Journal of Multivariate Analysis **81**, 189–204 (2002)
doi:10.1006/jmva.2001.2001, available online at http://www.idealibrary.com on IDEAL®

Geometric Generalization of the Exponential Law

V. Henschel and W.-D. Richter

University of Rostock, Rostock, Germany
E-mail: wolf-dieter.richter@mathematik.uni-rostock.de

Received October 1, 1997; published online December 12, 2001

For the multivariate ℓ_1-norm symmetric distributions, which are generalizations of the n-dimensional exponential distribution with independent marginals, a geometric representation formula is given, together with some of its basic properties. This formula can especially be applied to a new developed and statistically well motivated system of sets. From that the distribution of a t-statistic adapted for the two-parameter exponential distribution and its generalizations is determined. Asymptotic normality of this adapted t-statistic is shown under certain conditions.

AMS 1991 subject classifications: 60D05; 60E05; 62E15; 62F04; 62F25.

Key words and phrases: exponential distributions; ℓ_1-norm symmetric distributions; modified t-test; F-distribution; simplicially contoured distributions; intersection percentage function.

1. INTRODUCTION

In classical statistics, many estimators and tests are developed by explicitly using or at least orienting on nice analytical properties of the Gaussian law.

Problems arise if one leaves the normal sample distribution and still wants to use standard estimators or tests. Hotelling (1961) examined the behaviour of some standard statistical tests under non-standard conditions. For example, he studied the distribution of Student's t-statistic T in the case of exponentially distributed samples. The basic problem which appears is easily described if one regards the problem geometrically. The probability $P(T \geqslant t)$ is given by the sample measure of a cone of rotation with the apex in the origin. If $t < n-1$, this cone intersects with the boundary of the positive orthant $\mathbb{R}^n_+$ which is the support of the exponential sample measure and the geometrical intersection problem turns out to be difficult. Hoq *et al.* (1978) solve this problem for $t \geqslant \sqrt{(n-1)(n-2)/2}$, but say that the computations become tedious and unwieldy for smaller t.

0047-259X/01 $35.00

Davids and Richter (1991) modify the problem by modifying the statistic of interest to

$$T^E = \sqrt{n}\,\frac{\bar{X}_n - \mu_0}{\bar{X}_n - X_{(1)}}, \qquad \text{where} \quad \bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad X_{(1)} = \min_i X_i$$

and yield an exact test for the expected lifetime $\mu = \theta + \sigma$ in the two-parameter exponential model with unknown minimum lifetime, i.e., if the sample distribution is a multivariate exponential distribution with independent two-parameter exponential marginals. Its density is given by

$$f(\mathbf{w}) = \begin{cases} \dfrac{1}{\sigma^n}\, e^{-\frac{1}{\sigma}(\sum_{i=1}^{n} w_i - n\theta)}, & \text{if} \quad w_i > \theta, \quad i = 1, ..., n \\ 0, & \text{otherwise} \end{cases}, \qquad \theta \in \mathbb{R}, \quad \sigma > 0.$$

A natural way to generalize this sample distribution for $\theta = 0$ and $\sigma = 1$ is to consider multivariate ℓ_1-norm symmetric distributions. These distributions were introduced by Fang and Fang (1988) as a generalization of the exponential distribution analogous to the generalization of the Gaussian law to the class of spherical distributions.

The paper is organized as follows. In Section 2, a geometric measure representation formula will be given for the multivariate ℓ_1-norm symmetric distributions having a certain density generating function. A basic tool of investigation occurring therein is the so-called simplicial intersection percentage function of a given Borel set, a suitably defined counterpart to the spherical intersection percentage function considered in Richter (1991, 1995).

In Section 3 we shall deal with statistical inference with respect to the expectation of the identically distributed one-dimensional marginals. Recognize that these marginals have distributions which depend on the sample size and that these marginals are not independent, except for the exponential law.

2. GEOMETRIC MEASURE REPRESENTATION FORMULA

Let $\mathbf{Y}$ follow the density f for $\theta = 0$ and $\sigma = 1$, $\mathbf{Y} \sim E_{0_n, I_n}$, i.e., let $\mathbf{Y}$ be a random vector with independent exponentially distributed marginals Y_i, $Y_i \sim E_{0,1}$, $i = 1, ..., n$ and

$$R := \|\mathbf{Y}\|_1, \qquad \mathbf{U} := \frac{\mathbf{Y}}{R},$$

where $\|x\|_1 = |x_1| + \cdots + |x_n|$ denotes the ℓ_1-norm of $\mathbf{x} \in \mathbb{R}^n$ and I_n the identity matrix. Notice that R is a nonnegative, Gamma$(n, 1)$ distributed random variable, $P(R=0)=0$, and $\mathbf{U}$ is uniformly distributed on the regular simplex $S_n^1(1)$, where

$$S_n^1(r)=\{\mathbf{x}\in\mathbb{R}^n : \|x\|_1=r\}, \qquad r>0.$$

The distribution of $\mathbf{U}$ will be denoted by ν. R and $\mathbf{U}$ are independent.

According to Fang *et al.* (1990), denote by $\mathscr{R}$ the set of all nonnegative random variables defined on the same probability space $(\Omega, \mathfrak{A}, P)$ as $\mathbf{Y}$ and put

$$L_n(F):=\{\mathbf{X} : \mathbf{X}\overset{d}{=} R\mathbf{U},\ R\in\mathscr{R} \text{ has distribution function } F,\ \mathbf{U}\sim\nu,$$
$$R \text{ and } \mathbf{U} \text{ are stochastically independent}\}.$$

From now on let $\mathbf{X}$ denote an arbitrary element of $L_n(F)$. $\mathbf{X}$ is called multivariate ℓ_1-norm symmetric distributed and the corresponding R is called its generating variate.

