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Exact tests and confidence regions in nonlinear regression

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Nonlinear regression models with spherically symmetric error vectors and a single nonlinear parameter are considered. On the basis of a new geometric approach, exact one- and two-sided tests and confidence regions for the nonlinear parameter are derived in the cases of known and unknown error variances. A geometric measure representation formula is used to determine the power functions of the tests if the error variance is known and to derive different lower bounds for the power function of a one-sided test in the case of an unknown error variance. The latter can be done quite effectively by constructing and measuring several balls inside the critical region. A numerical study compares the results for different density generating functions of the error distribution.

Keywords: Nonlinear regression; Spherically symmetric errors; Exact tests; Confidence regions; Power functions; Exact distributions; Student distribution; Doubly non-central generalized t-distribution; Geometric representation formulae; Intersection percentage function; Heavy or light error tails

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1. Introduction

We consider the nonlinear regression model

$$\mathbf{Y} = \boldsymbol{\eta}(\gamma) + \sigma\mathbf{E}, \quad \gamma \in \Gamma = (a, b), \ -\infty \le a < b \le \infty, \ \sigma > 0,$$

where $\mathbf{Y}$ denotes the n-dimensional vector of the response variables and

$$\boldsymbol{\eta}(\gamma) = (f(x_1, \gamma), \ldots, f(x_n, \gamma))^{\mathrm{T}},$$

its mathematical expectation which depends on both the known experimental design point $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathcal{X}^n \subseteq \mathbb{R}^n$ and the unknown value of the one-dimensional nonlinear parameter γ of the nonlinear regression function $f(x, \gamma)$.

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Design points $\mathbf{x}$ including repeated measurements are assumed to be given as

$$\mathbf{x} = (\xi_1 \mathbf{1}_{n_1}^{\mathrm{T}}, \dots, \xi_m \mathbf{1}_{n_m}^{\mathrm{T}})^{\mathrm{T}}, \quad \mathbf{1}_{n_j} = (1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^{n_j}, \quad j = 1, \dots, m, \tag{1}$$

with $m \le n$, $\xi_1 < \cdots < \xi_m$, $\xi_j \in \mathcal{X} \subseteq \mathbb{R}$, $n_j \ge 1$, $j = 1, \dots, m$, and $n_1 + n_2 + \cdots + n_m = n$.

Given the experimental design point $\mathbf{x}$, let the nonlinear regression function $f(x, \gamma)$ satisfy the assumption:

(RA1) $\boldsymbol{\eta} \mid \Gamma \to \mathbb{R}^n$ is twice continuously differentiable with respect to γ.

The range of the mean value of the observation vector $\mathbf{Y}$, *i.e.*,

$$\mathfrak{M} = \{\boldsymbol{\eta}(\gamma) \colon \gamma \in \Gamma\}$$

is a curve in the sample space $\mathbb{R}^n$, which is called the expectation curve or the solution locus.

The error vector $\mathbf{E} \sim EC_n(\mathbf{0}_n, I_n, g)$ is assumed to follow a spherically symmetric distribution with a density

$$p(\mathbf{z}; g) = C(n, g) g(\|\mathbf{z}\|^2), \quad \mathbf{z} \in \mathbb{R}^n,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$, the density generating function $g \mid [0, \infty) \to [0, \infty)$ satisfies the condition

$$0 < I_{n+2,g} < \infty \tag{2}$$

with

$$I_{n,g} = \int_0^\infty r^{n-1} g(r^2)\,\mathrm{d}r,$$

and the norming constant $C(n, g)$ is defined as

$$C(n, g) = (\omega_n I_{n,g})^{-1}.$$

Here,

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

denotes the surface area of the unit sphere $S_n(\mathbf{0}_n, 1) = \{\mathbf{z} \in \mathbb{R}^n \colon \|\mathbf{z}\| = 1\}$ in $\mathbb{R}^n$.

Examples of density generating functions are the Gaussian density generator

$$g_{\mathrm{G}}(r) = \exp\left(\frac{-r}{2}\right), \quad r > 0,$$

the Kotz-type density generator

$$g_{\mathrm{K}}(r) = r^{M-1} \exp(-t r^s), \quad r > 0, \; t > 0, \; s > 0, \; 2M + n > 2,$$

and the Pearson-VII-type density generator

$$g_{\mathrm{P}}(r) = \left(1 + \frac{r}{m^*}\right)^{-M}, \quad r > 0, \; M > \frac{n}{2}, \; m^* > 0.$$

The corresponding norming constants are

$$C(n, g_{\mathrm{G}}) = (2\pi)^{-n/2}, \quad C(n, g_{\mathrm{K}}) = \frac{s t^{((2M+n-2)/(2s))} \Gamma(n/2)}{\pi^{n/2} \Gamma((2M+n-2)/(2s))}$$

and

$$C(n, g_{\mathrm{P}}) = \frac{\Gamma(M)}{(\pi m^*)^{n/2} \Gamma(M - n/2)}.$$

Spherically symmetric and more general elliptically symmetric distributions have been studied, *e.g.*, in refs. [1–9]. Nonlinear regression models with spherically symmetric error vectors are considered in ref. [10] and ref. [11, section 7.5.1].

The expectation vector and the covariance matrix of the elliptically symmetric distributed response vector **Y**,

$$\mathbf{Y} \sim EC_n(\boldsymbol{\eta}(\gamma), \sigma^2 I_n, g),$$

are due to condition (2) and Theorems 2.17 and 2.9 in ref. [6], and is given by

$$\mathbb{E}\mathbf{Y} = \boldsymbol{\eta}(\gamma) \quad \text{and} \quad \mathrm{Cov}(\mathbf{Y}) = \frac{C(n, g) I_{n+2,g}}{n} \sigma^2 I_n,$$

respectively.

Several methods of constructing size-α tests and confidence regions for the parameter $\gamma \in \Gamma \subset \mathbb{R}^p$, $p \geq 1$, can be found in the literature if the error vector **E** follows a Gaussian distribution. Local linear approximations to the curve $\mathfrak{M}$ are the basis of linear methods for the definition of confidence regions in refs. [12–15]. However, in ref. [16, p. 223], it is not recommended to use this methods.

The so-called almost exact confidence regions which are based on maximum likelihood estimators were proposed in ref. [17] for flat regression models. The words ‘almost exact’ mean that the regions under consideration are exact for a restricted sample space. In the present article, the whole sample space $\mathbb{R}^n$ is considered, but the regression functions are assumed to satisfy certain assumptions.

For the case of a known parameter σ and arbitrary density generating function g satisfying assumption (2), we derive exact α-tests for one-sided and two-sided alternatives in section 2. We determine the power-functions of all these tests and define the corresponding confidence regions. There are similarities to the considerations in refs. [10, 11, 17], concerning the methods of constructing confidence regions and evaluating the first-type error probabilities of the corresponding tests. The critical regions of the tests for known parameter σ are half-spaces in both approaches, but the respective normal vectors are chosen in different ways.

In Section 3, a new geometric approach will be developed to consider the more interesting case of unknown σ. We give size-α tests for the one-sided alternatives and also under some additional assumption, for the two-sided alternative and define the corresponding confidence regions. The critical regions of the new tests are modifications of the critical regions of the well-known Student test, which are rotationally symmetric single or double cones. These cones have symmetry axes that consist of the sets of all points from the sample space describing the one- or two-sided alternatives. The new critical regions are constructed in such a way that the tests reject the null hypotheses for sample points near to the respective alternatives. Several possibilities for the evaluation of lower bounds for the power function of one of these new tests are given in section 4.

There are also possibilities to derive a point estimator for the parameter γ by the help of the given two-sided confidence regions for γ, taking into account the parameter-effect curvature. If the two-sided confidence region for a concrete sample $\mathbf{y} \in \mathbb{R}^n$ is an interval (γ_l, γ_u), then one can choose that parameter value $\hat{\gamma}$ as a point estimator for γ which satisfies either the condition that the corresponding point $\boldsymbol{\eta}(\hat{\gamma})$ on the expected curve $\mathfrak{M}$ has the same euclidean distances from the points $\boldsymbol{\eta}(\gamma_l)$ and $\boldsymbol{\eta}(\gamma_u)$ on $\mathfrak{M}$ or the condition that the respective curve length is the same. In the case of unknown parameter σ, an estimator $\hat{\sigma}$ could be chosen as $\widehat{\sigma} := \|\mathbf{y} - \boldsymbol{\eta}(\hat{\gamma})\|$. To investigate the statistical properties of these point estimators, further work would be needed, which is not the aim of this article.

