PROBABILITIES AND LARGE QUANTILES OF NONCENTRAL GENERALIZED CHI-SQUARE DISTRIBUTIONS

C. ITTRICH, D. KRAUSE and W.-D. RICHTER
University of Rostock

Summary. Exact values of probability integrals for noncentral generalized chi-square distributions are numerically evaluated based upon new geometric representation formulae for these distributions. Using iterative numerical methods exact quantiles can be calculated then.
Explicit quantile approximation formulae are deduced from an asymptotic expansion for related probabilities of large deviations. Though this method is originally directed to the construction of starting values for determining exact large quantiles it is of benefit for simply approximating large quantiles and for obtaining quantiles from the central part of the distributions, too. The accuracy of the explicit asymptotic approximation method can be improved by combining it with the geometric measure representation formulae.
Several numerical studies compare the present results with results of other authors available in the special case of the classical noncentral chi-square distribution.
As an application, critical test points as well as power functions for expectation tests in elliptically contoured sample distributions are considered and certain problems of sensitivity and robustness type are discussed.

AMS subject classification: 62E15, 62E17, 62-04, 41-04

Key words: spherical samples, elliptically contoured distributions, geometric representation formulae, noncentral generalized chi-square distributions, exact distributions, exact quantiles, large quantiles, asymptotic expansions, large deviation quantile approximation, power functions, model correctness, robustness, sensitivity.

1 INTRODUCTION

The well known family of noncentral chi-square distributions has been considered by many authors. Several methods have been developed for approximating probability integrals and quantiles of these distributions. We mention here the methods of reduction to Bessel or Incomplete Gamma functions, asymptotic normal approximations, normal approximations after suitable transformations, series representations using central distributions, Edgeworth expansions, Cornish-Fisher expansions and continued fractions. For more details we refer to the results in Fisher (1928), Patnaik (1954), Abdel-Aty (1954), Sankaran (1959, 1963), Haynam et al. (1982), Farebrother (1987), Ashour and Abdel-Samad (1990) and Wang and Kennedy (1994).
The family of noncentral generalized chi-square distributions has been studied first in Cacoullos and Koutras (1984) and then in Hsu (1990). The noncentral g-generalized chi-square distribution function with $k \geq 2$ degrees of freedom (d.f.) and noncentrality

parameter (n.c.p.) $\delta^2 \geq 0$ can be defined as

$$CQ(k, \delta^2; g)(x) = P(\|X + \mu^*\|^2 < x), \ x \in \mathbf{R}$$

for arbitrary $\mu^* \in \mathbf{R}^k$ satisfying $\|\mu^*\|^2 = \delta^2$. The random vector X follows a k-dimensional spherically symmetric distribution having the density

$$p(x; g) = C(k, g)\, g(\|x\|^2), x \in \mathbf{R}^k$$

with the density generating function $g|\,[0, \infty) \to [0, \infty)$ satisfying

$$0 < I_{k,g} < \infty, \tag{1}$$

where

$$I_{k,g} = \int_0^\infty r^{k-1} g(r^2)\, dr. \tag{2}$$

The norming constant $C(k, g)$ is

$$C(k, g) = (\omega_k I_{k,g})^{-1}$$

with

$$\omega_k = 2\pi^{k/2}/\Gamma(k/2)$$

denoting the surface area of the unit sphere $S_k(1)$ in $\mathbf{R}^k$.
Spherically symmetric distributions, or more general elliptically contoured distributions have been studied by many authors beginning with Schoenberg (1938) and Kelker (1970). Anderson and Fang (1982) studied central distributions of quadratic forms for elliptically contoured random vectors. For the general theory of elliptically contoured distributions and their applications to statistics we refer to the monographs of Watson (1983), Fisher et al. (1987), Johnson (1987), Fang, Kotz and Ng (1990), Fang and Anderson (1990), Fang and Zhang (1990) as well as Gupta and Varga (1993). A geometric approach to this class of distributions has been developed in Richter (1991) and Richter (1995).
The aim of the present paper is to demonstrate new methods for numerically evaluating probability integrals and quantiles as well as for explicitly approximating large quantiles of noncentral generalized chi-square distributions. The basic idea behind numerically evaluating probability integrals is to use new geometric measure representation formulae. Such formulae will be derived in Section 2 and exploited in Section 3. The basic idea behind explicitly approximating large quantiles of a probability distribution is to exploit suitable theorems on probabilities of large deviations in a suitable way. Namely, one can use asymptotic expansions for large deviations on combining them with standard methods of handling asymptotic solutions of asymptotic equations to derive asymptotic approximations for large quantiles of probability distributions. This general method has been developed recently in Richter (in preparation). It will be used in Sections 4 and 5 for deriving explicit asymptotic quantile approximation formulae for the concrete distributions considered in this paper. A suitable asymptotic expansion for large deviations of

noncentral g-generalized chi-square distributions has been proved recently in Richter and Schumacher (in preparation). It turns out that this new method is of benefit when dealing with quantiles from the central part of the distributions under consideration here, too. The power of the explicit asymptotic quantile approximation method can of course be improved numerically by combining it with the geometric measure representation formulae from Section 2. Several numerical studies in Section 6 compare our related results with results of other authors available in the special case of the classical noncentral chi-square distribution.

Let $\chi_q^2 = \chi_q^2(k, \delta^2; g)$ denote the $100q$-percentage point of the noncentral g-generalized chi-square distribution with k d.f. and n.c.p. δ^2, i.e. the solution of the quantile equation

$$CQ(k, \delta^2; g)\,(\chi_q^2) = q,\ q \in (0,1).$$

Further, let $\chi_{q,N}^2$ be a corresponding solution of the N-th order approximative quantile equation

$$CQ_N(k, \delta^2; g)\,(\chi_{q,N}^2) = q,$$

where $1 - CQ_N(k, \delta^2; g)\,(\chi^2)$ denotes a finite sum including $N+1$ terms of an asymptotic expansion for the large deviation probabilities

$$1 - CQ(k, \delta^2; g)\,(c^2) \text{ as } c \to \infty.$$

It is known from the large deviation theory that under certain conditions the relative approximation error in terms of the quantile order,

$$\frac{|1 - CQ(k, \delta^2; g)\,(\chi_{1-\alpha,N}^2) - \alpha|}{\alpha} = r(\alpha) = r(\alpha; k, \delta^2, g, N)$$

tends to zero when α itself approaches zero. This means that asymptotic quantile approximations become better the larger the quantiles themselves are.

Typical practical applications of this asymptotic theory, however, do not only concern the asymptotic quantile behaviour but also fixed quantiles. Besides large quantiles, fixed quantiles from the central part of the distributions can be dealt with. A suitable method for controlling the number of approximating terms in $1 - CQ_N$ is then to minimize the relative error $r(\alpha; k, \delta^2, g, N)$ with respect to N. For doing this one needs to know the distribution function $CQ(k, \delta^2; g)(x), x \in \mathbf{R}$ at least in a neighborhood of $x = \chi_{1-\alpha}^2$. The geometric measure representation formulae from Section 2 apply again.

We shall prove in Section 2 a new representation formula for the distribution function $CQ(k, \delta^2; g)(x), x \in \mathbf{R}$ for arbitrary $k \geq 2$, $\delta > 0$ and g satisfying assumption (1). This formula will be the starting point as well for our numerical studies in Section 3 concerning the evaluation of the noncentral g-generalized chi-square distribution functions for several concrete spherical sample distributions as for controlling the relative approximation errors of our quantile approximations in Sections 4 and 5. Our geometric representation formula for the noncentral g-generalized chi-square distribution function in Section 2 is new both for the Gaussian case, i.e. for the case when the density generating function is

$$g_G(r) = e^{-\frac{r}{2}},\ r > 0$$

and for the general non Gaussian case. We shall make in Section 3 numerical studies of comparisons between our and other representation, respectively approximation formulae. As a result, it will be seen in Section 3 that the geometric representation formula from Section 2 is comparable with the best of other methods for approximating the distribution functions under consideration and it will be seen in Sections 5 and 6 that the geometric representation formula from Section 2 is comparable with the best of other methods to be used for controlling the relative errors when explicitly approximating the respective quantiles. Note that competing results to both directions of investigation seem to be known from the literature only in the special case $g = g_G$.

2 GEOMETRIC REPRESENTATION FORMULAE FOR NONCENTRAL g-GENERALIZED CHI-SQUARE DISTRIBUTION FUNCTIONS

Let $\Phi(A; g),\ A \in B^k$ denote the probability measure corresponding to the density function $p(x; g),\ x \in \mathbf{R}^k$. Then

$$CQ(k, \delta^2; g)(c^2) = \Phi(A(c); g),\ c > 0$$

where

$$A(c) = \{y\ \in \mathbf{R}^k : \parallel y + \mu^* \parallel^2 < c^2\},\ c > 0.$$

The geometric measure representation formula for elliptically contoured distributions in Richter (1991) says that

$$\Phi(A(c); g) = \frac{1}{I_{k,g}} \int_0^\infty \mathcal{F}(A(c), \nu)\nu^{k-1} g(\nu^2)\, d\nu \tag{3}$$

where $I_{k,g}$ is assumed to satisfy condition (1) and the intersection-percentage function $\mathcal{F}(A, \nu)$, $\nu > 0$ is defined as

$$\mathcal{F}(A, \nu) = U_k(\nu^{-1} A \cap S_k(1)),\ \nu > 0$$

with U_k being the uniform probability distribution on the unit sphere $S_k(1)$. Evaluating $\mathcal{F}(A(c), \nu),\ \nu > 0$, we obtain the main result of this section. We shall make use of the notations

$$\begin{aligned} \alpha^*(\nu) &= \arctan\left(\frac{4\delta^2\nu^2}{(\nu^2 + \delta^2 - c^2)^2} - 1\right)^{\frac{1}{2}} \quad \text{for} \quad \nu \geq 0,\, \nu \neq \sqrt{c^2 - \delta^2}, \\ \alpha^*(\nu) &= \pi/2 \quad \text{for} \quad \nu = \sqrt{c^2 - \delta^2} \end{aligned}$$

and

$$f(\nu) = \frac{\Gamma(\frac{k}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{k-1}{2})} \int_0^{\alpha^*(\nu)} (\sin\alpha)^{k-2} d\alpha.$$

Theorem 1. *If assumption* (1) *is satisfied then*

$$CQ(k,\delta^2;g)(c^2) = I_{k,g}^{-1}\int\limits_0^{\infty} \mathcal{F}(A(c),\nu)\nu^{k-1}g(\nu^2)\,d\nu,\ c>0 \tag{4}$$

with

$$\mathcal{F}(A(c),\nu) = I_{(\delta-c,\delta+c)}(\nu)f(\nu)$$

if $c \le \delta$ *and*

$$\mathcal{F}(A(c),\nu) = \begin{cases} 1 & : 0 \le \nu \le c-\delta, \\ 1-f(\nu) & : c-\delta < \nu < (c^2-\delta^2)^{1/2}, \\ f(\nu) & : (c^2-\delta^2)^{1/2} \le \nu < c+\delta, \\ 0 & : c+\delta \le \nu \end{cases}$$

if $c > \delta$.

Remark 1. *Using a recurrence relation in Zeitler (1996) the integral in the function* $f(\nu)$ *can be simplified for* $k \ge 4$ *:*

$$f(\nu) = \frac{\Gamma(\frac{k}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{k-1}{2})}\left(\left[-\frac{(\sin\alpha)^{k-3}\cos\alpha}{k-2}\right]_0^{\alpha^*(\nu)} + \frac{k-3}{k-2}\int\limits_0^{\alpha^*(\nu)}(\sin\alpha)^{k-4}d\alpha\right).$$

Note that $\mathcal{F}(A(c),\nu) = 0$ for $\nu > \delta + c$ so that the integration in (4) is restricted to the finite interval $(0, c+\delta)$.

Proof of Theorem 1. Without loss of generality, we put $\mu^* = -\mu \mathbb{1}_k,\ \mathbb{1}_k^T = (1,\ldots,1) \in \mathbf{R}^k,\ \mu > 0$. Define the half spaces

$$H_k(e,r) = \{x \in \mathbf{R}^k :\ \Pi_e x = \lambda e,\ \lambda > r\},\ \ r > 0$$

where Π_e denotes the orthogonal projection onto e, $e \in \mathbf{R}^k$. Let the k-dimensional sphere with radius ν and centre $0_k \in \mathbf{R}^k$ be

$$S_k(\nu) = \nu S_k(1),\ r > 0.$$

Case 1: $c \le \delta$*.* Following the notation in Richter (1995) for $\delta - c < \nu < \delta + c$ we define the so called direction type function

$$e_{A(c)}(\nu) = -\mathbb{1}_k$$

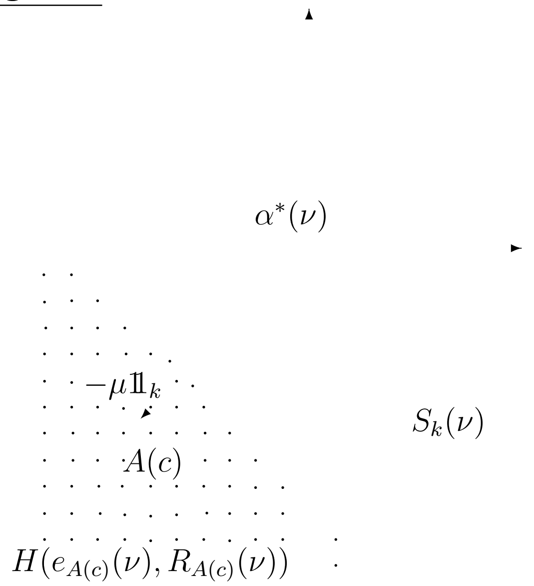
and the so called distance type function

$$R_{A(c)}(\nu) = \frac{|\nu^2+\delta^2-c^2|}{2\delta}.$$

Then, see Figure 1,

$$A(c) \cap S_k(\nu) = H(e_{A(c)}(\nu), R_{A(c)}(\nu)) \cap S_k(\nu). \tag{5}$$

<u>**Figure 1**</u>



Hence, the Borel sets $A(c)$ belong to a system of sets quite similar to that for which it has been proved in Richter (1995) that

$$\mathcal{F}(A(c), \nu) = \frac{\omega_{k-1}}{\omega_k} \int\limits_0^{\alpha^*(\nu)} (\sin \alpha)^{k-2} \, d\alpha,$$

where

$$\begin{aligned} \alpha^*(\nu) &= \arctan \left(\left(\frac{\nu}{R_{A(c)}(\nu)} \right)^2 - 1 \right)^{1/2} \\ &= \arctan \left(\frac{4\delta^2\nu^2}{(\nu^2 + \delta^2 - c^2)^2} - 1 \right)^{1/2}. \end{aligned} \tag{6}$$