The assumption $\mathbf{X}\in L_n(F)$ implies that $\mathbf{X}$ has a density on $\mathbb{R}_+^n := (0,\infty)^{\times n}$ iff R has a density on $(0,\infty)$. In this case, the density p of $\mathbf{X}$ is of the form

$$p(\mathbf{x})=\mathrm{I}_{\mathbb{R}_+^n}(\mathbf{x})\, c(n,g)\, g(\|\mathbf{x}\|_1), \qquad \mathbf{x}\in\mathbb{R}^n,$$

where

$$g \mid \mathbb{R}_+ \to \mathbb{R}_+$$

will be called density generating function satisfying

$$0<\int_0^\infty r^{n-1}g(r)\,\mathrm{d}r<\infty \tag{1}$$

and

$$c(n,g):=\frac{\Gamma(n)}{\int_0^\infty r^{n-1}g(r)\,\mathrm{d}r} \tag{2}$$

denotes a suitably chosen norming constant.

This means that the density of $\mathbf{X}$ is constant on the regular simplicia or simplicial spheres $S_n^1(r)$, $r>0$. That is why one could call the distribution of $\mathbf{X}\in L_n(F)$ a regular simplicial distribution to point out this basic geometric aspect. Recall for comparison that the densities of spherical distributions are constant on the Euclidean spheres $S_n^2(r)=\{\mathbf{x}\in\mathbb{R}^n : \|\mathbf{x}\|_2=r\}$, $r>0$.

Recognize that fixing the density generating function g gives the possibility of affecting the behaviour of the density p of $\mathbf{X}$ in its tails. The density generating function g is related to the density f of the random variable R through

$$f(r) = c(n, g)\,\frac{r^{n-1}}{\Gamma(n)}\,g(r)\,\mathrm{I}_{\mathbb{R}_+}(r), \qquad r \in \mathbb{R}. \tag{3}$$

For multivariate ℓ_1-norm symmetric distributions defined in the described way, we shall use the notation E_g. Note that $\mathrm{E}_g = \mathrm{E}_{0_n, \mathrm{I}_n}$, if $g = g_0$, where $g_0(r) = \mathrm{e}^{-r}\mathrm{I}_{(0,\infty)}(r)$, $r \in \mathbb{R}$.

In the following we will study some properties of the multivariate ℓ_1-norm symmetric measure $\mathrm{E}_g(A)$ of certain sets A, which are elements of the n-dimensional Borel σ-field $\mathfrak{B}^n$, where A, e.g., can be thought of as being generated by the distribution function of a statistic T,

$$A = A(t) = \{\mathbf{x} \in \mathbb{R}^n_+ : T(x) < t\}.$$

This geometric method allows a unified approach to statistical problems for spherical and multivariate ℓ_1-norm symmetric sample distributions; linear and nonlinear models can be treated and central and noncentral distributions can be derived; see Richter (1995). This exceeds the possibilities actually provided by the analytic method given in Fang *et al.* (1990).

DEFINITION 1. Let $A \in \mathfrak{B}^n$. The function

$$\mathscr{F}(A, r) = v\left(\frac{1}{r}\,A \cap S^1_n(1)\right) = v\left(\frac{1}{r}\,(A \cap S^1_n(r))\right), \qquad r > 0$$

will be called the simplicial intersection percentage function $\mathscr{F}$ of the set A, see Fig. 1.

THEOREM 2 (Geometric measure representation formula). *Multivariate ℓ_1-norm symmetric distributions with density generating function g allow the representation*

$$\mathrm{E}_g(A) = \int_0^\infty \mathscr{F}(A, r)\, r^{n-1} g(r)\,\mathrm{d}r \bigg/ \int_0^\infty r^{n-1} g(r)\,\mathrm{d}r, \qquad A \in \mathfrak{B}^n.$$

Proof. Denote the probability distribution induced by a random variable Y by P^Y. Then

$$\mathrm{E}_g(A) = \int_A P^{\mathrm{X}}(\mathrm{d}\mathbf{x}) = \iint \mathrm{I}_A(r\mathbf{u})\, P^{\mathrm{U}, R}(\mathrm{d}(\mathbf{u}, r))$$

$$= \int_0^\infty \int_{S^1_n(1)} \mathrm{I}_A(r\mathbf{u})\, v(\mathrm{d}\mathbf{u})\, f(r)\,\mathrm{d}r$$

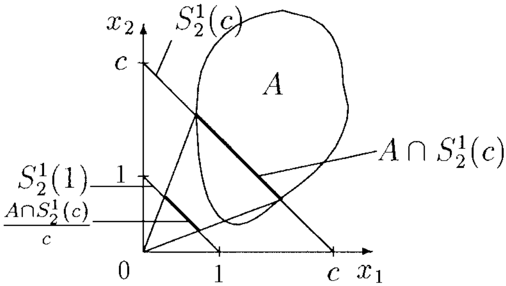

FIG. 1. Central projection onto the ℓ_1-norm unit sphere $S_2^1(1)$.

holds for $\mathbf{X} \sim \mathrm{E}_g$, where the latter equality is true because R and $\mathbf{U}$ are independent. Recognize that for $r > 0$

$$\int_{S_n^1(1)} \mathrm{I}_A(r\mathbf{u})\, \nu(\mathrm{d}\mathbf{u}) = \int_{S_n^1(1)} \mathrm{I}_{\frac{1}{r}A}(\mathbf{u})\, \nu(\mathrm{d}\mathbf{u}) = \nu\left(\frac{1}{r} A \cap S_n^1(1)\right) = \mathscr{F}(A, r).$$

Consequently,

$$P^{\mathbf{X}}(A) = \int_0^\infty \mathscr{F}(A, r)\, f(r)\, \mathrm{d}r$$

which leads with (3) to

$$\mathrm{E}_g(A) = \frac{c(n, g)}{\Gamma(n)} \int_0^\infty \mathscr{F}(A, r)\, r^{n-1} g(r)\, \mathrm{d}r \tag{4}$$

and with Eq. (2) follows the assertion. ∎

EXAMPLE 3. Assume that $\mathscr{F}(A, r)$ does not depend on r, i.e., $\mathscr{F}(A, r) = \mathscr{F}_0$, for all $r > 0$. This can be the case, e.g., when A is a cone in $\mathbb{R}_+^n$ with the apex in the origin. Then

$$\mathrm{E}_g(A) = \mathscr{F}_0 \frac{c(n, g)}{\Gamma(n)} \int_0^\infty r^{n-1} g(r)\, \mathrm{d}r = \mathscr{F}_0,$$

i.e., the measure of A does not depend on the concrete density generating function g.