2. Known parameter σ^2

When testing the hypothesis H_0: $\gamma = \gamma_0$ about the unknown nonlinear parameter versus one of the one-sided alternatives H_{A_1}: $\gamma > \gamma_0$ or H_{A_2}: $\gamma < \gamma_0$, or versus the two-sided alternative H_A: $\gamma \neq \gamma_0$ in the case of known parameter σ^2, we shall make use of one of the following additional model assumptions or their combination, respectively:

(RA21) The functions $f(\xi_j, \cdot) \mid \Gamma_i \to \mathbb{R}$ are monotonous for all j, $j \in \{1, \dots, m\}$ as $\gamma, \gamma > \gamma_0$, approaches b, and there is at least one index j such that this monotony holds in the strong sense.

(RA22) The functions $f(\xi_j, \cdot) \mid \Gamma_i \to \mathbb{R}$ are monotonous for all j, $j \in \{1, \dots, m\}$, as γ, $\gamma < \gamma_0$, approaches a, and there is at least one index j such that this monotony holds in the strong sense.

2.1 *One-sided test problem*

This section deals with the construction of exact one-sided tests for proving

$$H_0\colon \gamma = \gamma_0 \quad \text{versus} \quad H_{A_1}\colon \gamma \in \Gamma_1 = (\gamma_0, b) \quad \text{or} \quad H_{A_2}\colon \gamma \in \Gamma_2 = (a, \gamma_0)$$

if the parameter σ is known.

Let a class of half-spaces in the sample space $\mathbb{R}^n$ be defined by

$$H_n(\mathfrak{n}, d) = \{\mathbf{y} \in \mathbb{R}^n\colon \Pi_{\mathfrak{n}}\mathbf{y} = \lambda\mathfrak{n},\ \lambda \geq d\}, \quad \mathfrak{n} \in S_n(\mathbf{0}_n, 1),\ d \in \mathbb{R},$$

where $\Pi_{\mathfrak{n}}\mathbf{y} = \langle \mathbf{y}, \mathfrak{n}\rangle\, \mathfrak{n} = \mathbf{y}^{\mathrm{T}}\mathfrak{n}{\cdot}\mathfrak{n}$ denotes the orthogonal projection of $\mathbf{y}$ into the linear subspace spanned up by the vector $\mathfrak{n}$ and $S_n(\mathbf{0}_n, 1)$ was defined as mentioned earlier.

In what follows, we shall make use of the geometric measure representation formula for spherically symmetric distributions with density-generating function g. Basic results from this theory have been published for the first time in ref. [18] and are exploited in refs. [19–21]. According to this theory, it holds for a spherically symmetric distributed random vector $\mathbf{Z} \sim EC_n(\mathbf{0}_n, I_n, g)$ and all Borel sets $A \in \mathfrak{B}_n$

$$P(\mathbf{Z} \in A) = \Phi_{\mathbf{0}_n, I_n, g}(A) = I_{n,g}^{-1} \int_0^\infty \mathcal{F}(A, r) r^{n-1} g(r^2)\, \mathrm{d}r, \tag{3}$$

where the integral $I_{n,g}$ is assumed to satisfy $0 < I_{n,g} < \infty$ and the so-called intersection percentage function $\mathcal{F}(A, r)$, $r > 0$ is defined as

$$\mathcal{F}(A, r) = U_n(r^{-1}A \cap S_n(\mathbf{0}_n, 1)), \quad r > 0,$$

with U_n being the uniform probability distribution on the unit sphere $S_n(\mathbf{0}_n, 1)$.

The intersection percentage function of a half-space $H_n(\mathfrak{n}, d)$ is

$$\mathcal{F}(H_n(\mathfrak{n}, d), r) := \begin{cases} I_{[d,\infty)}(r)\dfrac{\omega_{n-1}}{\omega_n} \displaystyle\int_0^{\phi^*(r)} \sin^{n-2}\varphi\, \mathrm{d}\varphi, & d > 0, \\ \dfrac{1}{2}, & d = 0, \\ 1 - I_{(|d|,\infty)}(r)\dfrac{\omega_{n-1}}{\omega_n} \displaystyle\int_0^{\phi^*(r)} \sin^{n-2}\varphi\, \mathrm{d}\varphi, & d < 0, \end{cases} \tag{4}$$

where $\phi^*(r) := \arctan(r^2/d^2 - 1)^{1/2}$. For a derivation of this formula and its application, see refs. [22, 23], respectively.

For $\alpha \in (0, 1/2)$ and $\mathfrak{n} \in S_n(\mathbf{0}_n, 1)$, let $z_{1-\alpha}(n, g)$ denote an arbitrary value satisfying the equation

$$\Phi_{\mathbf{0}_n, I_n, g}(H_n(\mathfrak{n}, z_{1-\alpha}(n, g))) = \alpha.$$

Note that $z_{1-\alpha}(n, g)$ depends in no way on the vector $\mathfrak{n}$ and for the special case $g = g_{\mathrm{G}}$, the value $z_{1-\alpha}(n, g_{\mathrm{G}})$ equals the usual $(1-\alpha)$-quantile of the one-dimensional Gaussian distribution.

Assuming that the conditions (RA2i), $i \in \{1, 2\}$, are satisfied, the γ_0-related expected curve points distance function $\nu \mid \Gamma \to [0, \infty)$, defined by

$$\nu(\gamma) := \|\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)\|, \quad \gamma \in \Gamma, \tag{5}$$

increases strongly monotonously and unbounded as γ, $\gamma > \gamma_0$, approaches b or as γ, $\gamma < \gamma_0$, approaches a, respectively. Let the parameter value $\gamma = \gamma_{(i)} \in \Gamma_i$, $i \in \{1, 2\}$, denote the uniquely determined solution of the equation

$$\nu(\gamma) = \sigma z_{1-\alpha}(n, g)$$

if the corresponding alternative under consideration is H_{A_i}. Put

$$\mathfrak{b}_i := \frac{\boldsymbol{\eta}(\gamma_{(i)}) - \boldsymbol{\eta}(\gamma_0)}{\|\boldsymbol{\eta}(\gamma_{(i)}) - \boldsymbol{\eta}(\gamma_0)\|} \in S_n(\mathbf{0}_n, 1), \quad i \in \{1, 2\}$$

and let the test statistic be defined by

$$t_i(\mathbf{Y}) := \sigma^{-1}\langle \mathbf{Y} - \boldsymbol{\eta}(\gamma_0), \mathfrak{b}_i \rangle, \quad i \in \{1, 2\}.$$

The hypothesis H_0 will be rejected, if for a concrete sample $\mathbf{y} \in \mathbb{R}^n$ it holds

$$t_i(\mathbf{y}) \geq z_{1-\alpha}(n, g), \quad i \in \{1, 2\},$$

i.e., if the actual sample belongs to the critical region

$$\mathcal{K}_i(z_{1-\alpha}(n, g)) := \{\mathbf{y} \in \mathbb{R}^n \colon t_i(y) \geq z_{1-\alpha}(n, g)\}, \quad i \in \{1, 2\}.$$

THEOREM 2.1 *The tests $\psi_i \mid \mathbb{R}^n \to \{0, 1\}$, $i \in \{1, 2\}$, defined by*

$$\psi_i(\mathbf{y}) = \begin{cases} 1 & \text{if } t_i(\mathbf{y}) \geq z_{1-\alpha}(n, g), \\ 0 & \text{otherwise}, \end{cases}$$

are size-α tests.

Proof If the null hypothesis is true, then $\mathbf{Y} \sim EC_n(\boldsymbol{\eta}(\gamma_0), \sigma^2 I_n, g)$. If the transformation $V \mid \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$V(\mathbf{y}) := \sigma^{-1}(\mathbf{y} - \boldsymbol{\eta}(\gamma_0)), \quad \mathbf{y} \in \mathbb{R}^n,$$

then $V(\mathbf{Y}) \sim EC_n(\mathbf{0}_n, I_n, g)$ and $V(\mathcal{K}_i(z_{1-\alpha}(n, g))) = H_n(\mathfrak{b}_i, z_{1-\alpha}(n, g))$. Consequently,

$$\Phi_{\boldsymbol{\eta}(\gamma_0), \sigma^2 I_n, g}(\mathcal{K}_i(z_{1-\alpha}(n, g))) = \Phi_{\mathbf{0}_n, I_n, g}(V(\mathcal{K}_i(z_{1-\alpha}(n, g)))) = \alpha. \qquad \blacksquare$$

COROLLARY 2.1 *If the parameter σ is known, then the regions*

$$\{\gamma_0 \in \Gamma: t_i(\mathbf{y}) < z_{1-\alpha}(n, g)\}, \quad i = 1, 2,$$

are one-sided confidence regions for the parameter γ of size $1 - \alpha$.