Further, we have for the remaining values of ν

$$\mathcal{F}(A(c), \nu) = 0 \text{ if } \{\nu < \delta - c\} \text{ or } \{\nu > \delta + c\}.$$

Case 2: $c > \delta$. We consider the complement of the set $A(c)$, $A(c)^C$, for $c - \delta < \nu < (c^2 - \delta^2)^{1/2}$ and the set $A(c)$ for $(c^2 - \delta^2)^{1/2} < \nu < c + \delta$. We define the direction type functions

$$e_{A(c)^C}(\nu) = 1\!\!1_k \text{ and } e_{A(c)}(\nu) = -1\!\!1_k$$

and the distance type functions

$$R_{A(c)^C}(\nu) = R_{A(c)}(\nu) = \frac{|\nu^2 + \delta^2 - c^2|}{2\delta}.$$

Then it holds

$$A(c)^C \cap S_k(\nu) = H(e_{A(c)^C}(\nu), R_{A(c)^C}(\nu)) \cap S_k(\nu)$$

and

$$A(c) \cap S_k(\nu) = H(e_{A(c)}(\nu), R_{A(c)}(\nu)) \cap S_k(\nu),$$

in the respective intervals for ν. Analogously to Case 1, we have

$$\mathcal{F}(A(c), \nu) = 1 - \mathcal{F}(A(c)^C, \nu) = 1 - \frac{\omega_{k-1}}{\omega_k} \int\limits_0^{\alpha^*(\nu)} (\sin\alpha)^{k-2}\, d\alpha$$

for $c - \delta < \nu < (c^2 - \delta^2)^{1/2}$ and

$$\mathcal{F}(A(c), \nu) = \frac{\omega_{k-1}}{\omega_k} \int_0^{\alpha^*(\nu)} (\sin\alpha)^{k-2}\, d\alpha$$

for $(c^2 - \delta^2)^{1/2} < \nu < c + \delta$ with $\alpha^*(\nu)$ from (6).
Further we have for the remaining values of ν

$$\mathcal{F}(A(c), \nu) = \begin{cases} 1 & : 0 \leq \nu \leq c - \delta, \\ 1/2 & : \nu = (c^2 - \delta^2)^{1/2} \\ 0 & : \nu \geq c + \delta. \end{cases}$$

The assertion of Theorem 1 follows now from (3).

Corollary 1. *The derivation of $CQ(k, \delta^2; g)(c^2)$ with respect to c^2 yields the density function of the noncentral g-generalized chi-square distribution, which coincides with formula (2.4) in Fan (1990):*

$$\psi(c^2) = C(k, g)\, \omega_{k-1} \frac{1}{2\delta} \int_{|c-\delta|}^{c+\delta} \left(1 - \frac{(\nu^2 + \delta^2 - c^2)^2}{4\delta^2\nu^2}\right)^{(k-3)/2} \nu^{k-2} g(\nu^2) d\nu.$$

Corollary 2. *If the k-dimensional sample distribution has a Kotz-type density generator*

$$g_K(r) = r^{M-1} e^{-\beta r^\gamma}, \; r > 0$$

for certain constants $\beta > 0$, $\gamma > 0$, $2M + k > 2$ then

$$CQ(k, \delta^2; g_K)(c^2) = \frac{2\gamma\beta^{\frac{2M+k-2}{2\gamma}}}{\Gamma\left(\frac{2M+k-2}{2\gamma}\right)} \int\limits_0^\infty \mathcal{F}(A(c), \nu) \nu^{2M+k-3} e^{-\beta\nu^{2\gamma}}\, d\nu.$$

If the k-dimensional sample distribution has a Pearson-VII-type density generator

$$g_P(r) = (1 + r/m)^{-M}, \; r > 0$$

for certain constants $M > k/2, m > 0$ then

$$CQ(k,\delta^2;g_P)(c^2) = \frac{2\Gamma(M)}{m^{k/2}\Gamma(M-\frac{k}{2})\Gamma(\frac{k}{2})} \int\limits_0^\infty \mathcal{F}(A(c),\nu) \frac{\nu^{k-1}}{(1+\frac{\nu^2}{m})^M}\, d\nu$$

where $\mathcal{F}(A(c),\nu)$, $\nu > 0$ is both times precisely the same function as in Theorem 1.

Remark 2. *The norming constants which correspond to the density generating functions g_G, g_K and g_P are*

$$\begin{aligned} C_G(k,g) &= (2\pi)^{-k/2}, \\ C_K(k,g) &= \frac{\gamma\beta^{\frac{2M+k-2}{2\gamma}}\Gamma(k/2)}{\pi^{k/2}\Gamma\left(\frac{2M+k-2}{2\gamma}\right)} \end{aligned}$$

and

$$C_P(k,g) = \frac{\Gamma(M)}{(\pi m)^{k/2}\Gamma(M-k/2)},$$

respectively.

Examples:

1. The Kotz-type density generator g_K with $M = 1, \beta = 0.5, \gamma = 1$ generates the standard Gaussian law.

2. The Pearson-VII-type density generator g_P with $M = (k+m)/2$ generates the k-dimensional Student distribution with m degrees of freedom which is a k-dimensional Cauchy distribution if $m = 1$.

Results concerning the behaviour of statistics which were well studied for the multivariate Gaussian sample distribution can be interpreted as robustness type results as far as, e.g., error probabilities do not change rapidly when the underlying sample distribution changes within a certain subclass of the class of elliptically contoured distributions. On the other side, such results can be interpreted as sensitivity type results as far as one is interested in, e.g., how strong error probabilities for common statistical decisions change when the density generating function of the sample distribution changes in a well defined way within the class of admissible density generators. As the examples show, both heavy tails and light tails are allowed for the sample distribution to occur. Such robustness and sensitivity type problems can be considered as special questions of model correctness.

It is well known that in many statistical problems such as significance tests or confidence estimations basic quantities like critical test values or confidence interval limits are taken from or constructed with the help of respective chi-square distributions if the underlying sample distribution is a Gaussian one. Corresponding power functions are based upon the respective noncentral chi-square distributions. The usual assumption however, that the underlying multivariate sample distribution is a Gaussian one is not always satisfied and can often be substituted by the assumption that the multivariate sample vector follows an elliptically contoured distribution or a more special spherical distribution. One needs then to consider central and noncentral g-generalized chi-square distributions. This motivates the following applications of the above results to constructing critical points and power functions for expectation test in elliptically contoured distributions and to certain problems of model correctness.

Application 1 (Inference for the mean in elliptically contoured distributions)

Consider a k dimensional feature vector $X = (X_1, \ldots, X_k)^T$ and let us be given a sample of size n of identically distributed vectors $X_i = (X_{1i}, \ldots, X_{ki})^T$, $i = 1, \ldots, n$. Assume that

$$X_{(nk)} = (X_{11}, \ldots, X_{k1}, \ \ldots \ , X_{1n}, \ldots, X_{kn})^T = \mu_{(nk)} + \mathcal{E}_{(nk)}$$

with $\mu_{(nk)} = 1\!\!1_n \otimes \mu$ and $\mu = (\mu_1, \ldots, \mu_k)^T$ satisfies $\sigma^{-1}\mathcal{E}_{(nk)} \sim \Phi(.\,; g_{nk})$ for some $\sigma > 0$. Note that both the coordinates of the vectors X_i and the vectors $X_1, \ldots, X_n$ are uncorrelated but not necessarily independent. Without loss of generality assume $\sigma^2 = 1$.
The arithmetic mean of the sample $X_1, \ldots, X_n$ can be written as

$$\overline{X}_n = B^T X_{(nk)} \quad \text{where} \quad B^T = \frac{1}{n}\ 1\!\!1_n^T \otimes I_k\ .$$

Using Theorem 2.16 in Fang, Kotz, Ng (1990) and $B^T\mu_{(nk)} = \mu$, $B^T B = \frac{1}{n} I_k$ one gets for $X_{(k)} = \sqrt{n}(\overline{X}_n - \mu)$:

$$X_{(k)} \sim \Phi(.\,; g_{nk,k})$$

where due to formula (2.23) in Fang, Kotz and Ng (1990)

$$g_{nk,k}(u) = \frac{\pi^{k(n-1)/2}}{\Gamma(k(n-1)/2)}\, C(nk, g_{nk}) \int_u^\infty (y-u)^{(nk-k)/2-1}\, g_{nk}(y)\, dy,\ u \geq 0$$

is a density generating function satisfying

$$\int_{\mathbf{R^k}} g_{nk,k}(\|z\|^2)\, dz = 1.$$

Remark 3. *Recognize that the marginals of spherically symmetric distributed vectors have distributions which depend on the sample size and that these marginals are not independent except for the Gaussian case. For example, in the case of the Pearson-VII-type density generator $g_{nk} = g_P$, the marginal distribution depends on the sample size as follows:*

$$g_{nk,k}(u) = \frac{\Gamma(N - k(n-1)/2)}{(\pi m)^{k/2}\Gamma(N - nk/2)}\, (1 + u/m)^{-N+k(n-1)/2},$$

whereas with $g_{nk} = g_G$ it holds

$$g_{nk,k}(u) = (2\pi)^{-k/2}\, e^{-u/2}.$$

For testing the hypothesis $H_0 : \mu = \mu_{0,k}$ versus $H_A : \mu \neq \mu_{0,k}$ we use the test statistic

$$T^2 = n\|\overline{X}_n - \mu_{0,k}\|^2.$$

If H_0 is true then

$$T^2 = \|X_{(k)}\|^2 \sim CQ(k, 0; g_{nk,k})$$

and hence we reject H_0 with first kind error probability α whenever it holds

$$n\|\overline{x}_n - \mu_{0,k}\|^2 > \chi^2_{1-\alpha}(k, 0; g_{nk,k}).$$

If $H_A : \mu = \mu_{1,k}$ is true then

$$T^2 = \|\sqrt{n}(\overline{X}_n - \mu_{1,k}) + \sqrt{n}(\mu_{1,k} - \mu_{0,k})\|^2 \sim CQ(k, n\|\mu_{1,k} - \mu_{0,k}\|^2; g_{nk,k})$$

such that the power function of the test is

$$m(\mu_{1,k}) = P_{\mu_{1,k}}(\text{reject } H_0) = 1 - CQ(k, n\|\mu_{1,k} - \mu_{0,k}\|^2; g_{nk,k})(\chi^2_{1-\alpha}(k, 0; g_{nk,k})),$$

$\mu_{1,k} \in \mathbf{R^k}\backslash\{\mu_{\mathbf{0,k}}\}$.

Example: Let $k = 2l + 1$, $l \in \mathbf{N}$, $\mu_{0,k} = (\mu_0, \ldots, \mu_0)^T \in \mathbf{R^k}$ and consider the umbrella alternative

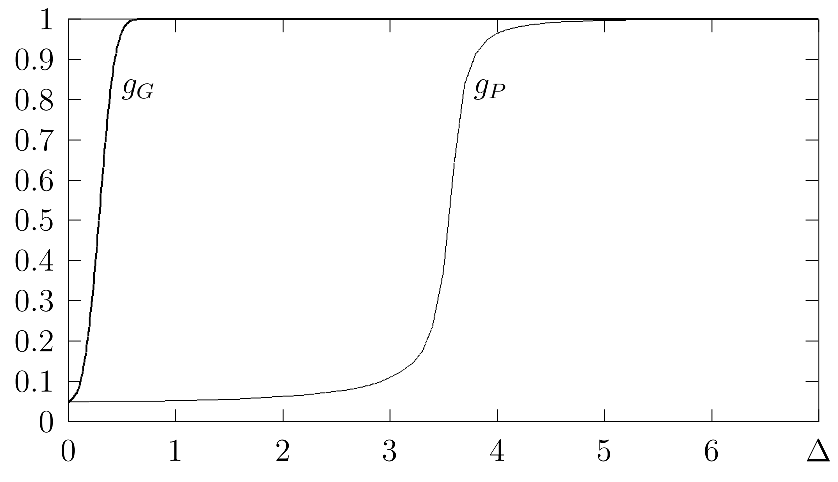
$$H_A : \mu_{1,k} = \mu_{1,k}(\Delta) = (\mu_0, \mu_0 + \Delta, \ldots, \mu_0 + l\Delta, \mu_0 + (l-1)\Delta, \ldots, \mu_0)^T,\ \Delta \in \mathbf{R}.$$

Then

$$\|\mu_{1,k} - \mu_{0,k}\|^2 = \Delta^2 \left(l(l-1)(2l-1)/3 + l^2\right)$$

and the power function $m(\mu_{1,k}(\Delta)) =: \widetilde{m}(\Delta; g_{nk})$ depends on the alternative only through Δ.
For a comparison of $\widetilde{m}(\Delta; g_{nk})$ in the cases of Gaussian and Pearson VII density generating functions g_{nk}, g.e. see Figure 2.