For the investigation of $\mathscr{F}(A, r)$ it is useful to define a new coordinate system which is in a certain sense an analogue to the spherical coordinate system. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard vector base of the Cartesian coordinate system,

$$\mathbf{e}_1 := (1, 0, \ldots, 0)', \ldots, \mathbf{e}_n := (0, \ldots, 0, 1)',$$

and put

$$\mathbf{x}_i := x_i \mathbf{e}_i, \qquad i = 1, \ldots, n, \qquad \text{for} \quad \mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n.$$

DEFINITION 4. The (almost one-to-one) map Z, $Z \mid \mathbb{R}_+^n \to \mathbb{R}_+ \times [0,1]^{n-1}$, defined by

$$r(x_1, ..., x_n) := \|\mathbf{x}\|_1$$

and

$$\lambda_i(x_1, ..., x_n) := \frac{\|\mathbf{x}_i\|_1}{\|\mathbf{x} - \mathbf{x}_1 - \cdots - \mathbf{x}_{i-1}\|_1} = \frac{\|\mathbf{x}_i\|_1}{\|\mathbf{x}_i + \cdots + \mathbf{x}_n\|_1}, \qquad i = 1, ..., n-1,$$

transforms Cartesian coordinates into new coordinates which will be called simplex coordinates.

For the Jacobian J_Z of the transformation Z one gets

$$J_Z := \frac{D(x_1, ..., x_n)}{D(r, \lambda_1, ..., \lambda_{n-1})} = (-r)^{n-1} \prod_{i=1}^{n-2} (1-\lambda_i)^{n-1-i}.$$

Remark 5. The random analogues R and Λ_i, $i = 1, ..., n-1$ are the known decomposition of the random vector $\mathbf{X} \sim \mathrm{E}_{g_0}$ into independent parts, where R is $\mathrm{Ga}(n, 1)$-distributed and the Λ_i are Beta distributed with parameters 1 and $n-i$, $\Lambda_i \sim \mathrm{Be}(1, n-i)$.

This statement also holds true in the more general case $\mathbf{X} \sim \mathrm{E}_g$ by means of Fang *et al.* (1990, Theorem 1.4), but R has distribution function F.

EXAMPLE 6. Let $\mathfrak{M} = \mathfrak{L}(\mathbf{e}_1, ..., \mathbf{e}_m)$ be the m-dimensional linear subspace of $\mathbb{R}^n$, $0 < m < n$, which is spanned by the first m standard base vectors of the Cartesian coordinate system and $\mathfrak{M}^\perp$ its orthogonal complement. For $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_1 = r = r(\mathbf{x})$, let

$$r_1 := r_1(\mathbf{x}) := \|\Pi_{\mathfrak{M}} \mathbf{x}\|_1, \qquad r_2 := r_2(\mathbf{x}) := \|\Pi_{\mathfrak{M}^\perp} \mathbf{x}\|_1$$

be the ℓ_1-norms of the orthogonal projections of $\mathbf{x}$ onto $\mathfrak{M}$ and $\mathfrak{M}^\perp$, respectively. Then $r = r_1 + r_2$ and the cone

$$C^1_{\mathfrak{M}}(\lambda_0) = \left\{ \mathbf{x} \in \mathbb{R}_+^n : \frac{r_1}{r} < \lambda_0 \right\}$$

satisfies for all $0 < \lambda_0 < 1$ the assumption of the preceding example. We now determine the actual value of $\mathscr{F}_0 = \mathscr{F}_0(\lambda_0)$. Just as it has been done for measuring double cones with Gaussian measures using spherical coordinates in Richter (1991), we now introduce new simplex coordinates separately in the subspaces $\mathfrak{M}$ and $\mathfrak{M}^\perp$ and in the two dimensional subspace $\mathfrak{L} = \mathfrak{L}(\Pi_{\mathfrak{M}} \mathbf{x}, \Pi_{\mathfrak{M}^\perp} \mathbf{x})$ connecting them.

Let

$$Z_1 \mid (x_1, \ldots, x_m) \to (r_1, \lambda_1, \ldots, \lambda_{m-1}),$$

$$J_{Z_1} = (-r_1)^{m-1} \prod_{i=1}^{m-2} (1-\lambda_i)^{m-1-i},$$

$$Z_2 \mid (x_{m+1}, \ldots, x_n) \to (r_2, \lambda_{m+1}, \ldots, \lambda_{n-1}),$$

$$J_{Z_2} = (-r_2)^{n-m-1} \prod_{i=1}^{n-m-2} (1-\lambda_{m+i})^{n-m-1-i}$$

and

$$Z_{1,2} \mid (r_1, r_2) \to (r, \lambda_m), \qquad J_{Z_{1,2}} = -r$$

denote the announced transformations in $\mathfrak{M}$, $\mathfrak{M}^\perp$ and $\mathfrak{L}$ with their Jacobians, respectively. Furthermore, let

$$Z \mid (x_1, \ldots, x_n) \to (r, \lambda_1, \ldots, \lambda_{m-1}, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{n-1})$$

be the composition of Z_1, Z_2 and $Z_{1,2}$. It can be checked that

$$J_Z = J_{Z_1} J_{Z_2} J_{Z_{1,2}}.$$

In the new coordinates we get with $r_1(\mathbf{x}) = r(\mathbf{x})\, \lambda_m(\mathbf{x})$

$$C^1_{\mathfrak{M}}(\lambda_0) = \left\{ \frac{r_1}{r} < \lambda_0 \right\} = \{\mathbf{x} \in \mathbb{R}^n_+ : \lambda_m(\mathbf{x}) < \lambda_0\}.$$