The power functions p_{ψ_i}, $i = 1, 2$, indicate the probabilities of rejecting the null hypothesis H_0: $\gamma = \gamma_0$ if the true parameter is some $\gamma \neq \gamma_0$, $\gamma \in \Gamma_i$.

THEOREM 2.2 *The power function p_{ψ_i}, $i \in \{1, 2\}$, satisfies the representation formula*

$$p_{\psi_i}(\gamma) = I_{n,g}^{-1} \int_0^\infty \mathcal{F}(H_n(\mathfrak{b}_i, d_i(\gamma)), r) r^{n-1} g(r^2)\,\mathrm{d}r, \quad \gamma \in \Gamma_i,$$

with $d_i(\gamma) := z_{1-\alpha}(n, g) - \sigma^{-1}\langle \boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0), \mathfrak{b}_i\rangle$, $z_{1-\alpha}(n, g)$ as defined earlier, and intersection percentage function $\mathcal{F}(H_n(\mathfrak{b}_i, d_i(\gamma)), \cdot)$ given in formula (4).

Proof Suppose $\mathbf{Y} \sim EC_n(\boldsymbol{\eta}(\gamma), \sigma^2 I_n, g)$ for some $\gamma \in \Gamma_i$, $i \in \{1, 2\}$, then it holds

$$p_{\psi_i}(\gamma) = \Phi_{\boldsymbol{\eta}(\gamma),\sigma^2 I_n, g}(\mathcal{K}_i(z_{1-\alpha}(n, g))).$$

Using the transformation $V_1 \mid \mathbb{R}^n \to \mathbb{R}^n$, defined by $V_1(\mathbf{y}) := \sigma^{-1}(\mathbf{y} - \boldsymbol{\eta}(\gamma))$, $\mathbf{y} \in \mathbb{R}^n$, we obtain

$$\begin{aligned} V_1(\mathcal{K}_i(z_{1-\alpha}(n, g))) &= \{\tilde{\mathbf{z}} \in \mathbb{R}^n: \sigma^{-1}\langle [\sigma\tilde{\mathbf{z}} + \boldsymbol{\eta}(\gamma)] - \boldsymbol{\eta}(\gamma_0), \mathfrak{b}_i\rangle \geq z_{1-\alpha}(n, g)\} \\ &= H_n(\mathfrak{b}_i, d_i(\gamma)), \end{aligned}$$

with

$$d_i(\gamma) := z_{1-\alpha}(n, g) - \sigma^{-1}\langle \boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0), \mathfrak{b}_i\rangle$$

and

$$\Phi_{\boldsymbol{\eta}(\gamma),\sigma^2 I_n, g}(\mathcal{K}_i(z_{1-\alpha}(n, g))) = \Phi_{\mathbf{0}_n, I_n, g}(V_1(\mathcal{K}_i(z_{1-\alpha}(n, g)))) = \Phi_{\mathbf{0}_n, I_n, g}(H_n(\mathfrak{b}_i, d_i(\gamma))).$$

■

LEMMA 2.1 *Under the assumption* (RA2*i*), $i \in \{1, 2\}$, *the power function p_{ψ_i} is monotonously increasing in the strong sense for $\gamma \in \Gamma_i$, $\gamma \to b$ or $\gamma \to a$, respectively.*

Proof We show the proposition for $i = 1$, and the considerations for the case $i = 2$ follow analogously. Using the representation formula for p_{ψ_1} given in Theorem 2.2, we first indicate that the function $d_1(\gamma)$ is monotonously falling in the strong sense for $\gamma \in \Gamma_1$, $\gamma \to b$. We have

$$\frac{\partial}{\partial\gamma} d_1(\gamma) = -\sigma^{-1} \sum_{j=1}^m n_j f'(\xi_j, \gamma) \frac{[f(\xi_j, \gamma_{(1)}) - f(\xi_j, \gamma_0)]}{\|\boldsymbol{\eta}(\gamma_{(1)}) - \boldsymbol{\eta}(\gamma_0)\|}.$$

For $f(\xi_j, \gamma)$ monotonously increasing in γ, it holds $f'(\xi_j, \gamma) \geq 0$ and $f(\xi_j, \gamma_{(1)}) - f(\xi_j, \gamma_0) \geq 0$ $(\gamma_{(1)} > \gamma_0)$, and for $f(\xi_j, \gamma)$ monotonously falling in γ, it holds $f'(\xi_j, \gamma) \leq 0$

and $f(\xi_j, \gamma_{(1)}) - f(\xi_j, \gamma_0) \leq 0$. If there is at least one index j such that this monotonicity holds in the strong sense as we assume in (RA21), it follows that

$$\frac{\partial}{\partial \gamma} d_1(\gamma) < 0, \quad \gamma \in (\gamma_0, b),$$

i.e., the function $d_1(\gamma)$ is monotonously falling in the strong sense for $\gamma \to b$. Hence, for the parameter values $\gamma, \gamma' \in \Gamma_1$, $\gamma < \gamma'$, and the half-spaces $H_n(\mathfrak{b}_1, d_1(\gamma))$ and $H_n(\mathfrak{b}_1, d_1(\gamma'))$, we obtain the relationship $H_n(\mathfrak{b}_1, d_1(\gamma)) \subset H_n(\mathfrak{b}_1, d_1(\gamma'))$. From this, it follows that

$$p_{\psi_1}(\gamma) = \mathbf{\Phi}_{\mathbf{0}_n, I_n, g}(H_n(\mathfrak{b}_1, d_1(\gamma))) < \mathbf{\Phi}_{\mathbf{0}_n, I_n, g}(H_n(\mathfrak{b}_1, d_1(\gamma')) = p_{\psi_1}(\gamma'),$$

i.e., $p_{\psi_1}(\gamma)$ is monotonously increasing in the strong sense. ■

2.2 *Two-sided test problem*

In this section, we consider the two-sided test problem

$$H_0\colon \gamma = \gamma_0 \quad \text{versus} \quad H_\mathrm{A}\colon \gamma \neq \gamma_0$$

for the case of known parameter σ^2. The critical region of a test ψ will be the union of the critical regions of two one-sided tests ψ_i of sizes α_1 and α_2, where $0 < \alpha_1 + \alpha_2 < 1$. Unlike the linear case, the critical regions of the one-sided tests in section 2.1 are not half-spaces with opposite normal vectors, rather their normal vectors depend on the concrete shape of the curve $\mathfrak{M}$. They are given by

$$\mathfrak{b}_i := \frac{\boldsymbol{\eta}(\gamma_{(i)}) - \boldsymbol{\eta}(\gamma_0)}{\|\boldsymbol{\eta}(\gamma_{(i)}) - \boldsymbol{\eta}(\gamma_0)\|}, \quad i = 1, 2,$$

where the parameters $\gamma = \gamma_{(i)} \in \Gamma_i$ denote the uniquely determined solutions of the equations

$$\nu(\gamma) = \sigma z_{1-\alpha_i}(n, g), \quad i = 1, 2.$$

The corresponding test statistics are

$$t_i(\mathbf{Y}) = \sigma^{-1}\langle \mathbf{Y} - \boldsymbol{\eta}(\gamma_0), \mathfrak{b}_i \rangle, \quad i = 1, 2.$$

We reject the null hypothesis $H_0\colon \gamma = \gamma_0$ if for a concrete sample vector $\mathbf{y} \in \mathbb{R}^n$ it holds $t_1(\mathbf{y}) \geq z_{1-\alpha_1}(n, g)$ or $t_2(\mathbf{y}) \geq z_{1-\alpha_2}(n, g)$. Consequently, the test $\psi \,|\, \mathbb{R}^n \to \{0, 1\}$ is defined by

$$\psi(\mathbf{y}) := \begin{cases} 1, & t_1(\mathbf{y}) \geq z_{1-\alpha_1}(n, g) \vee t_2(\mathbf{y}) \geq z_{1-\alpha_2}(n, g), \\ 0, & \text{otherwise}, \end{cases}$$

and its critical region $\mathcal{K}$ satisfies the representation

$$\mathcal{K} = \mathcal{K}_1(z_{1-\alpha_1}(n, g)) \cup \mathcal{K}_2(z_{1-\alpha_2}(n, g))$$

with $\mathcal{K}_i(z_{1-\alpha_i}(n, g)) = \{\mathbf{z} \in \mathbb{R}^n\colon t_i(\mathbf{z}) \geq z_{1-\alpha_i}(n, g)\}$, $i = 1, 2$.

