2 - \sigma_2^2(\lambda))}\right), \qquad \lambda \to \infty.$$

As a consequence, it is not possible, in general, to replace the dots in the assertion of Theorem 3.3 by the remainder term $\mathrm{o}(\rho^{-2})$. A higher-dimensional treatment needs a much more systematic study at this point.

The following lemma is a modification of Watson's lemma given in Richter and Schumacher (in print).

LEMMA 3.4 (modified Watson's lemma). *Let* $f\colon (0,\infty)^{\times 2} \to \mathbb{R}$ *satisfy the following assumptions*:

(i) $f(\lambda, .)$ *is locally integrable for every* $\lambda > 0$ *and uniformly (with respect to* λ*) bounded on finite intervals*;

(ii) $f(\lambda, y) = \mathrm{O}(\mathrm{e}^{ay})$, $y \to \infty$, *uniformly in* λ;

(ii) *for* $y \to 0+$, *the function* f *allows the expansion*

$$f(\lambda, y) = \sum_{j=0}^{m} c_j y^{a_j} + \mathrm{O}(y^{a_{m+1}})$$

uniformly with respect to λ, *where the sequence* (a_j) *increases monotonically to* $+\infty$ *as* $j \to \infty$, $a_0 > -1$ *and* $c_j = c_j(\lambda) = \mathrm{O}(1)$, $\lambda \to \infty$.

Then

$$\int_0^\infty f(\lambda, t)\mathrm{e}^{-\lambda t}\,\mathrm{d}t = \sum_{j=0}^{m} c_j \frac{\Gamma(a_j + 1)}{\lambda^{a_j+1}} + \mathrm{O}(\lambda^{-(a_{m+1}+1)}), \qquad \lambda \to \infty. \tag{8}$$

4. DISCUSSION OF THE LEADING TERM

If one suppresses the higher-order expansion terms in Theorem 3.3, one gets the following conclusion.

COROLLARY 4.1. *Under* (D1), (D2,0), *and* (5) *we have*

$$\Phi(\lambda A(\lambda); g) \sim \left(2\sqrt{\pi(1 - \sigma_2^2(\lambda)/\sigma_1^2)I_{\bar g}}\right)^{-1} \frac{\mathrm{e}^{-\bar g(\lambda^2/\sigma_1^2)}}{\left(\bar g'\left(\lambda^2/\sigma_1^2\right)\right)^{3/2}(\lambda/\sigma_1)}.$$

Remark. Under the assumptions of Corollary 4.1, we have
a) If $\sigma_2^2(\lambda) \to 0$, then $\sqrt{1-\sigma_2^2(\lambda)/\sigma_1^2} \sim 1$, $\lambda \to \infty$, and hence

$$\Phi(\lambda A(\lambda); g) \sim \frac{1}{2\sqrt{\pi} I_{\tilde{g}}} \frac{e^{-\tilde{g}(\lambda^2/\sigma_1^2)}}{\left(\tilde{g}'\left(\lambda^2/\sigma_1^2\right)\right)^{3/2}(\lambda/\sigma_1)} \quad \text{as} \quad \lambda \to \infty.$$

b) Assumption (5) implies

$$\Phi(\lambda A(\lambda); g) = o(1) \frac{\rho^{1/2}}{\lambda(\tilde{g}'(\lambda^2/\sigma_1^2))^{3/2}} e^{-\tilde{g}(\lambda^2/\sigma_1^2)} \quad \text{as} \quad \lambda \to \infty.$$

Proof. From Theorem 3.3 it follows that

$$\Phi(\lambda A(\lambda); g) \sim \frac{c_1^{3/2}\sigma_1}{2\sqrt{\pi} I_{\tilde{g}}} \frac{\lambda^2 \exp\{-\tilde{g}(\lambda^2/\sigma_1^2)\}}{\rho^{3/2}(1-\sigma_2^2(\lambda)/\sigma_1^2)^{1/2}}$$

as $\lambda \to \infty$. From the general representation formula for c_1 we have

$$c_1^{3/2} = \frac{\rho^{3/2}}{\lambda^3(\tilde{g}'(\lambda^2/\sigma_1^2))^{3/2}}.$$

Consequently,

$$\Phi(\lambda A(\lambda); g) \sim \frac{\sigma_1}{\lambda} \frac{(1-\sigma_2^2(\lambda)/\sigma_1^2)^{-1/2} \exp\{-\tilde{g}(\lambda^2/\sigma_1^2)\}}{2\sqrt{\pi}\ I_{\tilde{g}}\ (\tilde{g}'(\lambda^2/\sigma_1^2))^{3/2}}$$