The content of the surface area of $S^1_n(1)$ equals $1/\Gamma(n)$. Therefore and because of the above substitutions $r_1 = \lambda_m r$, $r_2 = (1-\lambda_m)\, r$ it follows for all $r > 0$

$$\frac{1}{\Gamma(n)} \mathscr{F}(C^1_{\mathfrak{M}}(\lambda_0), r) = \int_D \frac{J_{Z_1} J_{Z_2} J_{Z_{1,2}}}{(-r)^{n-1}} \, d\lambda_1 \cdots d\lambda_{n-1},$$

where the domain of integration is

$$D = \{(\lambda_1, \ldots, \lambda_{n-1}) \in [0, 1]^{\times(m-1)} \times [0, \lambda_0] \times [0, 1]^{\times(n-m-1)}\}.$$

Because of

$$\int \cdots \int_{[0,1]^{\times(m-1)}} \frac{J_{Z_1}}{(-r_1)^{m-1}} \, d\lambda_1 \cdots d\lambda_{m-1} = \frac{1}{\Gamma(m)},$$

an analogous result for the integral over J_{Z_2} and

$$r_1^{m-1} r_2^{n-m-1} = r^{n-2} \lambda_m^{m-1} (1-\lambda_m)^{n-m-1}$$

it follows that

$$\mathscr{F}_0(\lambda_0) = \frac{\Gamma(n)}{\Gamma(m)\,\Gamma(n-m)} \int_0^{\lambda_0} \lambda_m^{m-1}(1-\lambda_m)^{n-m-1}\,\mathrm{d}\lambda_m, \qquad 0<\lambda_0<1. \tag{5}$$

Remark 7. Formula (5) for $0<\lambda_0<1$ is the distribution function of a $\mathrm{Be}(m, n-m)$ distributed random variable; hence

$$\frac{\|\Pi_{\mathfrak{m}}\mathbf{X}\|_1/m}{\|\Pi_{\mathfrak{M}^\perp}\mathbf{X}\|_1/(n-m)}$$

is $\mathrm{F}(2m, 2(n-m))$ distributed. These facts illustrate known results in Fang *et al.* (1990) for so-called robust statistics, statistics which are independent of the density generating function g; i.e., if $\mathbf{X} \sim \mathrm{E}_{g_1}$ and $\mathbf{Y} \sim \mathrm{E}_{g_2}$ then $T(\mathbf{X}) \stackrel{d}{=} T(\mathbf{Y})$. This property coincides with a constant intersection percentage function of the sets $A(t) = \{T<t\}$, $t>0$.

In the following we will study possibly non-constant simplicial intersection percentage functions for sets from a certain system of Borel sets in $\mathbb{R}^n$. This system of sets is a suitable modification of the system of sets $\mathfrak{A}(\mathrm{dir}, \mathrm{dist})$ which has been studied for the spherical case in Richter (1995).

DEFINITION 8. Let $\mathfrak{M}$ be a m-dimensional subspace of $\mathbb{R}^n$ and $C^1_{\mathfrak{M}}(\lambda(r))$ for all $r>0$ a cone as described in Example 6.

$\mathfrak{A}^1_m \subset \mathfrak{B}^n$ is then defined to be a system of sets whose elements A satisfy the following conditions:

($\mathfrak{A}$1) $A \subset \mathbb{R}^n_+$,

($\mathfrak{A}$2) There is a function $\lambda^* \mid \mathbb{R}_+ \to [0, 1]$ such that

$$A \cap S^1_n(r) = C^1_{\mathfrak{M}}(\lambda^*(r)) \cap S^1_n(r), \qquad r>0.$$

The following theorem is determined by consideration, which is similar to those of Example 6, (5) and from (4).

THEOREM 9. *If $A \in \mathfrak{A}^1_m$ then for $\lambda^*(r)$ from assumption* ($\mathfrak{A}$2)

$$\mathscr{F}(A, r) = \frac{\Gamma(n)}{\Gamma(m)\,\Gamma(n-m)} \int_0^{\lambda^*(r)} \lambda^{m-1}(1-\lambda)^{n-m-1}\,\mathrm{d}\lambda, \qquad r>0,$$

holds and for all g satisfying condition (1)

$$\mathrm{E}_g(A) = \frac{c(n, g)}{\Gamma(n)} \int_0^\infty r^{n-1} g(r)\,\mathscr{F}(A, r)\,\mathrm{d}r$$

holds.

The step of generalizing spherical distributions to elliptically contoured distributions can be adapted to the present case of regular simplicial or ℓ_1-norm symmetric distributions in a somewhat restricted form; see Fang *et al.* (1990) Problem 5.8. There is no name provided for this generalization of multivariate ℓ_1-norm symmetric distributions, so we propose here to call the probability laws of random vectors $\mathbf{W} = \theta_n + \Sigma_n \mathbf{X}$ with $\theta_n \in \mathbb{R}^n$, $\Sigma_n = \operatorname{diag}(\sigma_1, ..., \sigma_n)$, $\sigma_i > 0$, $i = 1, ..., n$ and for which $\mathbf{X}$ has a ℓ_1-norm symmetric or regular simplicial distribution, $\mathbf{X} \in L_n(F)$, simplicially contoured and to denote it by $\mathrm{E}_{\theta_n, \Sigma_n; F}$. That means that we restrict our consideration to shifting and scaling with a diagonal matrix. If a density generating function g satisfying (1) exists, we will make use of the notation $\mathrm{E}_{\theta_n, \Sigma_n; g}$ instead of $\mathrm{E}_{\theta_n, \Sigma_n; F}$.