When evaluating the first-type error probability $\alpha = \Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n,g}(\mathcal{K})$ of the test ψ, we have to take into consideration that $\mathcal{K}_1(z_{1-\alpha_1}(n,g)) \cap \mathcal{K}_2(z_{1-\alpha_2}(n,g)) \neq \emptyset$ if the corresponding normal vectors $\mathfrak{b}_1$ and $\mathfrak{b}_2$ satisfy $|\langle \mathfrak{b}_1, \mathfrak{b}_2\rangle| \neq 1$. Consequently, α allows the decomposition

$$\alpha = \sum_{i=1}^{2} \Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n,g}(\mathcal{K}_i(z_{1-\alpha_i}(n,g))) - \Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n,g}(\mathcal{K}_1(z_{1-\alpha_1}(n,g)) \cap \mathcal{K}_2(z_{1-\alpha_2}(n,g))). \tag{6}$$

Now, we shall give a representation of $\mathcal{K}_1(z_{1-\alpha_1}(n,g)) \cap \mathcal{K}_2(z_{1-\alpha_2}(n,g))$, which allows to evaluate the elliptically symmetric measure of this set in terms of a known probability distribution.

LEMMA 2.2 *Let* $|\langle \mathfrak{b}_1, \mathfrak{b}_2\rangle| \neq 1$. *The set* $\mathcal{K}_1(z_{1-\alpha_1}(n,g)) \cap \mathcal{K}_2(z_{1-\alpha_2}(n,g))$ *satisfies the representation*

$$\mathcal{K}_1(z_{1-\alpha_1}(n,g)) \cap \mathcal{K}_2(z_{1-\alpha_2}(n,g)) = \mathcal{D}$$

for

$$\mathcal{D} = \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{y} - \boldsymbol{\eta}(\gamma_0), \mathfrak{x}\rangle + \mu \geq t^* |\langle \mathbf{y} - \boldsymbol{\eta}(\gamma_0), \mathfrak{y}\rangle + \lambda|\},$$

with

$$\mathfrak{x} := \frac{\mathfrak{b}_1 + \mathfrak{b}_2}{\|\mathfrak{b}_1 + \mathfrak{b}_2\|}, \quad \mathfrak{y} := \frac{\mathfrak{b}_1 - \mathfrak{b}_2}{\|\mathfrak{b}_1 - \mathfrak{b}_2\|}, \quad \mathfrak{x} \perp \mathfrak{y},$$

and

$$\mu = -\frac{\sigma(z_{1-\alpha_1}(n,g) + z_{1-\alpha_2}(n,g))}{\sqrt{2(1 + \langle \mathfrak{b}_1, \mathfrak{b}_2\rangle)}}, \quad \lambda = -\frac{\sigma(z_{1-\alpha_1}(n,g) - z_{1-\alpha_2}(n,g))}{\sqrt{2(1 - \langle \mathfrak{b}_1, \mathfrak{b}_2\rangle)}},$$

$$t^* = \sqrt{\frac{1 - \langle \mathfrak{b}_1, \mathfrak{b}_2\rangle}{1 + \langle \mathfrak{b}_1, \mathfrak{b}_2\rangle}}.$$

For the proof of this Lemma, we refer to ref. [24].

LEMMA 2.3 *The elliptically symmetric measure of the set* $\mathcal{D}$ *from Lemma* 2.2 *satisfies the representation*

$$\Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n,g}(\mathcal{D}) = 1 - F_{1,\mu/\sigma,\lambda^2/\sigma^2;g_{n,2}}(t^*),$$

where $F_{1,\mu/\sigma,\lambda^2/\sigma^2;g_{n,2}}(\cdot)$ *denotes the distribution function of the doubly non-central* $g_{n,2}$*-generalized* t*-distribution with one degree of freedom and non-centrality parameters* μ/σ *and* λ^2/σ^2. *The density generator* $g_{n,2}$ *is defined by*

$$g_{n,2}(r) = \frac{\pi^{n/2-1}}{\Gamma(n/2-1)} C(n,g) \int_r^\infty (u-r)^{n/2-2} g(u)\,\mathrm{d}u, \quad r \geq 0 \tag{7}$$

and satisfies the equation

$$\int_{\mathbb{R}^2} g_{n,2}(\|z\|^2)\,\mathrm{d}z = 1, \quad z \in \mathbb{R}^2.$$

For the evaluation of $F_{1,\mu/\sigma,\lambda^2/\sigma^2;g_{n,2}}(t^*)$, *see ref.* [24].

Proof Using the transformation V, one obtain $\widetilde{\mathbf{Y}} := V(\mathbf{Y}) \sim EC_n(\mathbf{0}_n, I_n, g)$ and

$$\widetilde{\mathcal{D}} := V(\mathcal{D}) = \{V(\mathbf{y})\colon \mathbf{y} \in \mathcal{D}\} = \left\{\widetilde{\mathbf{y}} \in \mathbb{R}^n\colon \langle \widetilde{\mathbf{y}}, \mathfrak{x}\rangle + \frac{\mu}{\sigma} \geq t^* \left|\langle \widetilde{\mathbf{y}}, \mathfrak{y}\rangle + \frac{\lambda}{\sigma}\right|\right\}.$$

Consider the $n \times 2$-matrix $B := (\mathfrak{x}\,\mathfrak{y})$ and let a random vector $\mathbf{Z}$ be defined by

$$\mathbf{Z} := B^{\mathrm{T}}\tilde{\mathbf{Y}} = (\langle \tilde{\mathbf{Y}}, \mathfrak{x}\rangle, \langle \tilde{\mathbf{Y}}, \mathfrak{y}\rangle)^{\mathrm{T}}.$$

On the basis of Theorem 2.16 in ref. [6] and $B^{\mathrm{T}}B = I_2$, it follows that

$$\mathbf{Z} \sim EC_2(\mathbf{0}_2, I_2, g_{n,2}),$$

where according to formula (2.23) in ref. [6], the density generator $g_{n,2}$ is defined by equation (7). Consequently, we obtain

$$P(\mathbf{Y} \in \mathcal{D}) = P(\widetilde{\mathbf{Y}} \in \widetilde{\mathcal{D}}) = P(\mathbf{Z} \in \widetilde{\mathcal{D}}^*),$$

where

$$\widetilde{\mathcal{D}}^* := \left\{\mathbf{z} \in \mathbb{R}^2\colon z_1 + \frac{\mu}{\sigma} \geq t^* \left|z_2 + \frac{\lambda}{\sigma}\right|\right\}.$$

Finally, Definition 1, for the doubly non-central g-generalized t-distribution, and relation (14) in ref. [24] yield

$$P(\mathbf{Z} \in \widetilde{\mathcal{D}}^*) = \Phi_{\mathbf{0}_2, I_2, g_{n,2}}(\widetilde{\mathcal{D}}^*) = 1 - F_{1,\mu/\sigma,\lambda^2/\sigma^2;g_{n,2}}(t^*).$$

■

Remark 2.1 Recognize that the marginals of spherically symmetric distributed vectors have distributions that depend on the dimensions of both the whole vector and the marginal vector and that the one-dimensional marginals are not independent except for the Gaussian case. For example, in the case of the Pearson-VII-type density generating function g_{P}, the two-dimensional marginal distribution depends on the overall dimension n as follows:

$$g_{n,2}(r) = \frac{\Gamma(M - (n-2)/2)}{\pi m^* \Gamma(M - n/2)} \left(1 + \frac{r}{m^*}\right)^{-M+(n-2)/2},$$

whereas in the case of the Gaussian density generating function g_{G} it holds

$$g_{n,2}(r) = (2\pi)^{-1}\mathrm{e}^{-r/2}.$$

THEOREM 2.3 *The two-sided test ψ is a size-α test with*

$$\alpha = \alpha_1 + \alpha_2 - (1 - F_{1,\mu/\sigma,\lambda^2/\sigma^2;g_{n,2}}(t^*)).$$

Proof Owing to formula (6), we obtain the assertion of the theorem by

$$\Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n, g}(\mathcal{K}_i(z_{1-\alpha_i}(n, g))) = \alpha_i, \quad i = 1, 2$$

and Lemma 2.3. ■

COROLLARY 2.2 *The two-sided test ψ is a level-α^* test with $\alpha^* = \alpha_1 + \alpha_2$ and*

$$\{\gamma_0 \in \Gamma\colon t_1(\mathbf{y}) < z_{1-\alpha_1}(n, g) \wedge t_2(\mathbf{y}) < z_{1-\alpha_2}(n, g)\}$$

is a confidence region for the parameter γ with confidence level $1 - \alpha^*$.