Figure 2 $\widetilde{m}(\Delta; g_{nk})$ for $g_{nk} = g_G$ and for $g_{nk} = g_P$ with $n = 15$, $k = 5$, $m = 1$, $M = 38$, $\chi^2_{0.95}(5, 0; g_{15,5}) = 1150.79$



Application 2 (Sensitivity and robustness)

Let us be given a data vector $(y_1, \ldots, y_k)^T = y_{(k)}$ and let us assume that the data have been generated by a random vector $Y_{(k)} = \mu_{0,k} + X_{(k)}$ where $X_{(k)}$ is distributed according to the standard Gaussian law $\Phi(\cdot; g_G)$. Alternatively, keep in mind that the expectation

$\mathbb{E}Y_{(k)}$ of the sample vector $Y_{(k)}$ could be another vector than $\mu_{0,k}$ and the distribution law of $X_{(k)}$ could be a spherical one with a density generating function g different from g_G and a form matrix which will be assumed for simplicity to be the unit matrix I_k. This means that we allow $X_{(k)}$ to be alternatively distributed according to $\Phi(\cdot; g)$ with $g \neq g_G$. Because we have not assumed that $\mathbb{E}Y_{(k)}$ takes values in a certain linear subspace of the sample space $\mathbf{R}^k$, deviations of $\mathbb{E}Y_{(k)}$ from $\mu_{0,k}$ are not preferred to act into certain prespecified directions from the unit sphere $S_k(1)$. A natural way to check the hypothesis $H_0 : \mathbb{E}Y_{(k)} = \mu_{0,k}$ is therefore to reject it whenever it holds $\|y_{(k)} - \mu_{0,k}\|^2 > \chi^2_{1-\alpha}(k, 0; g_G)$ for a certain given $\alpha \in (0, 1)$. One can think now about several types of sensitivity or of robustness statements concerning this decision. A first type robustness problem deals with the question in which way the density generating function g can deviate from g_G such that the probability of rejecting $H_{0,k}$ does not change more than $\alpha/5$. For solving such a problem one has to calculate quantities of the type

$$\Phi\left(\left\{x_{(k)} \in \mathbf{R}^k : \|x_{(k)}\|^2 > \chi^2_{1-\alpha}(k, 0; g_G)\right\}; g\right) = 1 - CQ(k, 0; g)(\chi^2_{1-\alpha}(k, 0; g_G))$$

for $g \neq g_G$. A second type robustness problem deals with the question how much g can deviate from g_G such that the probability of not rejecting $H_{0,k}$ does not change more than 20 percent of the corresponding value when $g = g_G$ and $\mathbb{E}Y_{(k)} = \mu_{1,k} \neq \mu_{0,k}$. For solving such a problem one has to calculate quantities of the type

$$\begin{aligned} &\Phi\left(\left\{x_{(k)} \in \mathbf{R}^k : \|x_{(k)} + \mu_{1,k} - \mu_{0,k}\|^2 \leq \chi^2_{1-\alpha}(k, 0; g_G)\right\}; g\right) \\ &\qquad = CQ(k; \|\mu_{1,k} - \mu_{0,k}\|^2; g)(\chi^2_{1-\alpha}(k, 0; g_G)). \end{aligned}$$

Remark 4. *The density generating functions g and g_G in the last two relations can be replaced in more general situations by two arbitrary density generating functions.*

3 TABLES OF PROBABILITY INTEGRALS FOR NONCENTRAL g-GENERALIZED CHI-SQUARE DISTRIBUTIONS

Various kinds of algorithms for producing tables of the usual noncentral chi-square distribution functions are available from the literature. No related tables and algorithms are known to us, however, for the general case of a noncentral g-generalized chi-square distribution function when g does not coincide with the density generating function g_G of the Gaussian sample distribution. Based upon the geometric representation formula in Theorem 1, we are in a position to present now a new unified approach to establishing probability integrals for all noncentral g-generalized chi-square distributions with arbitrary density generating function g satisfying assumption (1).

Our algorithm is mainly divided into two steps. The first one realizes the evaluation of the intersection-percentage function $\mathcal{F}(A(c), \nu)$, $\nu > 0$ for arbitrary $c > 0$. To this end, the representation formula for $\mathcal{F}$ from Theorem 1 has been implemented in a Borland Pascal program using the recursion relation for the function $f(\nu)$, $\nu > 0$, stated in Remark 1.

In the second step of our algorithm we use a standard numerical integration method for evaluating the integrals from the geometric representation formula for $CQ(k, \delta^2; g)(c^2)$ in

Theorem 1. To be more specific, the integration is performed by Simpson's rule with 100000 steps.
Computations for realizing our algorithm have been done on a Hewlett Packard Pentium computer.
Our numerical results following from Theorem 1 for the "ordinary" case $g = g_G$ will be compared in Table 1 with the four-digit exact and the Incomplete Gamma function approximative results in Patnaik (1949), see columns Pat.-ex. and Pat.-appr., respectively, with the Cornish-Fisher type approximation results in Abdel-Aty (1954), column Abd.-Aty, and with the approximation results using infinite series of Poissonian weighted central chi-square distributions in Ashour and Abdel-Samad (1990), Ash. & Abd.-Sam. Note that the upper 5% significance points for $k = 1(1)7$ and $\delta = 0(0.2)5.0$ have been first published in an implicit form by Fisher (1928).

Table 1 $CQ(k, \delta^2; g_G)(c^2)$-values

k	δ^2	c^2	Pat.-ex.	Pat.-appr.	Abd.-Aty	Ash.&Abd.-Sam.	Theorem 1
4	4	1.765	0.0500	0.0399	-	0.04999937329	0.04999937471
	4	10.000	0.7118	0.7191	0.7123	0.7117928156	0.71179281648
	4	17.309	0.9500	0.9492	-	0.9499957033	0.94999570938
	4	24.000	0.9925	0.9913	0.9925	0.9924603701	0.99246037447
	10	10.000	0.3148	0.3178	-	0.3148206466	0.31482065003
7	1	4.000	0.1628	0.1621	-	0.1628330056	0.16283300701
	1	16.004	0.9500	0.9499	-	0.9500015423	0.95000154258
	16	10.257	0.0500	0.0430	-	0.04999417662	0.04999418181
	16	24.000	0.5898	0.5947	0.5894	-	0.58633683948
	16	38.970	0.9500	0.9482	0.9500	0.9499992082	0.94999921449
12	6	24.000	0.8174	0.8187	-	0.8173526185	0.81735262555
	18	24.000	0.2901	0.2936	-	-	0.29004949596
16	8	30.000	0.7880	0.7895	-	0.788001461	0.78800147228
	8	40.000	0.9632	0.9626	0.9632	0.9632254713	0.96322547499
	32	30.000	0.0609	0.0590	-	-	0.06284204877
	32	60.000	0.8316	0.8329	-	0.8315634526	0.83156347739
24	24	36.000	0.1567	0.1556	-	0.1567110344	0.15671106200
	24	48.000	0.5296	0.5333	-	0.5296283918	0.52962840920
	24	72.000	0.9667	0.9656	-	0.9666953909	0.96669542296

For easy comparison, the parameters k, δ^2 and c^2 in Table 1 are chosen as in Patnaik (1949) and in the other papers of our comparison study. Let us remark, that at least 6 digits in the columns Ash. & Abd.-Sam. and Theorem 1 coincide and that the first four digits of these columns (after rounding) coincide with those of the column Pat.-ex. Exact results with more then four digits can also be obtained in the single case $g = g_G$ with the method of continued fractions as in the recent paper of Wang and Kennedy (1994). Respective intervals derived there are given in Table 2.

Table 2 Comparison with Wang and Kennedy (1994)

k	δ^2	c^2	Wang & Kennedy		Theorem 1
5	1	9.23636	0.8272918751175547	0.8272918751175549	<u>0.827291875117554810</u>
11	21	24.72497	0.2539481822183125	0.2539481822183127	<u>0.253948182218312659</u>
31	6	44.98534	0.81251987850649685	0.81251987850649695	0.812519878506496969
51	1	38.56038	0.08519497361859116	0.08519497361859120	<u>0.085194973618591190</u>
100	16	82.35814	0.01184348822747822	0.01184348822747826	<u>0.011843488227478234</u>
300	16	331.78852	0.73559567103067085	0.73559567103067095	0.735595671030670804
500	21	459.92612	0.02797023600800058	0.02797023600800062	0.027970236008000565

In the Maple V package one has the possibility to determine the number of floating point digits used for calculations. For the comparison with the results from Wang and Kennedy we decided therefore to use the Maple package with 60 digits instead of Borland Pascal to carry out the same numerical integration algorithm with 60000 steps. Note, however, that the computations become very time consuming when k or the number of steps increase. The results in Table 2 following from Theorem 1 coincide at least in 16 digits with the midpoint-approximations of Wang and Kennedy. In the four underlined cases the results of Theorem 1 belong to the intervals derived by Wang and Kennedy. By using the Borland Pascal programme with 100000 steps our results coincided with the values from Wang and Kennedy at least in 13 digits.

Let us turn now to the non Gaussian case, i.e. to the case $g \neq g_G$. The first step of our algorithm is not influenced in any way by the special choice of the density generating function g because in its first step our algorithm exploits only the pure geometric properties of the underlying spherical sampling distribution. Table 3 contains some typical values of the intersection-percentage function $\mathcal{F}(A(c), \nu)$ which have been used for calculating the values of the column Theorem 1 in Table 1.

Table 3 Intersection percentage function $\mathcal{F}(A(c), \nu)$

ν	$k = 7$, $\delta^2 = 16$ $c^2 = 10.257$	$k = 4$, $\delta^2 = 4$ $c^2 = 24$
0.00	0.000000000	1.000000000
0.80	0.000000000	1.000000000
1.00	0.004295470	1.000000000
1.40	0.029695285	1.000000000
1.80	0.049180979	1.000000000
2.00	0.054545474	1.000000000
2.40	0.058260564	1.000000000
2.80	0.055497901	1.000000000
3.00	0.052578892	0.985743021
3.40	0.044979398	0.868003984
3.80	0.036295444	0.727566204
4.00	0.031902881	0.657481179
4.40	0.023519620	0.523144707
4.80	0.016131808	0.399624626
5.00	0.012928522	0.342518821
5.40	0.007629064	0.238351701
5.80	0.003847733	0.148559576
6.00	0.002505124	0.109551019
6.40	0.000792158	0.045034599
6.80	0.000106013	0.003958628
7.00	0.000013900	0.000000000
7.20	0.000000000	0.000000000

Notice that exactly the same values of the intersection-percentage function $\mathcal{F}$ as in Table 3 are to be used for evaluating probability integrals of noncentral g-generalized chi-square distributions in all cases when the geometric representation formula from Theorem 1 applies, i.e. when the sample distribution has a density generating function g satisfying assumption (1).
In the second step of our algorithm we integrate the weighted intersection-percentage function $\mathcal{F}$ in the sense of Theorem 1. The answer to the question whether the underlying sample distribution has light or heavy tails becomes interesting to a certain extent in this second step of our algorithm. While the first step reflects a certain invariance or robustness property of our method with respect to changes of the sample distribution within a certain subclass of elliptically contoured distributions, the second step takes into account a certain necessarily existing sensitivity of the method with respect to these changes.

Application 2 (continued)

Table 4 deals with a comparison of results influenced by heavy and light tails of the underlying sampling distribution, respectively. Column g_K summarizes probability integrals $CQ(k, \delta^2; g_K)(c^2)$ for the case that the sampling distribution has the considerably light tail density generating function of Kotz-type from Corollary 2 with parameters $M = 1$,

$\gamma = 2$ and $\beta = 1$, i.e.

$$g_K(r) = e^{-r^2}, r > 0.$$

Column g_P summarizes probability integrals $CQ(k, \delta^2; g_P)(c^2)$ for the case that the underlying sampling distribution has the considerably heavy tails Pearson-VII-type with parameters $M = (k+1)/2$ and $m = 1$, i.e.

$$g_P(r) = (1+r)^{-(k+1)/2}, r > 0.$$

The last column in Table 4 coincides with that of Table 1.

Table 4 Sensitivity study

k	δ^2	c^2	g_K	g_P	g_G
4	4	1.765	0.034458	0.025625	0.04999937471
	4	10.000	0.993725	0.452918	0.71179281648
	4	17.309	1.000000	0.605000	0.94999570938
	4	24.000	1.000000	0.672473	0.99246037447
	10	10.000	0.409150	0.208772	0.31482065003
7	1	4.000	0.953115	0.178667	0.16283300701
	1	16.004	1.000000	0.515594	0.95000154258
	16	10.257	0.011781	0.024662	0.04999418181
	16	24.000	0.974122	0.379660	0.58633683948
	16	38.970	1.000000	0.587635	0.94999921449
12	6	24.000	1.000000	0.428322	0.81735262555
	18	24.000	0.907495	0.214391	0.29004949596
16	8	30.000	1.000000	0.405108	0.78800147228
	8	40.000	1.000000	0.488109	0.96322547499
	32	30.000	0.164262	0.044069	0.06284204877
	32	60.000	1.000000	0.455225	0.83156347739
24	24	36.000	0.998728	0.184009	0.15671106200
	24	48.000	1.000000	0.327561	0.52962840920
	24	72.000	1.000000	0.484215	0.96669542296

Recall that g_K and g_P generate distributions with lighter and heavier tails than g_G, respectively. One might expect therefore to find a certain monotony between the g_K-, g_P- and g_G-values of an arbitrary row in Table 4 which, however, in fact cannot be detected there. An explanation of this circumstance becomes obvious when studying the influence of the noncentrality parameter δ^2 onto the distributions under consideration. The graphs of the respective densities are drawn in Figures 3 up to 5 for selected cases.

Figure 3 $CQ(4,4;g)$-densities

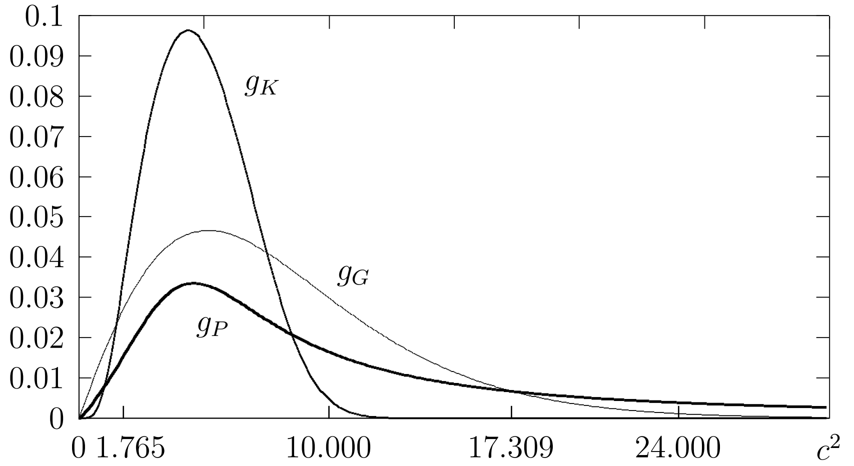


Figure 4 $CQ(4,10;g)$-densities

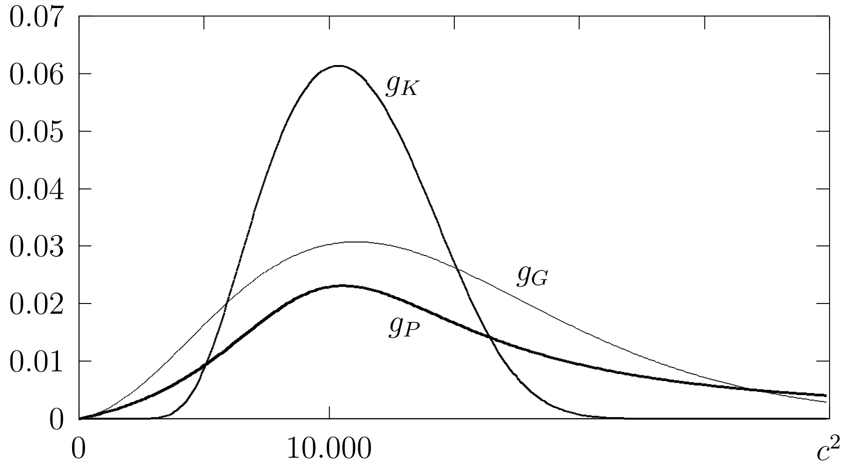
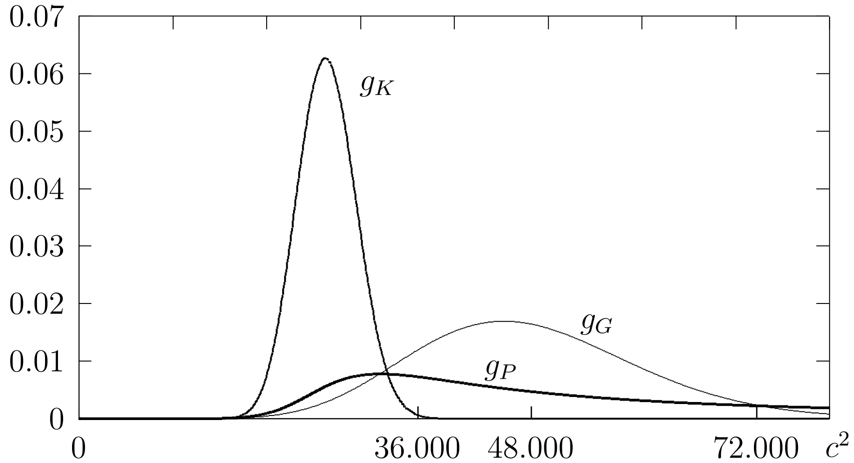


Figure 5 $CQ(24,24;g)$-densities



As an example for analyzing a first type robustness problem we change in Table 5a in each row one of the three parameters of the Kotz-type distribution as long as it holds

$$|1 - CQ(4,0;g_K)(\chi^2_{0.95}(4,0;g_G)) - 0.05| \leq 0.05/5$$

where $\chi^2_{0.95}(4,0;g_G) = 0.7107235$.