COROLLARY 10. *It holds that*

$$\mathrm{E}_{\theta_n, \Sigma_n; g}(A) = \frac{c(n, g)}{\Gamma(n)} \int_0^\infty \mathscr{F}(\Sigma_n^{-1}(A - \theta_n), r)\, r^{n-1} g(r)\, \mathrm{d}r, \qquad A \in \mathfrak{B}^n.$$

Because of this formula, the geometric method applies to quite different statistical problems in simplicially contoured sample distributions in the sense outlined before Definition 1. An example is given in the next section.

3. STATISTICAL INFERENCE FOR SIMPLICIALLY CONTOURED SAMPLE DISTRIBUTIONS

Let $\mathbf{X} = (X_1, ..., X_n)' \sim \mathrm{E}_g$. Then with $1_n = (1, ..., 1)'$ and $\mathbf{W} := \theta 1_n + \sigma \mathbf{X}$ it holds $\mathbf{W} \sim \mathrm{E}_{\theta 1_n, \sigma \mathrm{I}_n; g}$. In this section we will carry out statistical inference with respect to the expectation of the identically distributed, but dependent, except for the exponential case, one dimensional marginals W_i, $i = 1, ..., n$. The modified Student statistic $T^E \mid \mathbb{R}^n \to \mathbb{R}$,

$$T^E(\mathbf{W}) = \sqrt{n}\, \frac{\bar{W}_n - \mu_0}{\bar{W}_n - W_{(1)}}, \qquad \text{where} \quad \bar{W}_n = \frac{1}{n} \sum_{i=1}^n W_i, \qquad W_{(1)} = \min_i W_i,$$

was introduced in Davids and Richter (1991) for carrying out statistical inference with respect to the expectation in the exponential case $g = g_0$, solving a problem described in Hotelling (1961).

We assume for the further course that the expectation of the generating variate R exists, $\mathbb{E}R < \infty$. Let be $\kappa_g(n) := \mathbb{E}X_i$ the expectation of X_i; then by Fang *et al.* (1990)

$$\kappa_g(n) = \frac{\mathbb{E}R}{n}.$$

This implies that $\mathbb{E}W_i = \theta + \sigma \kappa_g(n)$, $i = 1, ..., n$.

In the following we shall make use of the notation $\mu(n) = \theta + \sigma\kappa_g(n)$.

DEFINITION 11. Let there be a function $g \mid \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$0 < \kappa = \kappa_g(n+1) = \frac{\int_0^\infty r^{n+1} g(r)\, dr}{(n+1) \int_0^\infty r^n g(r)\, dr} < \infty.$$

A random variable Y is said to be distributed according to the g-generalized modified Student distribution $t^E_{n,\,\delta;\,g}$ with n, $n \in \mathbb{N}$, degrees of freedom, noncentrality parameter δ and density generating function g if its density $f_Y(t)$ satisfies the representation

$$f_Y(t) = \frac{c(n+1, g)}{\Gamma(n)} \frac{\sqrt{n+1}^{\,n}}{t^{n+1}} \times \begin{cases} \int_0^\infty r^n g\left(r + (n+1)\left(\kappa - \frac{\delta}{\sqrt{n+1}}\right)\right) dr, \\ \qquad \sqrt{n+1} \leqslant t < \infty, \\ \int_0^{[t(n+1)(\kappa - (\delta/\sqrt{n+1}))]/[\sqrt{n+1} - t]} r^n g\left(r + (n+1)\left(\kappa - \frac{\delta}{\sqrt{n+1}}\right)\right) dr, \\ \qquad -\infty < t < \sqrt{n+1}, \qquad t \neq 0, \end{cases}$$

for $\delta/\sqrt{n+1} < \kappa$,

$$f_Y(t) = \begin{cases} n \dfrac{\sqrt{n+1}^{\,n}}{t^{n+1}}, & \sqrt{n+1} \leqslant t < \infty, \\ 0, & \text{otherwise}, \end{cases}$$

for $\delta/\sqrt{n+1} = \kappa$ and

$$f_Y(t) = \begin{cases} \dfrac{c(n+1, g)}{\Gamma(n)} \dfrac{\sqrt{n+1}^{\,n}}{t^{n+1}} \displaystyle\int_{(t(n+1)(\kappa - \frac{\delta}{\sqrt{n+1}}))/(\sqrt{n+1} - t)}^\infty r^n \\ \qquad \times g\left(r + (n+1)\left(\kappa - \dfrac{\delta}{\sqrt{n+1}}\right)\right) dr, \\ \qquad \sqrt{n+1} \leqslant t < \infty, \\ 0, \qquad \text{otherwise}, \end{cases}$$

for $\delta/\sqrt{n+1} > \kappa$.

In the exponential case, $g = g_0$, the random variable Y is said to be distributed according to the modified Student distribution $t^E_{n,\delta}$ with n degrees of freedom and noncentrality parameter δ; see Davids and Richter (1991).

Remark 12. For the limit of $f_Y(t)$ at $t = 0$ for $\delta/\sqrt{n+1} < \kappa$ one gets

$$\lim_{t\to 0} f_Y(t) = \frac{(n+1)^{n-\frac{1}{2}}\left(\kappa - \frac{\delta}{\sqrt{n+1}}\right)^{n+1}}{\Gamma(n)}\, g\left((n+1)\left(\kappa - \frac{\delta}{\sqrt{n+1}}\right)\right).$$

THEOREM 13. *Let* $\mathbf{W}$ *be simplicially contoured distributed,* $\mathbf{W} \sim \mathrm{E}_{\theta 1_n, \sigma \mathrm{I}_n; g}$, $\theta \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$. *Then* $T^E(\mathbf{W}) = \sqrt{n}\,(\bar{W}_n - \mu_0(n)/(\bar{W}_n - W_{(1)}))$ *is g-generalized modified Student distributed* $t^E_{n-1,\delta;g}$ *with* $n-1$ *degrees of freedom, noncentrality parameter* $\delta = \sqrt{n}\,(\mu(n) - \mu_0(n)/\sigma)$ *and density generating function g.*