THEOREM 2.4 *The power function of the two-sided test ψ satisfies the representation*

$$p_\psi(\gamma) = I_{n,g}^{-1} \sum_{i=1}^{2} \int_0^\infty \mathcal{F}(H_n(\mathfrak{b}_i, d_i(\gamma)), r) r^{n-1} g(r^2)\, \mathrm{d}r$$
$$- (1 - F_{1,\mu^*/\sigma,\lambda^{*2}/\sigma^2;g_{n,2}}(t^*)), \quad \gamma \in \Gamma,$$

where

$$d_i(\gamma) := z_{1-\alpha_i}(n, g) - \sigma^{-1}\langle \boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0), \mathfrak{b}_i\rangle, \quad i = 1, 2,$$

and the intersection percentage function $\mathcal{F}(H_n(\mathfrak{b}_i, d_i(\gamma)), \cdot)$ is given in formula (4).

The quantities μ^* and λ^* occurring in the non-centrality parameters of the doubly non-central $g_{n,2}$-generalized t-distribution satisfy the equations

$$\mu^* := \mu + \langle \boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0), \mathfrak{x}\rangle, \quad \lambda^* := \lambda + \langle \boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0), \mathfrak{y}\rangle, \tag{8}$$

and the density generating function $g_{n,2}$ is defined in formula (7).

Proof Let $\mathbf{Y} \sim EC_n(\boldsymbol{\eta}(\gamma), \sigma^2 I_n, g)$ for some $\gamma \in \Gamma$. We have to determine

$$\Phi_{\boldsymbol{\eta}(\gamma),\sigma^2 I_n,g}(\mathcal{K}) = \Phi_{\boldsymbol{\eta}(\gamma),\sigma^2 I_n,g}(\mathcal{K}_1(z_{1-\alpha_1}(n, g)) \cup \mathcal{K}_2(z_{1-\alpha_2}(n, g))).$$

The transformation V_1 yields $V_1(\mathbf{Y}) \sim EC_n(\mathbf{0}_n, I_n, g)$,

$$V_1(\mathcal{K}_i(z_{1-\alpha_i}(n, g))) = H_n(\mathfrak{b}_i, d_i(\gamma)), \quad i = 1, 2,$$

with $d_i(\gamma) = z_{1-\alpha_i}(n, g) - \sigma^{-1}\langle \boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0), \mathfrak{b}_i\rangle$ and

$$V_1(\mathcal{D}) = \{\widetilde{\mathbf{z}} \in \mathbb{R}^n\colon \langle [\sigma\widetilde{\mathbf{z}} + \boldsymbol{\eta}(\gamma)] - \boldsymbol{\eta}(\gamma_0), \mathfrak{x}\rangle + \mu \geq t^* |\langle [\sigma\widetilde{\mathbf{z}} + \boldsymbol{\eta}(\gamma)] - \boldsymbol{\eta}(\gamma_0), \mathfrak{y}\rangle + \lambda|\}$$
$$= \left\{\widetilde{\mathbf{z}} \in \mathbb{R}^n\colon \langle \mathbf{z}, \mathfrak{x}\rangle + \frac{\mu^*}{\sigma} \geq t^* \left|\langle \mathbf{z}, \mathfrak{y}\rangle + \frac{\lambda^*}{\sigma}\right|\right\},$$

with μ^* and λ^* given in equation (8). Consequently,

$$\Phi_{\boldsymbol{\eta}(\gamma),\sigma^2 I_n,g}(\mathcal{K}) = \sum_{i=1}^{2} \Phi_{\mathbf{0}_n,I_n,g}(H_n(\mathfrak{b}_i, d_i(\gamma))) - \Phi_{\mathbf{0}_n,I_n,g}(V_1(\mathcal{D})).$$

Making the same considerations as in the proof of Lemma 2.3, it follows that

$$\Phi_{\mathbf{0}_n,I_n,g}(V_1(\mathcal{D})) = \Phi_{\mathbf{0}_2,I_2,g_{n,2}}(\mathcal{D}^*) = 1 - F_{1,\mu^*/\sigma,\lambda^*2/\sigma^2;g_{n,2}}(t^*),$$

where

$$\mathcal{D}^* = \left\{\mathbf{z} \in \mathbb{R}^2\colon z_1 - \frac{\mu^*}{\sigma} \geq t^* \left|z_2 - \frac{\lambda^*}{\sigma}\right|\right\}.$$

■

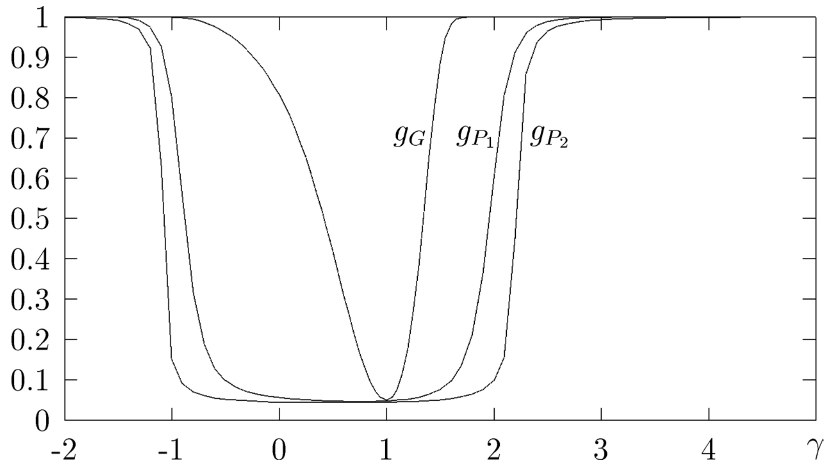

Figure 1. $p_\psi(\gamma)$ for $g = g_{\mathrm{G}}$, $g = g_{\mathrm{P}_1}$, and $g = g_{\mathrm{P}_2}$.

Example 2.1 Let $f(x, \gamma) = x^\gamma$, $\Gamma = (-\infty, \infty)$, $\gamma_0 = 1$, $n = 4$, $x_1 = 0.1$, $x_2 = 0.5$, $x_3 = 2$, $x_4 = 3.5$, and $\sigma = 1$.

We compare the first-type error probability α_{G} and the power function of the Gaussian density generator g_{G} with the first-type error probability α_{P} and the power function of the Pearson-VII-type density generator g_{P_1} with parameters $m^* = 10$ and $M = 3.1$, respectively. It holds $\mathrm{COV}(\mathbf{Y}) = I_4$ for $g = g_{\mathrm{G}}$ and according to formula (3.29) in ref. [6], $\mathrm{COV}(\mathbf{Y}) = m^*/(2M - n - 2)I_n = 50I_4$ for $g = g_{\mathrm{P}_1}$.

Notice that the existence of the moments is not necessary for the construction of the considered tests and the evaluation of the power functions. To illustrate this case, we choose another Pearson-VII-type density generator g_{P_2} with parameters $m^* = 1$ and $M = (1/2)(n + m^*) = 5/2$, *i.e.*, $\mathbf{Y}$ follows a multivariate Cauchy distribution.

Let $\alpha_1 = \alpha_2 = 0.025$. Then $z_{1-\alpha_i}(4, g_{\mathrm{P}_1}) = 8.419$, $z_{1-\alpha_i}(4, g_{\mathrm{P}_2}) = 12.705$, $i = 1, 2$, and $\alpha_{\mathrm{G}} = 0.050$, $\alpha_{\mathrm{P}_1} = 0.048$, and $\alpha_{\mathrm{P}_2} = 0.045$. For a comparison of the power functions, see figure 1.