Table 5a First type robustness study

N	β	γ	$1 - CQ(4,0;g_K)(0.7107235)$
1.110	0.5	1	0.04004512
0.908	0.5	1	0.05999714
1	0.442	1	0.04013002
1	0.554	1	0.05988153
1	0.5	0.940	0.04015277
1	0.5	1.057	0.05991291

For analyzing a second type robustness problem let $\|\mu_{1,k} - \mu_{0,k}\|^2 = 2$. Note that $CQ(4,2;g_G)(\chi^2_{0.95}(4,0;g_G)) = 0.02060116$. Results of changing the parameters of the Kotz-type distribution as long as it holds

$$|CQ(4,2;g_K)(\chi^2_{0.95}(4,0;g_G)) - 0.02060116| \leq 0.02060116/5$$

are given in the next table.

Table 5b Second type robustness study

N	β	γ	$CQ(4,2;g_K)(0.7107235)$
1.376	0.5	1	0.01648713
0.569	0.5	1	0.02471864
1	0.406	1	0.01648303
1	0.611	1	0.02470913
1	0.5	0.920	0.01648835
1	0.5	1.087	0.02472069

The results of a computational experience in Table 6 make the practical stability of our computer program evident as δ approaches zero. Note that the values for $\delta = 0$ were computed by means of the geometric representation formula for central g-generalized chi-square distributions as given in Richter (1991):

$$CQ(k,0;g)(c^2) = I^{-1}_{k,g} \int_0^c \nu^{k-1} g(\nu^2)\, d\nu.$$

Table 6 Stability study: small noncentrality parameters

δ^2	$k = 15$ $c^2 = 25$	$k = 20$ $c^2 = 31.41$	$k = 4$ $c^2 = 13.28$	$k = 10$ $c^2 = 23.21$	$k = 25$ $c^2 = 44.31$
1.0	0.925056	0.928948	0.971126	0.980030	0.984613
0.5	0.938208	0.939980	0.981702	0.985522	0.987501
0.1	0.947792	0.948073	0.988530	0.989184	0.989522
0.09	0.948021	0.948267	0.988682	0.989267	0.989570
0.08	0.948249	0.948461	0.988834	0.989351	0.989617
0.07	0.948477	0.948654	0.988984	0.989433	0.989664
0.06	0.948704	0.948847	0.989134	0.989516	0.989711
0.05	0.948931	0.949039	0.989283	0.989598	0.989758
0.04	0.949157	0.949231	0.989431	0.989680	0.989804
0.03	0.949383	0.949423	0.989578	0.989761	0.989851
0.02	0.949608	0.949614	0.989724	0.989842	0.989897
0.01	0.949832	0.949804	0.989870	0.989922	0.989943
0.001	0.950034	0.949976	0.990000	0.989995	0.989985
0.0001	0.950054	0.949993	0.990013	0.990002	0.989989
0.00001	0.950056	0.949995	0.990014	0.990003	0.989989
0.0	0.950056	0.949995	0.990014	0.990003	0.989989

4 LARGE DEVIATION APPROACH TO ASYMPTOTIC APPROXIMATIONS FOR LARGE QUANTILES

In Sections 2 and 3, we have presented a geometric approach to exact evaluating the distribution functions $CQ(k, \delta^2; g)$. Combining this approach with standard numerical methods like the secant method we can obtain quite satisfactory numerical approximations for the quantiles of these distributions. Besides these we are interested in explicit approximation formulae for the quantiles of interest. Such formulae yield in addition starting values for determining quantiles based upon our geometric measure representation formula in Section 2. Furthermore we are interested in estimations for the occurring approximation errors.

The type of explicit quantile approximations studied here is based upon exploiting an asymptotic expansion for large deviations. It turns out that large quantiles can be approximated in this way especially precisely and it is of some additional interest to know the properties of the method developed for quantile approximation in the tails when approximating quantiles from the central part of the distributions. To determine approximation errors when using explicit approximation formulae we use exact numerical approximations based upon the results of Sections 2 and 3. Notice that we use these exact numerical quantile approximations for both large and small quantiles and give tables for both types of quantiles.

In the theory of large deviations, amongst other things, one tries to describe quite precisely the asymptotic behaviour of tail probabilities of more or less complicated distributions when the argument of the distribution function approaches infinity. Assume that one could determine an explicitly known function

$$f = f_{k,\delta^2;g}|\mathbf{R}^+ \to [0,1]$$

such that

$$\frac{1 - CQ(k,\delta^2;g)(c^2)}{f_{k,\delta^2;g}(c^2)} \to 1 \text{ as } c \to \infty. \tag{7}$$

If the function $f_{k,\delta^2;g}$ would have an appropriate structure then one could hope to get a suitable asymptotic approximation for the quantile χ^2_q by dealing with χ^2_q as the asymptotic solution of the asymptotic equation

$$f_{k,\delta^2;g}(\chi^2_q) = 1 - q \text{ as } q \to 1.$$

For finding such a function $f_{k,\delta^2;g}$ we shall restrict our attention to the special class of sample distributions which are generated by the Kotz-type density generating function

$$g_K(r) = r^{M-1} e^{-\beta r^\gamma}, r > 0$$

with $\gamma > 0.5, \beta > 0$ and $2M + k > 2$. Exploiting a large deviation type geometric measure representation formula for $1 - CQ(k,\delta^2;g)(c^2)$ which can be derived from Theorem 1 and modifying standard asymptotic Laplace integral technique it has been shown in Richter and Schumacher (in preparation) that one can take

$$f_{k,\delta^2;g}(c^2) = \frac{\beta^{\frac{k}{2\gamma}+\frac{M-1}{\gamma}-\frac{k+1}{2}}\Gamma\left(\frac{k}{2}\right)}{2\sqrt{\pi}\gamma^{\frac{k-1}{2}}\delta^{\frac{k-1}{2}}\Gamma\left(\frac{k}{2\gamma}+\frac{M-1}{\gamma}\right)} c^{\frac{3k-1}{2}-\gamma(k+1)+2M-2} e^{-\beta(c-\delta)^{2\gamma}}. \tag{8}$$

Moreover, it has been proved there that the following asymptotic expansion formula for large deviations is a refinement of the asymptotic relation (7):

$$\begin{aligned} 1 - CQ(k,\delta^2;g)(c^2) = f_{k,\delta^2;g}(c^2)\Big[&1 + D_{1-2\gamma}c^{1-2\gamma} + D_{-2\gamma}c^{-2\gamma} \\ + D_{-1-2\gamma}c^{-1-2\gamma} + D_{2-4\gamma}c^{2-4\gamma} &+ D_{1-4\gamma}c^{1-4\gamma} + D_{-4\gamma}c^{-4\gamma} + D_{-1-4\gamma}c^{-1-4\gamma} \\ &+ D_{-2-4\gamma}c^{-2-4\gamma} + O(c^{3-6\gamma})\Big], \text{ as } \quad c \to \infty. \end{aligned} \tag{9}$$

For the asymptotic expansion coefficients D_i from (9) we refer to Appendix A.
This asymptotic expansion formula for large deviations is our starting point for first specifying the function $CQ_N(k,\delta^2;g)(.)$ occurring in the above mentioned N-th order approximative quantile equation (see Section 1) in such a way that

$$1 - CQ_N(k,\delta^2;g)(c^2) = f_{k,\delta^2;g}(c^2)\left[1 + \sum_{l=1}^{N} A_l(c)\right] \tag{10}$$

and second dealing with $\chi^2_{q,N}(k,\delta^2;g)$ as the asymptotic solution of the asymptotic equation

$$1 - CQ_N(k,\delta^2;g)(\chi^2_{q,N}(k,\delta^2;g)) = 1 - q \text{ as } q \to 1.$$

Here $A_l(c)$ denotes the l-th greatest term with respect to the power order of c, arising after the number one in the parentheses in formula (9). In the case $N = 0$ we put $\sum_{l=1}^{N} A_l(c) = 0$. In accordance with the choice of the parameter γ in the Kotz-type density generating function g_K we obtain different N-th order approximative quantile equations. We give some examples to illustrate this fact.

Examples:

1. For the Gaussian case $g_K = g_G$, i.e. for $\gamma = 1$, $M = 1$ and $\beta = 0.5$, it holds

$$1 - CQ_N(k, \delta^2; g_G)(c^2) = f_{k,\delta^2;g_G}(c^2)\Big[1+D_{1-2\gamma}c^{1-2\gamma} + D_{-2\gamma}c^{-2\gamma} + D_{2-4\gamma}c^{2-4\gamma} + O(c^{3-6\gamma})\Big]. \tag{11}$$

2. For the non Gaussian case of the Kotz-type density generating function g_K with the parameters $\gamma = 0.7$, $M = 1$ and $\beta = 0.5$ it holds

$$1 - CQ_N(k, \delta^2; g_K)(c^2) = f_{k,\delta^2;g_K}(c^2)\Big[1 + D_{1-2\gamma}c^{1-2\gamma}+D_{2-4\gamma}c^{2-4\gamma} + O(c^{3-6\gamma})\Big]. \tag{12}$$

3. For the non Gaussian case of the Kotz-type density generating function g_K with the parameters $\gamma = 1.5$, $M = 1$ and $\beta = 0.5$ it holds

$$1 - CQ_N(k, \delta^2; g_K)(c^2) =f_{k,\delta^2;g_K}(c^2)\Big[1 + D_{1-2\gamma}c^{1-2\gamma} + D_{-2\gamma}c^{-2\gamma} + D_{-1-2\gamma}c^{-1-2\gamma} + D_{2-4\gamma}c^{2-4\gamma} + D_{1-4\gamma}c^{1-4\gamma} + O(c^{3-6\gamma})\Big]. \tag{13}$$

Recall that a general approach to quantile approximations using asymptotic expansions for large deviations has been discussed in Richter (in preparation). In accordance with the approach developed there we pass over now from the N-th order approximative quantile equation to the following Large Deviation Iteration Procedure:
Define C_{n+1} from C_n by

$$\alpha = a_0 C_n^{\frac{3k-1}{2}-\gamma(k+1)+2M-2} e^{-\beta(C_{n+1}-\delta)^{2\gamma}} \left[1 + A_1(C_n) + \ldots + A_N(C_n)\right] \tag{14}$$

for $n = 0, 1, 2, \ldots$, with the starting value

$$C_0 = \delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right) - \frac{\theta}{2\gamma\beta}\ln\beta\right)^{\frac{1}{2\gamma}} \tag{15}$$

and the constants

$$a_0 = \frac{\beta^{\frac{k}{2\gamma}+\frac{M-1}{\gamma}-\frac{k+1}{2}}\Gamma\left(\frac{k}{2}\right)}{2\sqrt{\pi}\gamma^{\frac{k-1}{2}}\delta^{\frac{k-1}{2}}\Gamma\left(\frac{k}{2\gamma}+\frac{M-1}{\gamma}\right)}, \qquad \theta = \frac{3k-1}{2} - \gamma(k+1) + 2M - 2. \tag{16}$$

For $\theta \geq 0$ the starting value C_0 is defined for $\alpha \in (0, \min(a_0 e^{-x_1}, 1))$, where x_1 denotes the uniquely determined solution of the equation

$$\frac{\theta}{2\gamma} \ln \frac{x}{\beta} = -x \; .$$

For $\theta < 0$ we have to distinguish between two cases. If $1 - \ln(-\theta/(2\gamma\beta)) \geq 0$ then the starting value C_0 is defined for $\alpha \in (0, \min(a_0, 1))$. In the alternative case the starting value C_0 is defined for $\alpha \in [(0, a_0 e^{-x_2}) \cup (a_0 e^{-x_1}, a_0)] \cap (0, 1)$, where $x_1 < x_2$ are the uniquely determined solutions of the above mentioned equation.
The following Lemma 1 reflects our motivation for choosing the starting value C_0 as it has been just defined.

Lemma 1. *If $x \geq 1$ is a solution of the equation*

$$a_0 x^\theta e^{-\beta(x-\delta)^{2\gamma}} = \alpha$$

then for sufficiently small $\alpha > 0$ it holds

$$\delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right) - \frac{\theta}{2\gamma\beta}\ln\beta + \frac{\theta}{\beta}\ln\left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right)\right)^{\frac{1}{2\gamma}} \leq x \leq C_0^*$$

with

$$\begin{aligned} C_0^* = \delta + \Bigg(&-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right) - \frac{\theta}{2\gamma\beta}\ln\beta \\ &+ \frac{|\theta|^2}{(2\beta\gamma)^2}\frac{\ln(-\ln\frac{\alpha}{a_0})}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)} + \frac{|\theta|\theta(1)}{\beta(\ln\frac{\alpha}{a_0})^{\frac{1}{2\gamma}}}\Bigg)^{\frac{1}{2\gamma}}, \end{aligned} \tag{17}$$

where $\theta(1)$ denotes a certain positive constant.

For the proof of Lemma 1 see Appendix B.