Proof. We determine the distribution function.

$$\begin{aligned} F_{T^E(\mathbf{W})}(t) &= P^{\mathbf{W}}(T^E(\mathbf{W}) < t) \\ &= \int_\theta^\infty \cdots \int_\theta^\infty \mathrm{I}\left(\sqrt{n}\,\frac{\bar{w}_n - \mu_0(n)}{\bar{w}_n - w_{(1)}} < t\right) \\ &\quad \times \frac{c(n, g)}{\sigma^n}\, g\left(\frac{n}{\sigma}(\bar{w}_n - \theta)\right) \mathrm{d}w_1 \cdots \mathrm{d}w_n. \end{aligned}$$

Changing variables $v_i := (w_i - \theta)/\sigma$, $i = 1, \ldots, n$ gives

$$F_{T^E(\mathbf{W})}(t) = \int_0^\infty \cdots \int_0^\infty \mathrm{I}\left(\sqrt{n}\,\frac{\bar{v}_n - v}{\bar{v}_n - v_{(1)}} < t\right) c(n, g)\, g(n\bar{v}_n)\, \mathrm{d}v_1 \cdots \mathrm{d}v_n,$$

where

$$v := \frac{\mu_0(n) - \theta}{\sigma} = \frac{\kappa_g(n)\,\sigma - (\theta + \kappa_g(n)\,\sigma - \mu_0(n))}{\sigma} = \kappa_g(n) - \frac{\delta}{\sqrt{n}}.$$

Analogously to Sukhatme (1937) we change first to the ordered values $v_{(1)} \leqslant \cdots \leqslant v_{(n)}$ with the Jacobian $\frac{1}{n!}$ and then to the normalized spacings

$$z_i := (n - i + 1)(v_{(i)} - v_{(i-1)}), \qquad i = 1, \ldots, n, \qquad v_{(0)} := 0$$

with the Jacobian $n!$ and get

$$\begin{aligned} F_{T^E(\mathbf{W})}(t) &= \int_0^\infty \cdots \int_0^\infty \mathrm{I}\left(\sqrt{n}\,\frac{\sum_{i=1}^n z_i - nv}{\sum_{i=1}^n z_i - z_1} < t\right) c(n, g)\, g\left(\sum_{i=1}^n z_i\right) \mathrm{d}z_1 \cdots \mathrm{d}z_n \\ &= \mathrm{E}_g(A(t)), \end{aligned}$$

where

$$A(t) := \left\{ \mathbf{z} \in \mathbb{R}_+^n : \sqrt{n} \frac{\|\mathbf{z}\|_1 - nv}{\|\mathbf{z}\|_1 - z_1} < t \right\}.$$

We will consider first the case $\delta/\sqrt{n} < \kappa_g(n)$, i.e., $v > 0$.

From $A(0) = \{\mathbf{z} \in \mathbb{R}_+^n : \|\mathbf{z}\|_1 < nv\}$ it follows that $\mathscr{F}(A(0), r) = \mathrm{I}_{(0, nv)}(r)$. Theorem 2 applies and gives

$$F_{T^E(\mathbf{W})}(0) = \int_0^{nv} r^{n-1} g(r)\, dr \Big/ \int_0^{\infty} r^{n-1} g(r)\, dr.$$

For applying Theorem 9 in the case $t \neq 0$ we choose $\mathfrak{M} = \mathfrak{L}(\mathbf{e}_1)$, the linear subspace spanned by $\mathbf{e}_1$. With this choice we have that $z_1 = r_1 = \|\Pi_{\mathfrak{M}} \mathbf{z}\|_1$ and

$$A(t) = \left\{ \mathbf{z} \in \mathbb{R}_+^n : \sqrt{n} \left(1 - \frac{nv}{r} \right) < t \left(1 - \frac{r_1}{r} \right) \right\}.$$

Let now $t > 0$. With the notation $\lambda_t(r) = 1 - (\sqrt{n}/t)(1 - \frac{nv}{r})$ it follows that

$$A(t) = \left\{ \mathbf{z} \in \mathbb{R}_+^n : \frac{r_1}{r} < \lambda_t(r) \right\}.$$

Note that because of $v > 0$ in the case $0 < t < \sqrt{n}$ we have $\lambda_t(r) \leqslant 0$ for $r \geqslant r^0$ and $\lambda_t(r) \geqslant 1$ for $r \leqslant r^1$, where

$$0 < r^1 = nv < r^0 = \frac{\sqrt{n}}{\sqrt{n - t}} nv < \infty.$$

Consequently, if $0 < t < \sqrt{n}$ then $\mathscr{F}(A(t), r) = 0$ if $r \geqslant r^0$ and $\mathscr{F}(A(t), r) = 1$ if $r \leqslant r^1$; in the case $0 < \lambda_t(r) < 1$, i.e., if $r^1 < r < r^0$, we have for fixed t

$$A(t) \cap S_n^1(1) = C_{\mathfrak{M}}(\lambda_t(r)) \cap S_n^1(1),$$

such that $A(t) \in \mathfrak{A}_{1,n-1}^1$. For the set $A(t) \in \mathfrak{A}_{1,n-1}^1$ holds by Theorem 9

$$\mathscr{F}(A(t), r) = 1 - (1 - \lambda_t(r))^{n-1}.$$

Because of $v > 0$ for $0 < t < \sqrt{n}$ one gets

$$F_{T^E(\mathbf{W})}(t) = \frac{c(n, g)}{\Gamma(n)} \left(\int_0^{r^1} r^{n-1} g(r)\, dr \right.$$
$$\left. + \int_{r^1}^{r^0} r^{n-1} g(r) \left(1 - \left(\frac{\sqrt{n}}{t} \left(1 - \frac{nv}{r} \right) \right)^{n-1} \right) dr \right).$$

Using the fact that $\mathbb{R}_+^n \setminus A(t) \in \mathfrak{A}_{1,n-1}^1$ the formula for $-\infty < t < 0$ turns out to be just the same.