3. Unknown parameter σ^2

When testing the hypothesis $H_0\colon \gamma = \gamma_0$ about the unknown nonlinear parameter versus one of the one-sided alternatives $H_{\mathrm{A}_1}\colon \gamma > \gamma_0$ or $H_{\mathrm{A}_2}\colon \gamma < \gamma_0$ or versus the two-sided alternative $H_{\mathrm{A}}\colon \gamma \neq \gamma_0$ in the case of unknown parameter σ^2, we shall make use of one of the following additional model assumptions or their combination, respectively:

(RA31) The γ_0-related expected curve points distance function $\nu(\cdot)$ defined in equation (5) increases strongly monotonously and unbounded, as γ, $\gamma > \gamma_0$, approaches b.

(RA32) $\nu(\cdot)$ increases strongly monotonously and unbounded, as $\gamma < \gamma_0$ approaches a.

3.1 *One-sided test problem*

In this section, we propose an exact one-sided tests for proving

$$H_0\colon \gamma = \gamma_0 \quad \text{versus} \quad H_{\mathrm{A}_1}\colon \gamma \in \Gamma_1 = (\gamma_0, b) \quad \text{or} \quad H_{\mathrm{A}_2}\colon \gamma \in \Gamma_2 = (a, \gamma_0).$$

The construction of the tests is similar for the two alternatives H_{A_1} and H_{A_2} and is based on an idea of modification of Student's well-known test for the mean, if the variance in a Gaussian population is unknown.

For this reason, let us first rewrite the Student's test in a form, suitable for our further purposes. Let the sample vector $\mathbf{Z} \sim EC_n(\mu \mathbf{1}_n, \sigma^2 I_n, g_G)$ be given and consider the test problem H_0: $\mu = \mu_0$ versus H_A: $\mu > \mu_0$. The range $\{\mu \mathbf{1}_n: \mu \in \mathbb{R}\}$ of the mean value of the observation vector $\mathbf{Z}$ is a linear subspace of the sample space $\mathbb{R}^n$ and is usually called the model-space. The test statistic

$$T(\mathbf{Z}) := \sqrt{n}\frac{\overline{Z}_n - \mu_0}{\sqrt{(1/(n-1))\sum_{i=1}^n (Z_i - \overline{Z}_n)^2}} = \frac{\|\overline{Z}_n \mathbf{1}_n - \mu_0 \mathbf{1}_n\| \operatorname{sign}(\overline{Z}_n - \mu_0)}{\|\mathbf{Z} - \overline{Z}_n \mathbf{1}_n\| / \sqrt{n-1}}$$

follows Student's t-distribution with $n-1$ degrees of freedom if the null-hypothesis is true. Let $t_{n-1,1-\alpha}$ denote the $(1-\alpha)$-quantile of this Student distribution. Then, α is the probability under H_0 that $\mathbf{Z}$ falls into the critical region of this test, which is a rotationally symmetric cone around the middle line $\Lambda = \{\lambda \mathbf{1}_n: \lambda > \mu_0\}$:

$$K = \left\{ \mathbf{w} \in \mathbb{R}^n: \sqrt{n-1}\frac{\|\Pi_{\mathbf{1}_n/\sqrt{n}}(\mathbf{w} - \mu_0 \mathbf{1}_n)\| \operatorname{sign}(\overline{w}_n - \mu_0)}{\|\mathbf{w} - \Pi_{\mathbf{1}_n/\sqrt{n}} \mathbf{w}\|} \geq t_{n-1,1-\alpha} \right\}.$$

Note that for every density generating function g satisfying $0 < I_{n,g} < \infty$, it holds

$$\Phi_{\mu_0 \mathbf{1}_n, \sigma^2 I_n, g}(K) = \alpha.$$

Now, we want to adapt the decision rule for the nonlinear regression model and arbitrary density generating function g satisfying the relation (2). The critical region of this test can be interpreted as the result of a nonlinear transformation of a cone like K. The ray $\Lambda = \{\lambda \mathbf{1}_n, \lambda > \mu_0\}$ has been transformed into the curve $\{\boldsymbol{\eta}(\gamma): \gamma \in \Gamma_i\}$. Here, i indicates which of the alternatives H_{A_1} or H_{A_2} is actually under consideration. The critical region is constructed in such a way that the test rejects the null hypothesis for sample point $\mathbf{y}$ near to the point $\boldsymbol{\eta}(\gamma) \in \mathfrak{M}$ with parameter values γ belonging to the alternative Γ_i. Hence, the critical region is well adapted to the whole shape of the expected curve $\{\boldsymbol{\eta}(\gamma): \gamma \in \Gamma_i\}$. To this end, let

$$S_n(\boldsymbol{\eta}(\gamma), \nu) := \{\mathbf{y} \in \mathbb{R}^n: \|\mathbf{y} - \boldsymbol{\eta}(\gamma)\| = \nu\}, \quad \nu > 0,$$

denote the sphere with radius ν and center $\boldsymbol{\eta}(\gamma)$ in $\mathbb{R}^n$.

For a given $\mathbf{y} \in \mathbb{R}^n$, we find, because of the assumption (RA3i), an uniquely determined parameter $\gamma_i = \gamma_i(\mathbf{y}, \gamma_0) \in \Gamma_i \cup \{\gamma_0\}$ such that the uniquely determined point $\boldsymbol{\eta}(\gamma_i(\mathbf{y}, \gamma_0))$ on the curve $\mathfrak{M}$ has the same Euclidean distance from $\boldsymbol{\eta}(\gamma_0)$ as $\mathbf{y}$:

$$\|\mathbf{y} - \boldsymbol{\eta}(\gamma_0)\| = \|\boldsymbol{\eta}(\gamma_i(\mathbf{y}, \gamma_0)) - \boldsymbol{\eta}(\gamma_0)\|. \tag{9}$$

The function

$$\gamma_i \,|\, \mathbb{R}^n \times \Gamma \to \Gamma_i \cup \{\gamma_0\}$$

defined by equation (9) will play an essential role for defining a test statistic as in what follows. Let us therefore study first some of its properties.

LEMMA 3.1 *The mapping* $\gamma_i(\cdot, \gamma_0) \,|\, \mathbb{R}^n \to \Gamma_i \cup \{\gamma_0\}$, $i \in \{1, 2\}$, *is continuous.*

Proof The function $\gamma_i(\cdot, \gamma_0)$ is continuous if the following conditions holds: For any $\mathbf{y} \in \mathbb{R}^n$, there exists $\varepsilon = \varepsilon(\mathbf{y}) > 0$ for all $\delta, 0 < \delta \leq |\gamma_i(\mathbf{y}, \gamma_0) - \gamma_0|$, such that $|\gamma_i(\mathbf{y}, \gamma_0) - \gamma_i(\mathbf{z}, \gamma_0)| < \delta$ for all $\mathbf{z}$ satisfying $\|\mathbf{y} - \mathbf{z}\| < \varepsilon(\mathbf{y})$.

For a fixed $\mathbf{y} \in \mathbb{R}^n \backslash \{\boldsymbol{\eta}(\gamma_0)\}$, the value $\gamma_i(\mathbf{y}, \gamma_0)$ is uniquely determined by equation (9) if the condition (RA3i) is fulfilled. The set

$$\mathfrak{K} := \{\boldsymbol{\eta}(\gamma) \colon \gamma \in (\gamma_i(\mathbf{y}, \gamma_0) - \delta, \gamma_i(\mathbf{y}, \gamma_0) + \delta)\}$$

denotes the image of the interval $(\gamma_i(\mathbf{y}, \gamma_0) - \delta, \gamma_i(\mathbf{y}, \gamma_0) + \delta)$ under the mapping $\gamma \to \boldsymbol{\eta}(\gamma)$. Because of the strongly monotonic property (RA3i) of the γ_0-related expected curve points distance function $\nu(\cdot)$, it holds

$$\nu(\gamma) \in (\nu_1, \nu_2) \quad \text{for } \gamma \in (\gamma_i(\mathbf{y}, \gamma_0) - \delta, \gamma_i(\mathbf{y}, \gamma_0) + \delta)$$

with

$$\nu_1 := \nu(\gamma_i(\mathbf{y}, \gamma_0)) - \varepsilon_1(\mathbf{y}) \quad \text{and} \quad \nu_2 := \nu(\gamma_i(\mathbf{y}, \gamma_0)) + \varepsilon_2(\mathbf{y}),$$

for certain positive $\varepsilon_1(\mathbf{y})$ and $\varepsilon_2(\mathbf{y})$. The $\gamma_i(\cdot, \gamma_0)$-image of the set

$$\mathcal{R} := \{\mathbf{z} \in \mathbb{R}^n \colon \nu_1 < \|\mathbf{z} - \boldsymbol{\eta}(\gamma_0)\| < \nu_2\}$$

is the interval $(\gamma_i(\mathbf{y}, \gamma_0) - \delta, \gamma_i(\mathbf{y}, \gamma_0) + \delta)$.