Remark 5. *Obviously, $C_0^* \sim C_0$ as $\alpha \to 0$.*

Let us consider now the function

$$\phi(c) = \delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{\beta}\ln c + \frac{1}{\beta}\ln\left[1 + A_1(c) + \ldots + A_N(c)\right]\right)^{\frac{1}{2\gamma}} \tag{18}$$

for $c > 1$. The iteration procedure can be equivalently defined then by

$$C_{n+1} = \phi(C_n), \; n = 0, 1, 2, \ldots$$

and taking the starting value C_0 as above. From the fixed point theory it follows that the sequence (C_n) will converge to $\chi^2_{q,N}(k, \delta^2; g)$ if we approximate sufficiently large quantiles $\chi^2_q(k, \delta^2; g)$, because it then holds $|\phi'(c)| < 1$.

5 TABLES OF LARGE QUANTILES

The iteration procedure $C_{n+1} = \phi(C_n)$ with ϕ from (18) was used to implement a Borland Pascal Program for the evaluation of the N-th order approximative quantiles $\chi^2_{q,N}(k,\delta^2;g)$ of the noncentral g-generalized chi-square distribution with Kotz-type density generating function $g = g_K$. The iteration algorithm stops as soon as for the relative error

$$\varepsilon_n = \frac{|C_{n+1} - C_n|}{C_{n+1}}$$

it holds $\varepsilon_n \leq 10^{-8}$.
To check the accuracy of our numerical result additionally in terms of quantile orders we calculate the c.d.f. $CQ(k,\delta;g)(c^2)$ at the point $c^2 = C^2_{n+1}$ and the relative error for the quantile orders,

$$r_n(\alpha) = \frac{|1 - CQ(k,\delta^2;g)(C^2_{n+1}) - \alpha|}{\alpha}, \quad \alpha = 1 - q.$$

Table 7 gives a first impression of our computational results in the Gaussian case $g_K = g_G$. For the computations we used the specific representation of the function $CQ_N(k,\delta^2;g_G)(c^2)$ in the N-th order approximative quantile equation, given in (11). We choose always the result with the smallest relative error $r_n(\alpha)$ among all results of the Large Deviation Iteration Procedure for different orders N. In all tables the footnotes 0, 1, 2 and S at the results indicate whether the given value is the N-th order approximative quantile, $N \in \{0,1,2\}$, or the starting value C_0^2, respectively. For all numerical results given in Table 7 it holds $r_n(\alpha) \leq 0.35$. If no result is given at some place in Table 7 this means that our result did not satisfy the inequality $r_n(\alpha) \leq 0.35$.

Table 7 Large or small quantiles: approximated values for the N-th order approximative quantiles $\chi^2_{q,N}(k,1;g_G)$ when $r_n(\alpha) \leq 0.35$

$\alpha \setminus k$	2	3	4	5	6
0.1	6.272647_1	-	-	-	-
0.01	12.422780_S	13.802919_S	15.663581_0	17.643733_0	19.729094_0
0.001	18.289315_S	19.899377_S	21.874278_0	23.938525_0	26.084156_0
0.0001	23.970570_S	25.727201_S	27.787963_0	29.921488_0	32.122476_0
0.00001	29.529460_S	31.394053_S	33.524108_0	35.716181_0	37.966403_0

If we use a preliminary stage from the derivation of the asymptotic expansion (9) with

$$\tilde{f}_{k,\delta^2;g}(c^2) = \frac{\gamma\beta^{\frac{M-1}{\gamma}+\frac{k}{2\gamma}}\Gamma(k/2)}{2\sqrt{\pi}\delta^{\frac{k-1}{2}}\Gamma\left(\frac{M-1}{\gamma}+\frac{k}{2\gamma}\right)} \frac{(1-\delta/c)^{2M-2+k}c^{\frac{3k-1}{2}-\gamma(k+1)+2M-2}}{\left(\beta\gamma(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}\right)^{\frac{k+1}{2}}} e^{-\beta(c-\delta)^{2\gamma}} \quad (19)$$

instead of $f_{k,\delta^2;g}(c^2)$ and

$$\tilde{b}_1 = \frac{(1-\delta/c)^k}{(k-1)\delta^{\frac{k-1}{2}}\left(\gamma\beta(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}\right)^{\frac{k+1}{2}}} \quad (20)$$

instead of b_1 we obtain a slightly modified Large Deviation Iteration Procedure. With starting value C_0 from (15), we compute corrected values $\tilde{\chi}^2_{q,N}(k,\delta^2;g_G)$ for the N-th order approximative quantiles $\chi^2_{q,N}(k,\delta^2;g_G)$. For the values given in Table 8 it holds $r_n(\alpha) \le 0.05$.

Table 8 Large or small quantiles: corrected values for the N-th order approximative quantiles $\tilde{\chi}^2_{q,N}(k,1;g_G)$ when $r_n(\alpha) \le 0.05$

$\alpha \setminus k$	2	3	4	5
0.35	3.286958_2	-	-	-
0.3	3.675375_2	-	-	-
0.2	4.843447_2	-	-	-
0.1	6.796464_2	-	-	-
0.05	8.672945_2	10.326004_0	-	-
0.03	10.025478_2	11.710646_0	13.306141_2	-
0.025	10.503386_2	12.200220_0	13.811369_2	-
0.01	12.874054_2	14.627922_0	16.309541_2	-
0.005	14.638875_2	16.432776_0	18.160458_2	-
0.0025	16.383801_2	18.214753_0	19.983714_2	-
0.001	18.664860_2	20.540546_0	22.358242_2	24.187268_2
0.0005	20.373838_2	22.280522_0	24.131499_2	25.986788_2
0.0001	24.296394_2	26.267231_0	28.186259_2	30.098434_2
0.00001	29.815791_1	31.868323_0	33.868263_2	35.852319_2

The empty places in Tables 7 and 8 (as well as in Table 9 below) indicate that the asymptotic behind our formulae depend in some way on the interrelation between the dimension k and the quantile order α. Namely the results make it evident that we are dealing with large quantiles.
Note that $A_i(c) = 0$, $i = 1,2$ for $k = 3$. The 0-indexed values of the column for $k = 3$ coincide therefore with the respective 2-indexed values.
Our geometric representation formula for the distribution function $CQ(k,\delta^2;g_K)(x)$, $x \in \mathbf{R}$, given in Corollary 2, Section 2, enables us to calculate numerically more exact approximations $\chi^2_{q,*} = \chi^2_{q,*}(k,\delta^2;g_K)$ for the quantiles $\chi^2_q = \chi^2_q(k,\delta^2;g_K)$ by using the secant method for approximating the solution χ^2_q of the equation

$$1 - CQ(k,\delta^2;g_K)(\chi^2_q) = 1 - q.$$

In accordance with Press et al. (1989), p. 282 the algorithm of the secant method stops as soon as for the increment with respect to the latest value

$$dC^2_n := \frac{(C^2_n - C^2_{n+1})CQ(k,\delta^2;g_K)(C^2_{n+1})}{CQ(k,\delta^2;g_K)(C^2_{n+1}) - CQ(k,\delta^2;g_K)(C^2_n)}$$

it holds $|dC^2_n| < 10^{-13}$ or $1 - CQ(k,\delta^2;g_K)(C^2_{n+1}) = 1 - q$. The values of the distribution function $CQ(k,\delta^2;g_K)(\chi^2_{q,*}(k,\delta^2;g_K))$ coincide with q at least for 12 digits. In this way, we can compute the possibly more interesting relative approximation error

$$\varepsilon^*_n(\alpha,k) = \frac{|C^2_n - \chi^2_{q,*}(k,\delta^2;g_K)|}{\chi^2_{q,*}(k,\delta^2;g_K)}.$$

If we use this relative error instead of $r_n(\alpha)$ for controlling our corrected quantile approximations $\tilde{\chi}^2_q(k,\delta^2;g_G)$ we obtain Table 9. In this table it holds $\varepsilon^*_n(\alpha,k) \le 0.02$ for all given values.

Table 9 Large or small quantiles: corrected values for the N-th order approximative quantiles $\tilde{\chi}^2_{q,N}(k,1;g_G)$ when $\varepsilon^*_n \le 0.02$

$\alpha \setminus k$	2	3	4	5	6
0.35	3.286958_2	-	-	-	-
0.3	3.675375_2	-	-	-	-
0.2	4.843447_2	-	-	-	-
0.1	6.796464_2	-	-	-	-
0.05	8.672945_2	10.326004_0	11.874000_2	-	-
0.03	10.025478_2	11.710646_0	13.306141_2	-	-
0.025	10.503386_2	12.200220_0	13.811369_2	15.515107_2	-
0.01	12.874054_2	14.627922_0	16.309541_2	18.046690_2	-
0.005	14.638875_2	16.432776_0	18.160458_2	19.925361_2	-
0.0025	16.383801_2	18.214753_0	19.983714_2	21.776609_2	23.715079_2
0.001	18.664860_2	20.540546_0	22.358242_2	24.187268_2	26.131113_2
0.0005	20.373838_2	22.280522_0	24.131499_2	25.986788_2	27.938659_2
0.0001	24.296394_2	26.267231_0	28.186259_2	30.098434_2	32.076165_2
0.00001	29.815791_1	31.868323_0	33.868263_2	35.852319_2	37.874289_2

For a comparison of the N-th order approximative quantiles $\chi^2_{q,N}(k,1;g)$ with the corrected values $\tilde{\chi}^2_{q,N}(k,1;g)$ we give in Tables 10 up to 12 for several density generating functions g both values with their relative approximation errors $\varepsilon^*_n(\alpha,k)$ and the results of the application of the geometric measure representation formula combined with the secant method, $\chi^2_{q,*}(k,1;g_G)$, too. For the computations of the approximations we used the specific representations of the function $CQ_N(k,\delta^2;g_K)(c^2)$ in the N-th order approximative quantile equations, given in (12) for $\gamma = 0.7$ and in (13) for $\gamma = 1.5$, respectively. Notice that in the Gaussian case, restricted to the range of Table 10, the corrected values $\tilde{\chi}^2_{q,N}(k,1;g_G)$ have for d.f. smaller than 5 a smaller relative approximation error $\varepsilon^*_n(\alpha,k)$ than the N-th order approximative quantiles $\chi^2_{q,N}(k,1;g_G)$. For d.f. 5 and 6 we observed for large values of α that the asymptotic expansion is better and for small α that the preliminary stage is better. For d.f. 8 and 10 the asymptotic expansion has the smaller relative error.

Table 10 Comparison of approximations for the quantiles $\chi^2_{1-\alpha}(k,1;g_G)$

k	α	$\chi^2_{1-\alpha,N}(k,1;g_G)(\varepsilon^*_n(\alpha,k))$	$\tilde{\chi}^2_{1-\alpha,N}(k,1;g_G)(\varepsilon^*_n(\alpha,k))$	$\chi^2_{1-\alpha,*}(k,1;g_G)$
2	10^{-1}	$6.27264722_1(0.1986528)$	$6.79646412_2(0.0038897)$	6.77013045
	10^{-2}	$12.32348549_1(0.0408692)$	$12.87405417_2(0.0019813)$	12.84859769
	10^{-3}	$18.18213409_1(0.0249444)$	$18.66486018_2(0.0009428)$	18.64727974
	10^{-4}	$23.86645384_1(0.0171785)$	$24.29639463_2(0.0005265)$	24.28360971
	10^{-5}	$29.43002096_1(0.0127647)$	$29.81579086_1(0.0001761)$	29.81054242
	10^{-6}	$34.90547027_1(0.0099784)$	$35.25602920_0(0.0000355)$	35.25728201
	10^{-7}	$40.31330712_1(0.0080864)$	$40.63662063_0(0.0001312)$	40.64195334
	10^{-8}	$45.75411880_S(0.0048414)$	$45.98206797_2(0.0001165)$	45.97671294

k	α	$\chi^2_{1-\alpha,N}(k,1;g_G)(\varepsilon^*_n(\alpha))$	$\tilde{\chi}^2_{1-\alpha,N}(k,1;g_G)(\varepsilon^*_n(\alpha))$	$\chi^2_{1-\alpha,*}(k,1;g_G)$
3	10^{-1}	$7.09432971_0(0.4747291)$	$8.41418910_0(0.0215007)$	8.23708570
	10^{-2}	$13.80291938_0(0.0520500)$	$14.62792229_0(0.0046091)$	14.56081028
	10^{-3}	$19.89937706_0(0.0294528)$	$20.54054658_0(0.0018188)$	20.50325620
	10^{-4}	$25.72720143_0(0.0196455)$	$26.26723128_0(0.0009327)$	26.24275474
	10^{-5}	$31.39405313_0(0.0143369)$	$31.86832308_0(0.0005535)$	31.85069342
	10^{-6}	$36.95053428_0(0.0110757)$	$37.37784362_0(0.0003606)$	37.36437110
	10^{-7}	$42.42524749_0(0.0088998)$	$42.81693914_0(0.0002506)$	42.80621232
	10^{-8}	$47.83623295_0(0.0073606)$	$48.19975160_0(0.0001826)$	48.19094967
4	10^{-1}	$8.77957771_0(0.3272683)$	$9.88670591_2(0.0244904)$	9.65036473
	10^{-2}	$15.66358088_0(0.0341762)$	$16.30954149_2(0.0056541)$	16.21784458
	10^{-3}	$21.87427839_0(0.0194037)$	$22.35824209_2(0.0022918)$	22.30711968
	10^{-4}	$27.78796302_0(0.0129564)$	$28.18625958_2(0.0011913)$	28.15272218
	10^{-5}	$33.52410758_0(0.0094561)$	$33.86826331_2(0.0007127)$	33.84414107
	10^{-6}	$39.13892478_0(0.0073024)$	$39.44524050_2(0.0004669)$	39.42683364
	10^{-7}	$44.66414804_0(0.0058642)$	$44.94224815_2(0.0003258)$	44.92761241
	10^{-8}	$50.11975317_0(0.0048466)$	$50.37584342_2(0.0002382)$	50.36384829
5	10^{-1}	$10.66620456_0(0.1188386)$	$11.79048626_2(0.2157233)$	11.02543135
	10^{-2}	$17.64373258_0(0.0005866)$	$18.04668978_2(0.0121399)$	17.83023266
	10^{-3}	$23.93852494_0(0.0053071)$	$24.18726823_2(0.0050287)$	24.06624588
	10^{-4}	$29.92148807_0(0.0032500)$	$30.09843452_2(0.0026445)$	30.01904880
	10^{-5}	$35.71618091_0(0.0022094)$	$35.85231948_2(0.0015939)$	35.79526706
	10^{-6}	$41.38170573_0(0.0016060)$	$41.49176359_2(0.0010493)$	41.44827284
	10^{-7}	$46.95169491_0(0.0012233)$	$47.04374697_2(0.0007348)$	47.00920206
	10^{-8}	$52.44741260_0(0.0009646)$	$52.52633582_2(0.0005388)$	52.49805162
6	10^{-1}	$12.71181897_0(0.0275142)$	$13.74011540_0(0.1106328)$	12.37142944
	10^{-2}	$19.72909378_0(0.0166664)$	$20.44110682_2(0.0533571)$	19.40567026
	10^{-3}	$26.08415552_0(0.0115361)$	$26.13111319_2(0.0133571)$	25.78667865
	10^{-4}	$32.12247646_0(0.0086629)$	$32.07616452_2(0.0072087)$	31.84659272
	10^{-5}	$37.96640295_0(0.0068504)$	$37.87428860_2(0.0044076)$	37.70808815
	10^{-6}	$43.67588487_0(0.0056132)$	$43.55923787_2(0.0029275)$	43.43209098
	10^{-7}	$49.28548255_0(0.0047206)$	$49.15508660_2(0.0020624)$	49.05391984
	10^{-8}	$54.81722165_0(0.0040494)$	$54.67902937_2(0.0015183)$	54.59613810
8	10^{-1}	$17.15783304_0(0.1439895)$	$18.02504991_0(0.2018107)$	14.99824398
	10^{-2}	$21.57800150_S(0.0395892)$	$24.81099736_0(0.1043076)$	22.46747022
	10^{-3}	$28.60417087_S(0.0180614)$	$31.12590734_0(0.0685060)$	29.13030580
	10^{-4}	$35.05992212_S(0.0096482)$	$37.17427244_0(0.0500767)$	35.40148157
	10^{-5}	$41.20251459_S(0.0055576)$	$40.55870092_S(0.0210964)$	41.43278223
	10^{-6}	$47.14278971_S(0.0033045)$	$46.56631968_S(0.0154922)$	47.29908845
	10^{-7}	$52.93964427_S(0.0019591)$	$53.66395871_2(0.0116960)$	53.04356291
	10^{-8}	$58.72945312_2(0.0006102)$	$59.21034032_2(0.0088033)$	58.69364118
10	10^{-1}	$21.94703948_0(0.2497895)$	$22.69325840_0(0.2922834)$	17.56058924
	10^{-2}	$24.56655253_S(0.0341694)$	$29.48236919_0(0.1590953)$	25.43567384
	10^{-3}	$31.99185082_S(0.0116178)$	$35.84431678_0(0.1074034)$	32.36789385
	10^{-4}	$38.71601015_S(0.0033001)$	$37.69062343_0(0.0296976)$	38.84420165
	10^{-5}	$45.06083225_S(0.0004132)$	$44.16658036_0(0.0194404)$	45.04221993
	10^{-6}	$51.16339664_S(0.0022332)$	$50.36172343_0(0.0134707)$	51.04939559
	10^{-7}	$57.09583591_S(0.0031588)$	$56.36384031_0(0.0097021)$	56.91604755
	10^{-8}	$62.90118277_S(0.0036231)$	$62.22398762_0(0.0071819)$	62.67410804