Example 2. If $\tilde{g}(r) = \beta r^\gamma$ for some $\beta > 0$, $\gamma > 0$, then assumption (D1) is satisfied,

$$(\tilde{g}'(\lambda^2/\sigma_1^2))^{3/2} = (\beta\gamma)^{3/2} \frac{\lambda^{3(\gamma-1)}}{\sigma_1^{3(\gamma-1)}}, \qquad \tilde{g}^{-1}(t) = (t/\beta)^{1/\gamma},$$

assumption (D2,0) is satisfied with $\rho = \lambda^{2\gamma}$,

$$I_{\tilde{g}} = \frac{\Gamma(1/\gamma)}{2\gamma\beta^{1/\gamma}},$$

and therefore under (D2,m)

$$\Phi(\lambda A(\lambda); g) \sim \frac{\beta^{1/\gamma-3/2}\sigma_1^{3\gamma-2}\lambda^{2-3\gamma} \exp\left\{-\beta(\lambda^{2\gamma}/\sigma_1^{2\gamma})\right\}}{\sqrt{\gamma}\ \Gamma(1/\gamma)\sqrt{\pi}\sqrt{1-\sigma_2^2(\lambda)/\sigma_1^2}}$$

as $\lambda \to \infty$.

Note that in this case (5) is equivalent to

$$\sigma_2^2(\lambda) = \sigma_1^2 - \frac{R(\lambda)}{\lambda^{2\gamma}} \tag{9}$$

for some function $R(\lambda)$ with

$$R(\lambda) \longrightarrow \infty \quad \text{as} \quad \lambda \to \infty.$$

In the special case $g(r) = g_G(r) = e^{-r/2}$, it follows under

$$(\sigma_1^2 - \sigma_2^2(\lambda))\lambda^2 \longrightarrow \infty, \qquad \lambda \to \infty, \tag{10}$$

that

$$\Phi(\lambda A(\lambda); g_G) \sim \frac{\sqrt{2}}{\sqrt{1 - \sigma_2^2(\lambda)/\sigma_1^2}} \frac{\exp\{-\lambda^2/(2\sigma_1^2)\}}{\sqrt{\pi}(\lambda/\sigma_1)}$$

as $\lambda \to \infty$. Hence, under (10)

$$\Phi(\lambda A(\lambda); g_G) = o(1)e^{-\lambda^2/(2\sigma_1^2)}, \qquad \lambda \to \infty, \tag{11}$$

and if $\sigma_2^2(\lambda) \to 0$, then

$$\Phi(\lambda A(\lambda);\ g_G) \sim 2\frac{\exp\{-\lambda^2/(2\sigma_1^2)\}}{\sqrt{2\pi}(\lambda/\sigma_1)}, \qquad \lambda \to \infty. \tag{12}$$

Note that $\exp\{-\lambda^2/(2\sigma_1^2)\}$ and $(2\pi)^{-1/2}(\lambda/\sigma)^{-1}\exp\{-\lambda^2/(2\sigma_1^2)\}$ describe the order and the exact speed at which the Gaussian measure of the complement of a large ball and of a half-space with large distance from the origin tend to zero, respectively. Related results for higher dimensions and the general Kotz-type density-generating function can be found in Richter and Schumacher (in print).

Example 1 (*continued*). From Example 1 we know that the function

$$\tilde{g}(r) = \beta r^\gamma \ln r - (N-1)\ln r$$

satisfies assumption (D2,0) with $\rho = \lambda^{2\gamma}\ln\lambda^2$ and

$$\tilde{g}'(\lambda^2/\sigma_1^2) = \beta\gamma\left(\frac{\lambda^2}{\sigma_1^2}\right)^{\gamma-1}\ln\left(\frac{\lambda^2}{\sigma_1^2}\right) + \beta\left(\frac{\lambda^2}{\sigma_1^2}\right)^{\gamma-1} - \frac{N-1}{\lambda^2/\sigma_1^2}.$$

Hence, if

$$(\sigma_1^2 - \sigma_2^2(\lambda))\lambda^{2\gamma}\ln\lambda \longrightarrow \infty, \qquad \lambda \to \infty,$$

then

$$\Phi(\lambda A(\lambda; g) \sim \frac{\sigma_1^{3\gamma-2N}}{2^{5/2}\sqrt{\pi}I_{\tilde{g}}(\beta\gamma)^{3/2}} \frac{\lambda^{2N-3\gamma}\exp\left\{-2\beta\frac{\lambda^{2\gamma}}{\sigma_1^{2\gamma}}\ln\left(\frac{\lambda}{\sigma_1}\right)\right\}}{\left[\ln\left(\frac{\lambda}{\sigma_1}\right)\right]^{3/2}}, \qquad \lambda \to \infty.$$

Our final considerations concern a further interpretation of condition (5) stated above.