Let $\varepsilon(\mathbf{y}) > 0$ satisfy

$$\varepsilon(\mathbf{y}) < \min(\varepsilon_1(\mathbf{y}), \varepsilon_2(\mathbf{y}))$$

and put $U_{\varepsilon(\mathbf{y})}(\mathbf{y}) := \{\mathbf{z} \in \mathbb{R}^n \colon \|\mathbf{z} - \mathbf{y}\| < \varepsilon(\mathbf{y})\}$. If $\mathbf{z} \in U_{\varepsilon(\mathbf{y})}(\mathbf{y})$ then

$$\|\mathbf{z} - \boldsymbol{\eta}(\gamma_0)\| \leq \|\mathbf{z} - \mathbf{y}\| + \|\mathbf{y} - \boldsymbol{\eta}(\gamma_0)\| < \varepsilon(\mathbf{y}) + \nu(\gamma_i(\mathbf{y}, \gamma_0))$$

and

$$\nu(\gamma_i(\mathbf{y}, \gamma_0)) = \|\mathbf{y} - \boldsymbol{\eta}(\gamma_0)\| \leq \|\mathbf{y} - \mathbf{z}\| + \|\mathbf{z} - \boldsymbol{\eta}(\gamma_0)\| < \varepsilon(\mathbf{y}) + \|\mathbf{z} - \boldsymbol{\eta}(\gamma_0)\|.$$

Hence $\mathbf{z} \in \mathcal{R}$, because $(\nu(\gamma_i(\mathbf{y}, \gamma_0)) - \varepsilon(\mathbf{y}), \nu(\gamma_i(\mathbf{y}, \gamma_0)) + \varepsilon(\mathbf{y})) \subset (\nu_1, \nu_2)$. Consequently, $U_{\varepsilon(\mathbf{y})}(\mathbf{y}) \subset \mathcal{R}$. It follows that $\boldsymbol{\eta}(\gamma_i(\mathbf{z}, \gamma_0)) \in \mathfrak{K}$ for all $\mathbf{z} \in U_{\varepsilon(\mathbf{y})}(\mathbf{y})$ and therefore

$$\gamma_i(\mathbf{z}, \gamma_0) \in (\gamma_i(\mathbf{y}, \gamma_0) - \delta, \gamma_i(\mathbf{y}, \gamma_0) + \delta).$$

The continuity of $\gamma_i(\cdot, \gamma_0)$ in the point $\boldsymbol{\eta}(\gamma_0)$ follows analogously. ∎

Now, we continue with the construction of the test. Let a mapping $\mathbf{e} \mid \Gamma \backslash \{\gamma_0\} \to S_n(\mathbf{0}_n, 1)$ be defined by

$$\mathbf{e}(\gamma) := \frac{\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)}{\|\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)\|}, \quad \gamma \in \Gamma \backslash \{\gamma_0\}.$$

As in Student's test for linear models, we want to compare

$$\|\Pi_{\mathbf{e}(\gamma_i(\mathbf{y}, \gamma_0))}(\mathbf{y} - \boldsymbol{\eta}(\gamma_0))\|$$

with the square root of a corresponding variance-type estimator

$$\|\mathbf{y} - \boldsymbol{\eta}(\gamma_0) - \Pi_{\mathbf{e}(\gamma_i(\mathbf{y}, \gamma_0))}(\mathbf{y} - \boldsymbol{\eta}(\gamma_0))\|.$$

Then, the test is defined as to reject the hypothesis $H_0 \colon \gamma = \gamma_0$ if the first of the two norm-terms is larger than a certain multiple of the second one.

The test statistic

$$T_i(\mathbf{Y}) := \sqrt{n-1}\frac{\|\Pi_{\mathbf{e}(\gamma_i(\mathbf{y},\gamma_0))}(\mathbf{Y}-\boldsymbol{\eta}(\gamma_0))\|\operatorname{sign}(\langle \mathbf{Y}-\boldsymbol{\eta}(\gamma_0), \mathbf{e}(\gamma_i(\mathbf{y},\gamma_0))\rangle)}{\|\mathbf{Y}-\boldsymbol{\eta}(\gamma_0)-\Pi_{\mathbf{e}(\gamma_i(\mathbf{y},\gamma_0))}(\mathbf{Y}-\boldsymbol{\eta}(\gamma_0))\|},$$

is a measurable function because of the continuity of $\gamma_i(\cdot, \gamma_0)$. The critical region of the test

$$\mathcal{K}_{f,i} = \mathcal{K}_{f,i}(t_{n-1,1-\alpha}) := \{\mathbf{y} \in \mathbb{R}^n \colon T_i(\mathbf{y}) \geq t_{n-1,1-\alpha}\}, \quad i \in \{1, 2\}$$

is, therefore, a measurable set. It will be called a curved transformed cone-type set. An Example is given in figure 2.

In this way, we arrived at the following decision rule:

Reject the hypothesis $H_0\colon \gamma = \gamma_0$ if for a concrete sample $\mathbf{y}$, it holds

$$T_i(\mathbf{y}) > t_{n-1,1-\alpha}, \quad i \in \{1, 2\}.$$

Note that initial considerations concerning this test, as well as the following theorem, have been made in ref. [25].

THEOREM 3.1 *The tests $\Psi_i \mid \mathbb{R}^n \to \{0, 1\}$, $i \in \{1, 2\}$, defined by*

$$\Psi_i(\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \in \mathcal{K}_{f,i}(t_{n-1,1-\alpha}), \\ 0 & \text{otherwise}, \end{cases}$$

are exact size-α tests.

Proof If the null hypothesis $H_0\colon \gamma = \gamma_0$ is true, then $\mathbf{Y}_n \sim EC_n(\boldsymbol{\eta}(\gamma_0), \sigma^2 I_n, g)$ and the first-type error probability is equal to

$$\Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n, g}(\mathcal{K}_{f,i}(t_{n-1,1-\alpha})), \quad i \in \{1, 2\}.$$

If the transformation $V \mid \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$V(\mathbf{y}) := \sigma^{-1}(\mathbf{y} - \boldsymbol{\eta}(\gamma_0)), \quad \mathbf{y} \in \mathbb{R}^n,$$

then it holds $V(\mathbf{Y}) \sim EC_n(\mathbf{0}_n, I_n, g)$ and

$$\mathcal{K}_i^* := V(\mathcal{K}_{f,i}(t_{n-1,1-\alpha})) = \{\mathbf{z}_n \in \mathbb{R}^n \colon T_i(\sigma \mathbf{z}_n + \boldsymbol{\eta}(\gamma_0)) \geq t_{n-1,1-\alpha}\}, \quad i \in \{1, 2\}.$$

For the application of the geometric measure representation formula for the system $\mathfrak{A}$(dir, dist) of Borel sets, first introduced in ref. [25] and later on studied, *e.g.*, in ref. [21], we have to

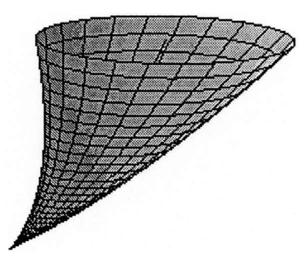

Figure 2. $\mathcal{K}_{f,1}$ for $f(x, \gamma) = \exp(\gamma x)$, $n = 3$, $x_1 = 0.01$, $x_2 = 0.2$, $x_3 = 1$, $\gamma_0 = 0.03$.

determine two functions. The first one is the so-called direction-type function for the set $\mathcal{K}_i^*$,

$$\mathbf{e}_{\mathcal{K}_i^*}(\widetilde{\nu}(\gamma)) := \frac{\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)}{\sigma\widetilde{\nu}(\gamma)}, \quad \gamma \in \Gamma_i,$$

and the second one is the distance type function for the same set $\mathcal{K}_i^*$,

$$R_{\mathcal{K}_i^*}(\tilde{\nu}(\gamma)) := \frac{\tilde{\nu}(\gamma)}{(1 + (n-1)/t_{n-1,1-\alpha}^2)^{1/2}}, \quad \gamma \in \Gamma_i,$$

where

$$\tilde{\nu}(\gamma) := \|\sigma^{-1}(\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0))\|, \quad \gamma \in \Gamma.$$