Let now $t \geqslant \sqrt{n}$. Then $r^0 < 0 < r^1$ and $\lambda_t(r) \geqslant 1$ iff $0 < r \leqslant r^1$; in the case $0 < \lambda_t(r) < 1$, i.e., if $r > r^1$ we have for fixed t

$$A(t) \cap S_n^1(1) = C_{\mathfrak{M}}(\lambda_t(r)) \cap S_n^1(1),$$

such that $A(t) \in \mathfrak{A}_{1,n-1}^1$. In the case $0 < r \leqslant r^1$ it holds that $\mathscr{F}(A(t), r) = 1$.

In this way the distribution function of the statistic $T^E(\mathbf{W})$ is received for $v > 0$.

$$F_{T^E(\mathbf{W})}(t) = \begin{cases} 1 - \dfrac{c(n,g)}{\Gamma(n)} \left(\dfrac{\sqrt{n}}{t}\right)^{n-1} \displaystyle\int_{nv}^{\infty} (r-nv)^{n-1} g(r)\, \mathrm{d}r, \\ \qquad \sqrt{n} \leqslant t < \infty \\ \dfrac{c(n,g)}{\Gamma(n)} \displaystyle\int_0^{r^0} r^{n-1} g(r)\, \mathrm{d}r + \dfrac{c(n,g)}{\Gamma(n)} \left(\dfrac{\sqrt{n}}{t}\right)^{n-1} \displaystyle\int_{r^0}^{nv} (r-nv)^{n-1} g(r)\, \mathrm{d}r, \\ \qquad -\infty < t < \sqrt{n}, \qquad t \neq 0 \\ \dfrac{c(n,g)}{\Gamma(n)} \displaystyle\int_0^{nv} r^{n-1} g(r)\, \mathrm{d}r, \\ \qquad t = 0. \end{cases}$$

If one takes the derivative of the distribution function with respect to t one gets for $v > 0$ the density formula

$$f_{T^E(\mathbf{W})}(t) = \begin{cases} \dfrac{c(n,g)}{\Gamma(n-1)} \dfrac{1}{\sqrt{n}} \left(\dfrac{\sqrt{n}}{t}\right)^{n} \displaystyle\int_{nv}^{\infty} (r-nv)^{n-1} g(r)\, \mathrm{d}r, \\ \qquad \sqrt{n} \leqslant t < \infty \\ \dfrac{c(n,g)}{\Gamma(n-1)} \dfrac{1}{\sqrt{n}} \left(\dfrac{\sqrt{n}}{t}\right)^{n} \displaystyle\int_{nv}^{r^0} (r-nv)^{n-1} g(r)\, \mathrm{d}r, \\ \qquad -\infty < t < \sqrt{n}, \qquad t \neq 0 \\ \dfrac{c(n,g)}{\Gamma(n-1)} \dfrac{1}{\sqrt{n}} (nv)^n \dfrac{g(nv)}{n}, \\ \qquad t = 0. \end{cases}$$

The case $v = 0$ is considered now. One has a fixed $\lambda_t^0 = 1 - \sqrt{n}/t$ for which $0 < \lambda_t^0 < 1$ holds if $t > \sqrt{n}$. Hence one has by Example 3

$$F_{T^E(\mathbf{W})}(t) = \begin{cases} \mathscr{F}_0 = 1 - \left(\dfrac{\sqrt{n}}{t}\right)^{n-1}, & \sqrt{n} < t < \infty \\ 0, & \text{otherwise.} \end{cases}$$

The case $v < 0$ is similar to the case $v > 0$ and is hence left to the reader. ▮

Remark 14. The T^E-statistic can be considered in the following way. For the orthogonal projection of $\mathbf{w}$ onto the linear space generated by the vector 1_n one gets

$$\Pi_{\mathfrak{L}(1_n)} \mathbf{w} = \bar{w}_n 1_n.$$

Hence $\mathbf{w}$ can be decomposed into $\bar{w}_n 1_n$ and $\mathbf{w} - \bar{w}_n 1_n$. For the ℓ_1-norm of $\bar{w}_n 1_n$ it holds that

$$\|\bar{w}_n 1_n\|_1 = n\bar{w}_n = \|\mathbf{w}\|_1 = \sum_{i=1}^{n} w_i = \langle \mathbf{w}, 1_n \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. Let us consider now the Minkowski functional

$$h_{K^E}(\mathbf{x}) := \inf\{\lambda > 0 : \mathbf{x} \in \lambda K^E\}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where $K^E := K^* - 1_n$ and

$$K^* := \left\{ \mathbf{x} \in \mathbb{R}^n : x_i > 0, i = 1, ..., n; \sum_{i=1}^{n} x_i = n \right\}.$$

At the point $\mathbf{w} - \bar{w}_n 1_n$ one gets

$$h_{K^E}(\mathbf{w} - \bar{w}_n 1_n) := \bar{w}_n - w_{(1)}.$$

Consequently,

$$T^E(\mathbf{W}) = \frac{\|\mathbf{W} - \mu_0 1_n\|_1}{\sqrt{n}\, h_{K^E}(\mathbf{W} - \bar{W}_n 1_n)}.$$

For details see Richter (1995).