Table 11 Heavy tail effects: comparison of the approximations for the quantiles $\chi^2_{1-\alpha}(k,1;g_K)$ with parameters $M=1$, $\gamma=0.7$ and $\beta=0.5$

k	α	$\chi^2_{1-\alpha,N}(k,1;g_K)(\varepsilon^*_n(\alpha))$	$\tilde{\chi}^2_{1-\alpha,N}(k,1;g_K)(\varepsilon^*_n(\alpha))$	$\chi^2_{1-\alpha,*}(k,1;g_K)$
2	10^{-1}	$14.86493108_2(0.0138557)$	$15.03970710_2(0.0044661)$	15.07378935
	10^{-2}	$34.19873621_2(0.0114173)$	$34.34522854_2(0.0071826)$	34.59370194
	10^{-3}	$56.86025772_2(0.0074745)$	$56.86974403_2(0.0073089)$	57.28846018
	10^{-4}	$82.13730812_2(0.0052510)$	$82.25693756_2(0.0038022)$	82.57088609
	10^{-5}	$109.65677689_2(0.0039086)$	$109.76886436_2(0.0028904)$	110.08706075
	10^{-6}	$139.16973174_2(0.0030313)$	$139.27604534_2(0.0022697)$	139.59287967
	10^{-7}	$170.49336517_2(0.0023906)$	$170.59505395_2(0.0017955)$	170.90191646
	10^{-8}	$203.48575659_2(0.0016130)$	$203.58361901_2(0.0011328)$	203.81450020
3	10^{-1}	$20.48896227_0(0.0728529)$	$19.97599483_0(0.0960652)$	22.09893365
	10^{-2}	$44.61580028_0(0.0220916)$	$44.26005612_0(0.0298889)$	45.62369965
	10^{-3}	$70.87841451_0(0.0108747)$	$70.57753289_0(0.0150736)$	71.65767114
	10^{-4}	$99.24700884_0(0.0065600)$	$98.97691909_0(0.0092424)$	99.90236631
	10^{-5}	$129.54627391_0(0.0043943)$	$129.29678540_0(0.0063227)$	130.11804641
	10^{-6}	$161.61916804_0(0.0031652)$	$161.38481453_0(0.0046106)$	162.13234805
	10^{-7}	$195.33528895_0(0.0023427)$	$195.11272269_0(0.0034795)$	195.79397925
	10^{-8}	$230.58629918_0(0.0013308)$	$230.37328617_0(0.0022533)$	230.89356607
4	10^{-1}	$30.99758965_0(0.0488901)$	$29.77961958_0(0.0076767)$	29.55275239
	10^{-2}	$55.01652743_1(0.0318210)$	$58.10434137_0(0.0225182)$	56.82474969
	10^{-3}	$84.76452186_1(0.0140760)$	$87.70180328_0(0.0200885)$	85.97470181
	10^{-4}	$116.09481535_1(0.0076020)$	$118.98183151_0(0.0170767)$	116.98413171
	10^{-5}	$149.04874406_1(0.0045731)$	$148.24652675_1(0.0099308)$	149.73350183
	10^{-6}	$183.56415836_1(0.0029344)$	$182.82286252_1(0.0069609)$	184.10439473
	10^{-7}	$219.56307306_1(0.0019022)$	$218.86790295_1(0.0050624)$	219.98153002
	10^{-8}	$256.97007013_1(0.0006834)$	$256.31146862_1(0.0032446)$	257.14581454
5	10^{-1}	$44.44110650_0(0.1888110)$	-	37.38281904
	10^{-2}	$75.79376247_0(0.1110184)$	$74.43577764_0(0.0911125)$	68.22007379
	10^{-3}	$108.20870185_0(0.0786034)$	$107.03985864_0(0.0669526)$	100.32297939
	10^{-4}	$129.46034399_1(0.0335465)$	$141.01234963_0(0.0526921)$	133.95403325
	10^{-5}	$165.32686432_1(0.0223372)$	$176.42262992_0(0.0432778)$	169.10418118
	10^{-6}	$203.03064098_S(0.0130377)$	$213.24332827_0(0.0366077)$	205.71266184
	10^{-7}	$240.79659339_1(0.0118989)$	$238.94491736_1(0.0194972)$	243.69630082
	10^{-8}	$280.40996409_1(0.0085487)$	$278.70840807_1(0.0145649)$	282.82777037
8	10^{-2}	$140.55409451_0(0.3564826)$	$137.95315889_0(0.3313811)$	103.61658162
	10^{-3}	$180.53144584_0(0.2544111)$	$178.21045679_0(0.2382838)$	143.91728970
	10^{-4}	$221.42216889_0(0.1981045)$	$219.28654894_0(0.1865488)$	184.81039331
	10^{-5}	$263.34965678_0(0.1621323)$	$261.34909395_0(0.1533040)$	226.60902167
	10^{-6}	$306.34960125_0(0.0220107)$	$304.45350311_0(0.0280638)$	313.24433489
	10^{-7}	$350.42255838_0(0.0207536)$	$348.61064224_0(0.0258170)$	357.84923716
	10^{-8}	$395.55378917_0(0.0135255)$	$393.81176533_0(0.0178700)$	400.97721882

Recall that we choose always the result with the smallest relative error $r_n(\alpha)$ among all results of the Large Deviation Iteration Procedure for different orders N. The footnotes 0, 1, 2 and S at the results indicate whether the given value is the N-th order approximative quantile, $N \in \{0,1,2\}$, or the starting value C_0^2, respectively.

Table 12 Light tail effects: comparison of the approximations for the quantiles $\chi^2_{1-\alpha}(k,1;g_K)$ with parameters $N=1$, $\gamma=1.5$ and $\beta=0.5$,

k	α	$\chi^2_{1-\alpha,N}(k,1;g_K)(\varepsilon^*_n(\alpha,k))$	$\tilde\chi^2_{1-\alpha,N}(k,1;g_K)(\varepsilon^*_n(\alpha,k))$	$\chi^2_{1-\alpha,*}(k,1;g_K)$
2	10^{-2}	$6.87185579_S(0.0439408)$	$7.00401848_2(0.0255534)$	7.18768859
	10^{-3}	$9.08949622_S(0.0253340)$	$9.28401156_2(0.0044761)$	9.32575474
	10^{-4}	$10.95907778_S(0.0168096)$	$11.13001266_2(0.0014742)$	11.14644503
	10^{-5}	$12.61004396_S(0.0121966)$	$12.75760974_2(0.0006370)$	12.76574211
	10^{-6}	$14.10932345_S(0.0093829)$	$14.23838034_2(0.0003218)$	14.24296372
	10^{-7}	$15.49570554_S(0.0075143)$	$15.61030277_2(0.0001744)$	15.61302644
	10^{-8}	$16.79387346_S(0.0061534)$	$16.89699995_2(0.0000504)$	16.89785246
3	10^{-2}	$6.97295539_S(0.0624126)$	$7.70944597_1(0.0366163)$	7.43712605
	10^{-3}	$9.17244144_S(0.0371421)$	$9.47454759_2(0.0054291)$	9.52626679
	10^{-4}	$11.03103680_S(0.0252909)$	$11.29734857_2(0.0017594)$	11.31726070
	10^{-5}	$12.67467647_S(0.0187121)$	$12.90685583_2(0.0007365)$	12.91636851
	10^{-6}	$14.16865014_S(0.0146180)$	$14.37371688_2(0.0003563)$	14.37883986
	10^{-7}	$15.55096657_S(0.0118625)$	$15.73468410_2(0.0001887)$	15.73765414
	10^{-8}	$16.84589118_S(0.0099000)$	$17.01252963_2(0.0001060)$	17.01433284
4	10^{-2}	$7.12728617_S(0.0731333)$	$7.92854649_1(0.0310665)$	7.68965615
	10^{-3}	$9.29926645_S(0.0451198)$	$9.65864734_2(0.0082173)$	9.73867307
	10^{-4}	$11.14131720_S(0.0314827)$	$11.47203477_2(0.0027333)$	11.50347734
	10^{-5}	$12.77386654_S(0.0236982)$	$13.06857978_2(0.0011734)$	13.08393280
	10^{-6}	$14.25978045_S(0.0187522)$	$14.52388030_2(0.0005789)$	14.53229278
	10^{-7}	$15.63590590_S(0.0153565)$	$15.87505211_2(0.0002967)$	15.87976350
	10^{-8}	$16.92588286_S(0.0127833)$	$17.14461813_2(0.0000254)$	17.14505369
5	10^{-2}	$7.32027088_S(0.0780942)$	$8.07641354_1(0.0171334)$	7.94036778
	10^{-3}	$9.45864078_S(0.0499704)$	$10.05698513_1(0.0101275)$	9.95615395
	10^{-4}	$11.28032661_S(0.0357057)$	$11.64054052_2(0.0049130)$	11.69801286
	10^{-5}	$12.89912969_S(0.0273258)$	$13.23219154_2(0.0022109)$	13.26151169
	10^{-6}	$14.37500650_S(0.0218898)$	$14.67979190_2(0.0011514)$	14.69671418
	10^{-7}	$15.74339645_S(0.0180964)$	$16.02329053_2(0.0006396)$	16.03354653
	10^{-8}	$17.02717648_S(0.0151686)$	$17.28583394_2(0.0002081)$	17.28943217
6	10^{-2}	$7.54228771_S(0.0787755)$	$8.02635655_1(0.0196508)$	8.18724210
	10^{-3}	$9.64307315_S(0.0522953)$	$10.17998657_1(0.0004716)$	10.17518772
	10^{-4}	$11.44176859_S(0.0382587)$	$11.92194900_1(0.0021030)$	11.89692968
	10^{-5}	$13.04492550_S(0.0297637)$	$13.47204137_1(0.0020036)$	13.44510213
	10^{-6}	$14.50931410_S(0.0241365)$	$14.89327338_1(0.0016877)$	14.86818063
	10^{-7}	$15.86881551_S(0.0201652)$	$16.21795999_1(0.0013931)$	16.19539818
	10^{-8}	$17.14545438_S(0.0172286)$	$17.43047865_2(0.0008912)$	17.44602584
7	10^{-2}	$7.78646831_S(0.0762815)$	$9.10773947_0(0.0804625)$	8.42948218
	10^{-3}	$9.84724617_S(0.0525889)$	$10.15886791_1(0.0226076)$	10.39384741
	10^{-4}	$11.62118942_S(0.0394037)$	$12.01881527_1(0.0065363)$	12.09789079
	10^{-5}	$13.20734770_S(0.0311663)$	$13.59840267_1(0.0024802)$	13.63221395
	10^{-6}	$14.65917783_S(0.0255830)$	$15.02884453_1(0.0010107)$	15.04404979
	10^{-7}	$16.00892023_S(0.0215510)$	$16.35565137_1(0.0003592)$	16.36152802
	10^{-8}	$17.27769331_S(0.0183154)$	$17.60304998_1(0.0001707)$	17.60004528

k	α	$\chi^2_{1-\alpha,N}(k,\delta^2;g_K)(\varepsilon^*_n(\alpha))$	$\tilde{\chi}^2_{1-\alpha,N}(k,\delta^2;g_K)(\varepsilon^*_n(\alpha))$	$\chi^2_{1-\alpha,*}(k,\delta^2;g_K)$
10	10^{-2}	$8.60675843_S(0.0570174)$	$10.20971753_0(0.1186077)$	9.12716560
	10^{-3}	$10.54298037_S(0.0449209)$	$11.81618653_0(0.0704177)$	11.03885538
	10^{-4}	$12.23775648_S(0.0364740)$	$13.30083891_0(0.0472266)$	12.70101285
	10^{-5}	$13.76845536_S(0.0304727)$	$14.68604087_0(0.0341405)$	14.20120506
	10^{-6}	$15.17875087_S(0.0260393)$	$15.51864268_3(0.0042298)$	15.58456242
	10^{-7}	$16.49590208_S(0.0226211)$	$16.82532551_3(0.0031028)$	16.87769376
	10^{-8}	$17.73821587_S(0.0764145)$	$18.05753414_3(0.0597884)$	19.20581971
12	10^{-2}	$9.19765585_S(0.0388563)$	$9.19765585_S(0.0388563)$	9.56949106
	10^{-3}	$11.05293767_S(0.0352125)$	$11.05293767_S(0.0352125)$	11.45634398
	10^{-4}	$12.69436168_S(0.0307786)$	$13.89980288_0(0.0612575)$	13.09748329
	10^{-5}	$14.18672787_S(0.0269595)$	$15.23910727_0(0.0452212)$	14.57979118
	10^{-6}	$15.56781098_S(0.0238264)$	$16.50537737_0(0.0349634)$	15.94778862
	10^{-7}	$16.86175081_S(0.0212618)$	$17.70982512_0(0.0279645)$	17.22805070
	10^{-8}	$18.08504230_S(0.0191438)$	$18.86126139_0(0.0229551)$	18.43801525

Restricted to the ranges of Tables 11 and 12 and the concrete Kotz type distributions dealt with there, one can observe the following effects of heavy and light tails.
Heavy tails:
In Table 11 we can see that for 2 d.f. the relative approximation errors $\varepsilon^*_n(\alpha,k)$ of the corrected values $\tilde{\chi}^2_{q,N}(k,1;g_K)$ are smaller than the relative approximation errors for the N-th order approximative quantiles $\chi^2_{q,N}(k,1;g_K)$. For 3 d.f. the asymptotic expansion is the better one. For d.f. 4, 5, 8 it turns out that for large α the preliminary stage is better and for small α the asymptotic expansion is better. One suggestion turning out from this table is therefore to use in the asymptotic formula for large deviations the function $f_{k,\delta^2;g}$ instead of $\widetilde{f}_{k,\delta^2;g}$ when working in the far tails and vice versa when dealing with quantiles from the central part of the distribution.
Light tails:
From Table 12 it can be seen that for 2 up to 6 d.f. the preliminary stage is better. For 7, 10, 12 d.f. the asymptotic expansion is better for large α, whereas for small α the preliminary stage is better. Roughly spoken, the observed effects under light tails are contrary to those under heavy tails. Note furthermore that with increasing d.f. the asymptotic expansion becomes better for decreasing α.