LEMMA 4.2. *The boundary of the large-deviation domain $A(\lambda)$ has the same curvature,*

$$\kappa(x_{0i}) = \sigma_2^2(\lambda)/\sigma_1^2, \qquad i = 1, 2,$$

at its two dominating points, $x_{01} = (1/\sigma_1, 0)^T$ and $x_{02} = (-1/\sigma_1, 0)^T$.

Proof. Recall the definition of the curvature of the boundary $\partial A(\lambda)$ of the set $A(\lambda)$,

$$A(\lambda) = \left\{(x_1, x_2)^T : g(x_1, x_2) \leqslant 0\right\},$$
$$g(x_1, x_2) = 1 - \sigma_1^2 x_1^2 - \sigma_2^2(\lambda)x_2^2, \qquad \sigma_2(\lambda) < \sigma_1,$$

at the point x_0:

$$\kappa(x_0) = \frac{-g_1^2 g_{22} + 2g_1 g_2 g_{12} - g_2^2 g_{11}}{(g_1^2 + g_2^2)^{3/2}},$$

where

$$g_i = \frac{\partial}{\partial x_i} g(x_1, x_2)|_{(x_1,x_2)^T = x_0}, \qquad g_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} g(x_1, x_2)|_{(x_1,x_2)^T = x_0}.$$

The assertion of the lemma now follows immediately.

Condition (5) can be reformulated because of Lemma 4.2 as

$$\big(1 - \kappa(x_{0i})\big)\rho(\tilde{g}, \lambda) \longrightarrow \infty. \tag{13}$$

This means that the dominating points can be allowed to degenerate asymptotically as λ approaches infinity, but "not too fast." Concerning the role of the curvatures of the boundaries of large-deviation domains, for our purposes we refer to [4–6]. Note that in the cited works it is not allowed that the dominating points degenerate asymptotically. In Richter and Schumacher (in print), the authors deal with an asymptotic expansion for large-deviations concerning a certain type of asymptotically degenerated dominating points. Here we deal with an asymptotic expansion for another type of dominating points. Recall that similar asymptotically degenerated dominating points have been already considered in [2] and [11] but only for deriving simple zero-order asymptotics. The asymptotic degeneracy of the dominating points has been expressed in these papers in terms of $g''(0)$, where $g''(f) = \frac{d^2}{df^2} R^2(f)$ and $\{[f, R(f)]^T, f \in \Theta\}$ describes the boundary $\partial A(\lambda)$ of $A(\lambda)$ with $[0, R(0)]^T$ corresponding to the dominating point $(1/\sigma_1, 0)$.

LEMMA 4.3.

$$g''(0) = \frac{2}{\sigma_1^2}(1 - \kappa(x_{01})).$$

Proof. Put

$$x_1(f) = R(f)\cos f, \qquad x_2(f) = R(f)\sin f$$

and consider the boundary

$$\partial A(\lambda) = \big\{(x_1, x_2)^T\colon\ \sigma_1^2 x_1^2 + \sigma_2^2(\lambda) x_2^2 = 1\big\}$$

of $A(\lambda)$. The points from $\partial A(\lambda)$ satisfy the equation

$$1 = \sigma_2^2(\lambda) R^2(f) + (\sigma_1^2 - \sigma_2^2(\lambda)) R^2(f)\cos^2 f.$$

By differentiating this equation twice and using the relation $R(0) = 1/\sigma_1$, we get $R'(0) = 0$ and then

$$R''(0) = \frac{R(0)(\sigma_1^2 - \sigma_2^2(\lambda))}{\sigma_1^2} = \left(1 - \frac{\sigma_2^2(\lambda)}{\sigma_1^2}\right)/\sigma_1.$$

Thus,

$$g''(0) = \frac{2}{\sigma_1^2}\left(1 - \frac{\sigma_2^2(\lambda)}{\sigma_1^2}\right).$$

In a slightly different situation than the one considered above, in [2], following a way of Fedoryuk [8], it had still been assumed that $\lambda^2(1 - \kappa(x_{01}))/\ln\lambda \to \infty$, whereas this assumption could be weakened in [11] to $\lambda^2(1 - \kappa(x_{01})) \to \infty$ due to the geometric approach created there.

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