Asymptotic statistical inference in simplicially contoured sample distributions starts with considering the asymptotic behaviour of the T^E-statistic. For simplicity we restrict our consideration to the case of multivariate ℓ_1-norm symmetric distributions with a density generating function of Kotz-type; i.e., $g(r) = r^{k-1} e^{-tr^s} \mathrm{I}_{(0,\infty)}(r)$, $s, t > 0$, $k + n > 1$. Let us assume that the null hypothesis

$$H_0 : \mu(n) = \mu_0(n)$$

is true for all n of interest. One has for the variance $\iota_g^2(n)$ of X_i,

$$\iota_g^2(n) = 2\frac{\mathbb{E}R^2}{n(n+1)} - \frac{(\mathbb{E}R)^2}{n^2}$$

and $\operatorname{Var}(W_i) = \sigma^2 \imath_g^2(n)$. It holds that

$$\frac{\mathbb{E}R^2}{(\mathbb{E}R)^2} = 1 + \frac{1}{s(k+n)} + O\left(\frac{1}{n^2}\right).$$

From that it follows that

$$\lim_{n\to\infty} \left(\frac{\imath_g(n)}{\kappa_g(n)}\right)^2 = \lim_{n\to\infty} 2\frac{n}{n+1}\frac{\mathbb{E}R^2}{(\mathbb{E}R)^2} - 1 = 1$$

and

$$\lim_{n\to\infty} \frac{\operatorname{Var} R}{n \operatorname{Var} X_1} = \lim_{n\to\infty} \left(\frac{\kappa_g(n)}{\imath_g(n)}\right)^2 n \left(\frac{\mathbb{E}R^2}{(\mathbb{E}R)^2} - 1\right) = \frac{1}{s}.$$

Remark 15. In the exponential case, $g = g_0$, one has that μ and μ_0 do not depend on n and that $\kappa_g(n) = 1$ and $\imath_g^2(n) = 1$ hold.

THEOREM 16. *If the above assumptions are satisfied, then the distribution of the T^E-statistic converges weakly to the normal distribution with mean* 0 *and variance* $\frac{1}{s}$.

Proof. The T^E-statistic can be decomposed under H_0 in the following way:

$$T^E(\mathbf{W}) = \sqrt{n}\frac{\bar{W}_n - \mu}{\imath_g(n)\,\sigma}\frac{\kappa_g(n)\,\sigma}{\bar{W}_n - W_{(1)}}\frac{\imath_g(n)}{\kappa_g(n)}.$$

Consider the characteristic function of the first term. It can be written as

$$\phi_{\sqrt{n}\frac{\bar{W}_n-\mu}{\imath_g(n)\sigma}}(t) = \exp\left\{-it\sqrt{n}\frac{\kappa_g(n)}{\imath_g(n)}\right\}\phi_R\left(\frac{t}{\imath_g(n)\sqrt{n}}\right), \qquad t \in \mathbb{R}$$

with

$$\phi_R\left(\frac{t}{\imath_g(n)\sqrt{n}}\right) = \exp\left\{it\sqrt{n}\frac{\kappa_g(n)}{\imath_g(n)} - \frac{t^2 \operatorname{Var}(R)}{2\imath_g^2(n)\,n} + R_n(t)\right\}$$

and

$$R_n(t) \to 0 \qquad \text{as} \quad n \to \infty.$$

Consequently,

$$\phi_{\sqrt{n}\frac{\bar{W}_n-\mu}{\imath_g(n)\sigma}}(t) \xrightarrow{n\to\infty} \exp\left\{-\frac{t^2}{2}\frac{1}{s}\right\}.$$

One can write the reciprocal of the second term in the following form

$$\frac{\bar{W}_n - W_{(1)}}{\kappa_g(n)\,\sigma} = 1 + \frac{\bar{W}_n - \mu(n)}{\kappa_g(n)\,\sigma} - \frac{W_{(1)} - \theta}{\kappa_g(n)\,\sigma}.$$

Because of $\mathbf{W} = \theta\,1_n + \sigma\mathbf{X}$ and $\sum_{i=1}^n X_i = R$ one gets

$$\mathbb{E}\bar{W}_n = \theta + \frac{\sigma}{n}\,\mathbb{E}R, \qquad \operatorname{Var}\bar{W}_n = \frac{\sigma^2}{n^2}\operatorname{Var}R.$$

By $nX_{(1)} \stackrel{d}{=} X_1$, see Fang *et al.* (1990), one has

$$\mathbb{E}W_{(1)} = \theta + \frac{\sigma}{n}\,\kappa_g(n), \qquad \operatorname{Var}W_{(1)} = \frac{\sigma^2}{n^2}\,\iota_g^2(n).$$

By means of Tchebycheff's inequality one gets that $(\bar{W}_n - \mu(n))/(\kappa_g(n)\,\sigma)$ and $(W_{(1)} - \theta)/(\kappa_g(n)\,\sigma)$ tend in probability to 0. ∎

ACKNOWLEDGMENTS

The authors are grateful to the referees whose comments led to a more comprehensive and richer in content version of this paper.

REFERENCES

J. Davids and W.-D. Richter, Exakter Test für erwartete Lebensdauer bei unbekannter Mindestlebensdauer, *Z. Klin. Med.* **46** (1991), 789–791.

B.-Q. Fang and K.-T. Fang, Some families of multivariate symmetric distributions related to exponential distributions, *J. Multivariate Anal.* **24** (1988), 109–122.

K.-T. Fang, S. Kotz, and K. W. Ng, "Symmetric Multivariate and Related Distributions," Chapman and Hall, London/New York, 1990.

A. K. M. S. Hoq, H. M. Ali, and J. G. C. Templeton, The distribution of student's ratio for samples from an exponential population, *Comm. Statist. Theory Meth. A* **7** (1978), 837–850.

H. Hotelling, The behaviour of some standard statistical tests under non-standard conditions, *in* "Proc. 4th Berkeley Symp. Math. Stat. Prob.," Vol. 1, pp. 319–360, University of California Press, Berkeley, 1961.

W.-D. Richter, Eine geometrische Methode in der Stochastik, *Rostock. Math. Kolloquium* **44** (1991), 63–72.

W.-D. Richter, A geometric approach to the Gaussian law, *in* "Symposia Gaussiana, Conf. B" (V. Mammitsch and H. Schneeweiß, Eds.), pp. 25–45, de Gruyter, Berlin/New York, 1995.

P. V. Sukhatme, Tests of significance for samples of the χ^2-population with two degrees of freedom, *Ann. Eugenics* **8** (1937), 52–66.