6 ORDINARY CHI-SQUARE QUANTILES: COMPARISON WITH OTHER RESULTS

We could not find any results in the literature concerning the general case discussed in Section 5, but there are various results for the ordinary chi-square distribution.
Fisher (1928) published the upper 5% significance points of the noncentral chi-square distribution function in an implicit form for $k = 1(1)7$ and $\delta = 0(0.2)5.0$. Several methods for approximating the quantiles of the usual noncentral chi-square distribution function have been developed by Patnaik (1949), Abdel-Aty (1954) and Sankaran (1963). In these papers the upper and the lower 5% points for several parameters are given. In Table 13 we follow the choice of parameters studied by other authors to compare our upper 5% points with their results. In the column Fisher we quoted the exact results of Fisher

(1928). Modified central chi-square-approximated percentage points from Patnaik (1949) are given in the column Patnaik. The next column contains the Cornish-Fisher type approximations in Abdel-Aty (1954). In the columns Sankaran ('closer' and 'normal') the results of a modification of Abdel-Aty's method and a normal approximation by Sankaran were listed. The last columns contain our corrected explicit approximation values $\tilde{\chi}^2_{q,N}(k,\delta^2;g_G)$ based upon the Large Deviation Iteration Procedure from Section 4, indexed by the actual value of N, and our exact 'implicit' results from the application of the secant method $\chi^2_{q,*}(k,\delta^2;g_G)$, respectively.

Table 13 Comparison study for the upper 5% points of $CQ(k,\delta^2;g_G)$

k	δ^2	Fisher	Patnaik	Abd.-Aty	Sankaran 'closer'	Sankaran 'normal'	$\tilde{\chi}^2_{0.95,N}$	$\chi^2_{0.95,*}$
2	1	8.642	8.63	8.38	8.38	8.87	8.67_2	8.64220388
	4	14.641	14.72	14.62	14.62	14.68	14.61_2	14.64021080
	16	33.054	33.35	33.08	33.08	33.057	32.95_2	33.05421469
	25	45.308	45.66	45.33	45.33	45.309	45.17_2	45.30823043
4	1	11.707	11.72	11.67	11.67	11.96	11.87_2	11.70722775
	4	17.309	17.38	17.24	17.27	17.39	17.57_0	17.30932288

In Dinges (1989) a Lugannani-Rice type formula called Wiener Germ approximation is used to approximate the distribution function of the noncentral chi-square distribution. We exploited the respective formula based on a second order approximation from example 4.5 in Dinges (1989). The resulting quantities $C^2_{D,1-\alpha}(k)$ given in column Dinges of Table 14 are evaluated by the root finding algorithm 'fsolve' of the formulae manipulating system Maple V.

Certain z_1 and z_2 transformations for approximating the quantiles of the noncentral chi-square distribution are given in Sankaran (1963). It is mentioned there that the approximation based on z_1 is not as good as that based on z_2, in general. We evaluated both types of values and confirmed them. In column Sankaran the values $C^2_{s,1-\alpha}(k)$ nearest to the exact quantiles are tabled. They are indexed by 'z_1' if the z_1 approximation was used, otherwise the z_2 approximation was used. In the last column of Table 14 our exact values of the quantiles are given, again evaluated by combining the exact formula (4) for the distribution function with the secant method. The relative errors are computed according to

$$\varepsilon^*_S(\alpha,k) = \frac{|C^2_{S,1-\alpha}(k) - \chi^2_{1-\alpha,*}(k,\delta^2;g_G)|}{\chi^2_{1-\alpha,*}(k,\delta^2;g_G)}$$

and

$$\varepsilon^*_D(\alpha,k) = \frac{|C^2_{D,1-\alpha}(k) - \chi^2_{1-\alpha,*}(k,\delta^2;g_G)|}{\chi^2_{1-\alpha,*}(k,\delta^2;g_G)},$$

respectively, where 'S' stands for Sankaran and 'D' for Dinges.

Table 14 Comparison study for large quantiles ($\delta = 1$)

k	α	Sankaran ($\varepsilon_S^*(\alpha,k)$)		Dinges ($\varepsilon_D^*(\alpha,k)$)		$\chi^2_{1-\alpha,*}(k,1;g_G)$
2	10^{-1}	6.78151398_{z_1}	(0.0016814)	6.78427274	(0.0020889)	6.770130446893
	10^{-2}	13.11027479_{z_1}	(0.0203662)	12.86014207	(0.0008985)	12.848597693670
	10^{-3}	19.11900814_{z_1}	(0.0252974)	18.65699390	(0.0005209)	18.647279743060
	10^{-4}	24.85364100	(0.0234739)	24.29193303	(0.0003428)	24.283609705167
	10^{-5}	30.43250001	(0.0208637)	29.81781012	(0.0002438)	29.810542419159
	10^{-6}	35.90979020	(0.0185071)	35.26373314	(0.0001830)	35.257282009222
	10^{-7}	41.30881727	(0.0164083)	40.64775757	(0.0001428)	40.641953336007
	10^{-8}	46.64480567	(0.0145311)	45.98198422	(0.0001147)	45.976712945780
5	10^{-1}	11.08178983_{z_1}	(0.0051117)	11.02741413	(0.0001798)	11.025431349567
	10^{-2}	18.21014455	(0.0213072)	17.83212578	(0.0001062)	17.830232659777
	10^{-3}	24.52924605	(0.0192386)	24.06830591	(0.0000856)	24.066245884333
	10^{-4}	30.48325316	(0.0154637)	30.02120647	(0.0000719)	30.019048804997
	10^{-5}	36.20303657	(0.0113917)	35.79746130	(0.0000613)	35.795267059696
	10^{-6}	41.75619172	(0.0074290)	41.45046464	(0.0000529)	41.448272843317
	10^{-7}	47.18302639	(0.0036977)	47.01136799	(0.0000461)	47.009202057374
	10^{-8}	52.50951253	(0.0002183)	52.50016107	(0.0000402)	52.498051624489
10	10^{-1}	17.65337272_{z_1}	(0.0052836)	17.56096416	(0.0000213)	17.560589244157
	10^{-2}	25.80446516	(0.0144990)	25.43588219	(0.0000082)	25.435673840466
	10^{-3}	32.84500575	(0.0147403)	32.36810544	(0.0000065)	32.367893850640
	10^{-4}	39.35010019	(0.0130238)	38.84444734	(0.0000063)	38.844201654265
	10^{-5}	45.51698223	(0.0105404)	45.04250619	(0.0000064)	45.042219934156
	10^{-6}	51.44559170	(0.0077610)	51.04972156	(0.0000064)	51.049395594493
	10^{-7}	57.19478292	(0.0048973)	56.91640987	(0.0000064)	56.916047546689
	10^{-8}	62.80227849	(0.0020450)	62.67448487	(0.0000060)	62.674108040217

The relative errors indicate that among the approximation formulae that of Dinges yields especially good results in the present special Gaussian situation.
Recognize, however, that it would not be possible to check the relative approximation errors ε_S^* and ε_D^* if we would not know the exact values $\chi^2_{1-\alpha,*}(k,1;g_G)$. The information about the differences with respect to the accuracy of the methods compared in Table 14 seem us therefore to be new. The finally Table 15 illustrates that Dinges' formula yields also good results in the central part of the distribution.

Table 15 Small Quantiles: Wiener Germ approximation

k	α	Dinges ($\varepsilon_D^*(\alpha,k)$)	$\chi^2_{1-\alpha,*}(k,1;g_G)$
2	0.45	2.570550010832(0.030952734)	2.493373290173
	0.35	3.265723130277(0.009013084)	3.236551817807
	0.3	3.708074346704(0.006152537)	3.685399787656
	0.2	4.861431555275(0.003446869)	4.844732394966
4	0.45	4.670783894531(0.005554594)	4.644982900987
	0.35	5.581361592328(0.001545554)	5.572748608575
	0.3	6.121704695126(0.001047945)	6.115296201591
	0.2	7.479410831986(0.000574271)	7.475118088576
5	0.45	5.715859549369(0.003042738)	5.698520445432
	0.35	6.709647601532(0.000839715)	6.704018135348
	0.3	7.291063435846(0.000566234)	7.286937324439
	0.2	8.738020360443(0.000302691)	8.735376242672

References

Abdel-Aty, S.H. (1954). *Approximate formulae for the percentage points and the probability integral of the noncentral* χ^2 *distribution.* Biometrika, 41, 538–540.

Abramowitz, M., Stegun, I.A. (1965). *Handbook of mathematical functions.* Dover Publications, Inc., New York.

Anderson, T.W, Fang, K.T. (1982). *On the theory of multivariate elliptically contoured distributions and their applications.* In: Fang, K.T. and Anderson, T.W. (eds.) (1990). *Statistical inference in elliptically contoured and related distributions.* Allerton Press Inc., N.Y., 1–23.

Ashour, S.A., Abdel-Samad, A.I. (1990). *On the computation of non-central chi-square distribution function.* Commun. Statist.- Simula., 19 (4), 1279–1291.

Cacoullous, T., Koutras, M. (1984). *Quadratic forms in spherical random variables: generalized noncentral* χ^2 *distribution.* Naval-Research-Logistics-Quarterly, 31, No.3, 447–461.

Dinges, H. (1989). *Special Cases of Second Order Wiener Germ Approximations.* Probability Theory and Related Fields, 83, 5–57

Fan, J. (1990). *Distributions of quadratic forms and non-central Cochran's Theorem* . In: Fang, K.T. and Anderson, T.W. (eds.) (1990). *Statistical inference in elliptically contoured and related distributions.* Allerton Press Inc., N.Y., 177–191.

Fang, K.T., Anderson, T.W. (1990). *Statistical Inference in Elliptically Contoured and Related Distributions.* Allerton Press Inc., N.Y.

Fang, K.T., Kotz, S., Ng, K.W. (1990). *Symmetric Multivariate and Related Distributions.* Chapman & Hall, London, N.Y.

Fang, K.T., Zhang, Y.T. (1990). *Generalized Multivariate Analysis.* Springer Verlag, N.Y.

Farebrother, R.W. (1987). *The Distribution of a Noncentral* χ^2 *Variable with Nonnegative Degrees of Freedom* . Applied Statistics, Vol. 36, No. 3, 402–405.

Fisher, R.A. (1928). *The general Sampling distribution of multiple correlation coefficient.* Proc. Roy. Soc. London, 121 A, 654–673.

Fisher, R.A., Lewis, T., Embleton, B.J.J. (1987). *Statistical analysis of spherical data.* Cambridge, University Press.

Gupta, A.K., Varga, T. (1993). *Elliptically contoured models in statistics.* Kluwer Academic Publishers, Dordrecht, Boston, London.

Haynam, G.E., Gorwindarajulu, Z., Leone, F.C., Siefert, P. (1982). *Tables of the Cumulative Non-Central Chi-Square Distribution.* Math. Operationsforsch. Statist., Ser. Statistics, Vol. 13, No.3, 413–443.

Hsu, H. (1990). *Noncentral distributions of quadratic forms for elliptically contoured distributions.* In: Fang, K.T. and Anderson, T.W. (eds.) (1990). *Statistical inference in elliptically contoured and related distributions.* Allerton Press Inc., N.Y., 97–102.

Johnson, M.E. (1987). *Multivariate statistical simulation.* J. Wiley & Sons, New York.

Kelker, D. (1970). *Distribution theory of spherical distributions and a location-scale parameter generalization.* Sankhya, A, 32, 419–430.

Lugannani, R., Rice, S. (1980). *Saddlepoint approximation for the distribution of the sum of independent random variables.* Adv. Appl. Probab. 12, 475–490.

Patnaik, P.B. (1949). *The non-central χ^2- and F-distributions and their applications.* Biometrika, 36, 202–232.

Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T. (1989). *Numerical Recipes in Pascal.* Cambridge University Press, Cambridge.

Richter, W.-D. (1991). *Eine geometrische Methode in der Stochastik.* Rostock. Math. Kolloq., 44, 63–72.

Richter, W.-D. (1995). *A geometric approach to the Gaussian law.* In: Symposia Gaussiana, Conf. B, Eds.. Mammitzsch/ Schneewei, Walter de Gruyter & Co., Berlin, New York, 25–45.

Richter, W.-D. (in preparation). *Approximating large Quantiles.*

Richter, W.-D., Schumacher, J. (in preparation). *Asymptotic expansions for large deviation probabilities of noncentral generalized chi-square distributions.*

Sankaran, M. (1959). *On the non-central chi-square distribution.* Biometrika, 46, 235–237.

Sankaran, M. (1963). *Approximations to the non-central chi-square distribution.* Biometrika, 50, 199–204.

Schoenberg, I. J. (1938). *Metric spaces and completely monotone functions.* Ann. Math., 39, 811–841.

Wang, M.C., Kennedy, J. (1994). *Self-Validating Computations of Probabilities for Selected Central and Noncentral Univariate Probability Functions.* JASA, 89, 878–887.

Watson, G.S. (1983). *Statistics on spheres.* J. Wiley & Sons, New York.

Zeitler, E. (ed.) (1996). *Teubner-Taschenbuch der Mathematik.* B.G. Teubner, Stuttgart-Leipzig.

Acknowledgement The authors are grateful to the referees whose hints led to improvements in the presentation of the material. The discussion of the Lugannani-Rice type formula coming out from the Wiener germ theory in Dinges (1989) was suggested by one of the referees.

APPENDIX A

Asymptotic expansion coefficients

The following quantities complete formula (9) and have been derived in Richter and Schumacher (in preparation):

$$
\begin{aligned}
D_{1-2\gamma} &= -\frac{c_1^{\frac{k+3}{2}}}{b_1\delta^{\frac{k+1}{2}}(1-\frac{\delta}{c})^3}\frac{k-3}{16}, \\
D_{-2\gamma} &= \frac{c_1^{\frac{k+3}{2}}}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})^3}\frac{(k+1)(k-3)}{8(k-1)} + \frac{c_1^{\frac{k-1}{2}}c_2}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})}\frac{(k+1)(k+3)}{4(k-1)}, \\
D_{-1-2\gamma} &= -\frac{c_1^{\frac{k+3}{2}}}{b_1\delta^{\frac{k-3}{2}}(1-\frac{\delta}{c})^3}\frac{k+1}{16}, \\
D_{2-4\gamma} &= \frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k+3}{2}}(1-\frac{\delta}{c})^5}\frac{(k-5)(k-3)(k+1)}{512},
\end{aligned}
$$

$$
\begin{aligned}
D_{1-4\gamma} &= -\frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k+1}{2}}(1-\frac{\delta}{c})^5}\frac{(k-5)(k-3)(k+3)}{128} + \frac{c_1^{\frac{k+1}{2}}c_2}{b_1\delta^{\frac{k+1}{2}}(1-\frac{\delta}{c})^3}\frac{(k-3)(k+3)(k+5)}{64}, \\
D_{-4\gamma} &= \frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})^5}\frac{3(k-5)(k-3)(k+1)(k+3)}{256(k-1)} \\
&\quad -\frac{c_1^{\frac{k+1}{2}}c_2}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})^3}\frac{(k-3)(k+1)(k+3)(k+5)}{32(k-1)} \\
&\quad +\frac{c_1^{\frac{k-1}{2}}c_3}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})}\frac{(k+1)(k+3)(k+5)}{8(k-1)} + \frac{c_1^{\frac{k-3}{2}}c_2^2}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})}\frac{(k+1)(k+3)(k+5)}{32},
\end{aligned}
$$

$$
\begin{aligned}
D_{-1-4\gamma} &= -\frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k-3}{2}}(1-\frac{\delta}{c})^5}\frac{(k-5)(k+1)(k+3)}{128} + \frac{c_1^{\frac{k+1}{2}}c_2}{b_1\delta^{\frac{k-3}{2}}(1-\frac{\delta}{c})^3}\frac{(k+1)(k+3)(k+5)}{64}, \\
D_{-2-4\gamma} &= \frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k-5}{2}}(1-\frac{\delta}{c})^5}\frac{(k-3)(k+1)(k+3)}{512}
\end{aligned}
$$

with

$$
b_1 = \frac{1}{(k-1)(\gamma\beta)^{\frac{k+1}{2}}\delta^{\frac{k-1}{2}}} \tag{21}
$$

and

$$\begin{aligned}
c_1 &= \frac{(1-\delta/c)^2}{\beta\gamma(1-\delta/c)^{2\gamma} - \frac{M-1}{c^{2\gamma}}}\\
c_2 &= -\frac{1}{2}\frac{(1-\delta/c)^2\big(\beta\gamma(\gamma-1)(1-\delta/c)^{2\gamma} + \frac{M-1}{c^{2\gamma}}\big)}{\big(\beta\gamma(1-\delta/c)^{2\gamma} - \frac{M-1}{c^{2\gamma}}\big)^3}\\
c_3 &= -\frac{1}{12}\frac{(1-\delta/c)^2\big(\beta\gamma(\gamma-1)(\gamma-2)(1-\delta/c)^{2\gamma} - 2\frac{M-1}{c^{2\gamma}}\big)}{\big(\beta\gamma(1-\delta/c)^{2\gamma} - \frac{M-1}{c^{2\gamma}}\big)^4}\\
&\quad +\frac{1}{4}\frac{(1-\delta/c)^2\big(\beta\gamma(\gamma-1)(1-\delta/c)^{2\gamma} + \frac{M-1}{c^{2\gamma}}\big)^2}{\big(\beta\gamma(1-\delta/c)^{2\gamma} - \frac{M-1}{c^{2\gamma}}\big)^5}.
\end{aligned}$$

APPENDIX B

Proof of Lemma 1 : *Case 1:* $\theta \leq 0$. We consider

$$a_0 x^\theta e^{-\beta(x-\delta)^{2\gamma}} = \alpha \tag{22}$$

with

$$\theta \leq 0, \; x > 0.$$

In (22) holds $\alpha \to +0$ if and only if $x \to \infty$.
Let $x \geq 1$. Then it follows from (23) that

$$a_0 e^{-\beta(x-\delta)^{2\gamma}} \geq \alpha.$$

This is equivalent to

$$x \leq \delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}.$$

Inserting the last inequality into (22) at the place of x^θ it follows

$$\alpha \geq a_0 e^{-\beta(x-\delta)^{2\gamma}}\left(\delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}\right)^\theta$$

or

$$\begin{aligned}
\ln\frac{\alpha}{a_0} &\geq -\beta(x-\delta)^{2\gamma} + \theta\ln\left[\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}\left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right)\right],\\
\beta(x-\delta)^{2\gamma} &\geq -\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma}\ln\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right) + \theta\ln\left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right),\\
\beta(x-\delta)^{2\gamma} &\geq -\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma}\ln\left(-\ln\frac{\alpha}{a_0}\right) + \frac{\theta}{2\gamma}\ln\frac{1}{\beta} + \theta\ln\left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right),
\end{aligned}$$

or

$$x \geq \delta + \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta} \ln \left(-\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\gamma\beta} \ln \beta \right.$$
$$\left. + \frac{\theta}{\beta} \ln \left(1 + \delta \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{-\frac{1}{2\gamma}} \right) \right)^{\frac{1}{2\gamma}}. \tag{23}$$

Inserting the last inequality again into (22) for x^θ supplies

$$\alpha \leq a_0 e^{-\beta(x-\delta)^{2\gamma}} \left[\delta + \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta} \ln \left(-\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\gamma\beta} \ln \beta \right. \right.$$
$$\left. \left. + \frac{\theta}{\beta} \ln \left(1 + \delta \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{-\frac{1}{2\gamma}} \right) \right)^{\frac{1}{2\gamma}} \right]^\theta .$$

It follows with $\theta(1) = O(1)$, $\alpha \to 0$ that

$$\alpha \leq a_0 e^{-\beta(x-\delta)^{2\gamma}} \left[\left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta} \ln \left(-\ln \frac{\alpha}{a_0} \right) \right)^{\frac{1}{2\gamma}} \left(1 + \frac{\theta(1)}{\left(-\ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}}} \right) \right]^\theta ,$$

respectively

$$\ln \frac{\alpha}{a_0} \leq -\beta(x-\delta)^{2\gamma}$$
$$+ \frac{\theta}{2\gamma} \ln \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta} \ln \left(-\ln \frac{\alpha}{a_0} \right) \right) + |\theta| \ln \left(1 + \frac{\theta(1)}{\left(-\ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}}} \right) ,$$

$$\ln \frac{\alpha}{a_0} \leq -\beta(x-\delta)^{2\gamma}$$
$$+ \frac{\theta}{2\gamma} \ln \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \left[1 + \frac{\theta}{2\gamma\beta} \frac{\ln \left(-\ln \frac{\alpha}{a_0} \right)}{\left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)} \right] \right) + |\theta| \left(\frac{\theta(1)}{\left(-\ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}}} \right) ,$$

or

$$x \leq \delta + \left[-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta} \ln \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right) \right.$$
$$\left. + \frac{|\theta|}{2\beta\gamma} \ln \left(1 + \frac{|\theta|}{2\gamma\beta} \frac{\ln \left(-\ln \frac{\alpha}{a_0} \right)}{\left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)} \right) + \frac{|\theta|\theta(1)}{\beta \left(-\ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}}} \right]^{\frac{1}{2\gamma}} ,$$

$$x \leq \delta + \left[-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\beta\gamma} \ln \left(-\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\beta\gamma} \ln \beta \right.$$
$$\left. + \frac{|\theta|^2}{(2\beta\gamma)^2} \frac{\ln \left(-\ln \frac{\alpha}{a_0} \right)}{\left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)} + \frac{|\theta|\theta(1)}{\beta \left(-\ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}}} \right]^{\frac{1}{2\gamma}} . \tag{24}$$

From the inequality (24) follows the upper bound

$$C_0^* = \delta + \left[-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\beta\gamma} \ln \left(-\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\beta\gamma} \ln \beta \right.$$
$$\left. + \frac{|\theta|^2}{(2\beta\gamma)^2} \frac{\ln\left(-\ln\frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)} + \frac{|\theta|\theta(1)}{\beta\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right]^{\frac{1}{2\gamma}} \tag{25}$$

for the case $\theta \leq 0$.
Case 2: $\theta > 0$. Let $x \geq 1$ then follows from (22) for $\theta > 0$

$$a_0 e^{-\beta(x-\delta)^{2\gamma}} \leq \alpha.$$

This is equivalent to

$$x \geq \delta + \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}}. \tag{26}$$

If we insert the last inequality (26) for x^θ into (22), we obtain

$$\begin{aligned}
\alpha &\geq a_0 \left(\delta + \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}} \right)^{\theta} e^{-\beta(x-\delta)^{2\gamma}}, \\
\ln \frac{\alpha}{a_o} &\geq -\beta(x-\delta)^{2\gamma} + \theta \ln \left(\delta + \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}} \right), \\
\ln \frac{\alpha}{a_o} &\geq -\beta(x-\delta)^{2\gamma} + \theta \ln \left[\left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}} \left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right) \right],
\end{aligned}$$

$$\begin{aligned}
\ln \frac{\alpha}{a_o} &\geq -\beta(x-\delta)^{2\gamma} + \frac{\theta}{2\gamma} \ln \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right) + \theta \ln \left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right) \\
\ln \frac{\alpha}{a_o} &\geq -\beta(x-\delta)^{2\gamma} + \frac{\theta}{2\gamma} \ln \left(-\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\gamma} \ln \beta + \theta \ln \left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right),
\end{aligned}$$

or

$$x \geq \delta + \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta} \ln \left(-\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\gamma\beta} \ln \beta \right.$$
$$\left. + \frac{\theta}{\beta} \ln \left(1 + \delta \left(-\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{-\frac{1}{2\gamma}} \right) \right)^{\frac{1}{2\gamma}}. \tag{27}$$

Finally, we prove $x \le C_0^*$. We show $x < (1+\varepsilon)C_0^*$, $\forall \varepsilon > 0$, leading the now stated assumption $x \ge (1+\varepsilon)C_0^*$ for $\varepsilon > 0$ to a contradiction. Because in (22) $\alpha \to +0$ if and only if $x \to \infty$, it holds for a sufficiently small α or a sufficiently large x, respectively,

$$\alpha = a_0 x^\theta e^{-\beta(x-\delta)^{2\gamma}} \le a_0(1+\varepsilon)^\theta C_0^{*\theta} e^{-\beta((1+\varepsilon)C_0^*-\delta)^{2\gamma}}$$

or

$$\frac{\alpha}{a_0} \le (1+\varepsilon)^\theta C_0^{*\theta} \exp\Bigg\{-\beta\Bigg((1+\varepsilon)\delta + (1+\varepsilon)\Bigg[-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\beta\gamma}\ln\big(-\ln\frac{\alpha}{a_0}\big) - \frac{\theta}{2\beta\gamma}\ln\beta + \frac{|\theta|^2}{(2\beta\gamma)^2}\frac{\ln\Big(-\ln\frac{\alpha}{a_0}\Big)}{\Big(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\Big)} + \frac{|\theta|\theta(1)}{\beta\Big(-\ln\frac{\alpha}{a_0}\Big)^{\frac{1}{2\gamma}}}\Bigg]^{\frac{1}{2\gamma}} - \delta\Bigg)^{2\gamma}\Bigg\}.$$

This is equivalent to

$$\begin{aligned}(1+\varepsilon)^{-\theta}\frac{\alpha}{a_0} \le& C_0^{*\theta}\exp\Bigg\{-\beta(1+\varepsilon)^{2\gamma}\Bigg[-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\beta\gamma}\ln\big(-\ln\frac{\alpha}{a_0}\big) - \frac{\theta}{2\beta\gamma}\ln\beta \\ &+ \frac{|\theta|^2}{(2\beta\gamma)^2}\frac{\ln\Big(-\ln\frac{\alpha}{a_0}\Big)}{\Big(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\Big)} + \frac{|\theta|\theta(1)}{\beta\Big(-\ln\frac{\alpha}{a_0}\Big)^{\frac{1}{2\gamma}}}\Bigg]\Bigg\} \\ =& C_0^{*\theta}\left(\frac{\alpha}{a_0}\right)^{(1+\varepsilon)^{2\gamma}} \beta^{\frac{\theta}{2\gamma}(1+\varepsilon)^{2\gamma}} \frac{\exp\left\{-\frac{|\theta|^2}{\beta(2\gamma)^2}\frac{\ln\left(-\ln\frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)} - \frac{|\theta|\theta(1)}{\beta(-\ln\frac{\alpha}{a_0})^{\frac{1}{2\gamma}}}\right\}}{\big(-\ln\frac{\alpha}{a_0}\big)^{\frac{\theta}{2\gamma}(1+\varepsilon)^{2\gamma}}} \\ \asymp& \frac{\big(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\big)^{\frac{\theta}{2\gamma}}}{\big(-\ln\frac{\alpha}{a_0}\big)^{\frac{\theta}{2\gamma}(1+\varepsilon)^{2\gamma}}}\alpha^{(1+\varepsilon)^{2\gamma}}.\end{aligned}$$

This, however, doesn't hold for $\alpha \to 0$ and so we have for sufficiently small α the upper bound C_0^* for x.

C. Ittrich, D. Krause, W.-D. Richter, Fachbereich Mathematik, Universität Rostock, Universitätsplatz 1, 18051 Rostock, Germany.