Note that the transformed critical region $\mathcal{K}_i^*$ satisfies, for all $\gamma \in \Gamma_i$, the equation

$$\mathcal{K}_i^* \cap S_n(\mathbf{0}_n, \widetilde{\nu}(\gamma)) = H_n(\mathbf{e}_{\mathcal{K}_i^*}(\widetilde{\nu}(\gamma)), R_{\mathcal{K}_i^*}(\widetilde{\nu}(\gamma))) \cap S_n(\mathbf{0}_n, \widetilde{\nu}(\gamma)).$$

Further the curve

$$\begin{aligned}
\gamma \to \mathfrak{C}(\tilde{\nu}(\gamma)) &= R_{\mathcal{K}_i^*}(\tilde{\nu}(\gamma)) \cdot \mathbf{e}_{\mathcal{K}_i^*}(\tilde{\nu}(\gamma)) \\
&= \frac{\tilde{\nu}(\gamma)}{\sqrt{1 + (n-1)/t_{n-1,1-\alpha}^2}} \cdot \frac{\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)}{\|\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)\|} \\
&= \frac{1}{\sigma\sqrt{1 + (n-1)/t_{n-1,1-\alpha}^2}} \cdot (\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0))
\end{aligned}$$

is continuous in γ and thus, after reparameterization, the curve $\tilde{\nu} \to \mathfrak{C}(\tilde{\nu})$ is continuous in $\tilde{\nu}$. Recall that from Lemma 3.1, it follows that $\mathcal{K}_i^*$ is a measurable set and therefore belongs to the Borel σ-algebra $\mathfrak{B}_n$. Consequently, $\mathcal{K}_i^*$ belongs to the system $\mathfrak{A}$(dir, dist) in ref. [25] and for the intersection percentage function defined there it holds

$$\mathcal{F}(\mathcal{K}_i^*, \tilde{\nu}) = \frac{\omega_{n-1}}{\omega_n} \int_0^{\alpha^*(\tilde{\nu})} (\sin\psi)^{n-2}\, \mathrm{d}\psi,$$

where

$$\alpha^*(\tilde{\nu}) := \arctan\left(\frac{\tilde{\nu}(\gamma)^2}{R_{\mathcal{K}_i^*}(\tilde{\nu}(\gamma))^2} - 1\right)^{1/2} = \arctan\left(\frac{\sqrt{n-1}}{t_{n-1,1-\alpha}}\right).$$

Note that $\alpha^*(\tilde{\nu})$ actually does not depend on $\tilde{\nu}$. Thus, for the first type error probability from the geometric measure representation formula (3) for spherical measures, we have

$$\Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n,g}(\mathcal{K}_{f,i}(t_{n-1,1-\alpha})) = \boldsymbol{\Phi}_{\mathbf{0}_n,I_n,g}(\mathcal{K}_i^*) = \frac{\omega_{n-1}}{\omega_n} \int_0^{\arctan(\sqrt{n-1}/t_{n-1,1-\alpha})} (\sin\psi)^{n-2}\, \mathrm{d}\psi. \tag{10}$$

Let T_{n-1} be a random variable following Student's t-distribution with $n-1$ degrees of freedom and put

$$\mathcal{A}_t := \left\{\mathbf{z} \in \mathbb{R}^n \colon \frac{z_1}{\sqrt{(z_2^2 + \cdots + z_n^2)/\sqrt{n-1}}} > t\right\} \quad \text{for all } t > 0.$$

It has been shown in ref. [25, sections 3 and 8] that

$$P(T_{n-1} > t) = \mathbf{\Phi}_{\mathbf{0}_n, I_n, g}(\mathcal{A}_t) = \frac{\omega_{n-1}}{\omega_n}\int_0^{\arctan(\sqrt{n-1}/t)} (\sin\psi)^{n-2}\mathrm{d}\psi.$$

Hence, if $t = t_{n-1,1-\alpha}$ then

$$\frac{\omega_{n-1}}{\omega_n}\int_0^{\arctan(\sqrt{n-1}/t_{n-1,1-\alpha})} (\sin\psi)^{n-2}\mathrm{d}\psi = P(T_{n-1} > t_{n-1,1-\alpha}) = \alpha.$$

From this equation and the relation (10), it follows that

$$\Phi_{\boldsymbol{\eta}(\gamma_0),\sigma^2 I_n, g}(\mathcal{K}_{f,i}(t_{n-1,1-\alpha})) = \alpha,$$

i.e., Ψ_i is a size-α test. ■

COROLLARY 3.1 *If the parameter σ is unknown, then*

$$\{\gamma_0 \in \Gamma\colon T_i(\mathbf{y}) < t_{n-1,1-\alpha}\}, \quad i = 1, 2,$$

are one-sided confidence regions for the parameter γ with a confidence level exactly equal to $1 - \alpha$.

3.2 *Two-sided test problem*

Now, let us consider the two-sided problem of testing

$$H_0\colon \gamma = \gamma_0 \quad \text{versus} \quad H_{\mathrm{A}}\colon \gamma \neq \gamma_0. \tag{11}$$

The critical region of a test Ψ will be defined as the union of the critical regions of two one-sided tests Ψ_i of sizes α_1 and α_2, respectively, where $0 < \alpha_1 + \alpha_2 < 1$. To this end, recall the definitions of the functions $\gamma_i \mid \mathbb{R}^n \times \Gamma \to \Gamma_i \cup \{\gamma_0\}$, $i = 1, 2$, as well as the definition of the test statistics T_i based upon them. Let $T_1(\mathbf{y})$ and $T_2(\mathbf{y})$ denote the realizations of these test statistics and define a decision rule to reject the hypothesis $H_0\colon \gamma = \gamma_0$ iff $T_1(\mathbf{y}) > t_{n-1,1-\alpha_1}$ or $T_2(\mathbf{y}) > t_{n-1,1-\alpha_2}$ holds true. Hence, the test $\Psi_3 \mid \mathbb{R}^n \to \{0, 1\}$ has been defined by

$$\Psi_3(\mathbf{y}) = \begin{cases} 1, & T_1(\mathbf{y}) > t_{n-1,1-\alpha_1} \vee T_2(\mathbf{y}) > t_{n-1,1-\alpha_2}, \\ 0, & \text{otherwise} \end{cases}$$

and its critical region $\mathcal{K}_{f,3}$ satisfies the representation

$$\mathcal{K}_{f,3} = \mathcal{K}_{f,1}(t_{n-1,1-\alpha_1}) \cup \mathcal{K}_{f,2}(t_{n-1,1-\alpha_2}).$$

For evaluating the first-type error probability of the test Ψ_3, we try to ensure that the intersection of $\mathcal{K}_{f,1}(t_{n-1,1-\alpha_1})$ with $\mathcal{K}_{f,2}(t_{n-1,1-\alpha_2})$ is empty. The condition given in what follows should ensure this. Let $\mathbf{z}(\nu)$ denote an arbitrary but fixed chosen point from $S_n(\boldsymbol{\eta}(\gamma_0), \nu)$ and define an angle type function $\nu \to \zeta(\nu)$ as

$$\zeta(\nu) := \angle\left(\frac{\boldsymbol{\eta}(\gamma_1(\mathbf{z}(\nu), \gamma_0)) - \boldsymbol{\eta}(\gamma_0)}{\nu}, \frac{\boldsymbol{\eta}(\gamma_2(\mathbf{z}(\nu), \gamma_0)) - \boldsymbol{\eta}(\gamma_0)}{\nu}\right), \quad \forall \nu \in (0, \infty).$$

LEMMA 3.2 *If the regression function f and the experimental design point $\mathbf{x}$ satisfy the condition*

$$\cos\zeta(\nu) < \frac{t_{n-1,1-\alpha_1}t_{n-1,1-\alpha_2} - (n-1)}{\sqrt{(n-1)+t^2_{n-1,1-\alpha_1}}\sqrt{(n-1)+t^2_{n-1,1-\alpha_2}}}, \quad \forall\nu \in (0,\infty), \tag{12}$$

then it holds

$$\mathcal{K}_{f,1}(t_{n-1,1-\alpha_1}) \cap \mathcal{K}_{f,2}(t_{n-1,1-\alpha_2}) = \emptyset.$$

Proof We prove this assertion indirectly. Suppose condition (12) is satisfied and there exists nevertheless any $\mathbf{y} \in \mathcal{K}_{f,1}(t_{n-1,1-\alpha_1}) \cap \mathcal{K}_{f,2}(t_{n-1,1-\alpha_2})$ with

$$\|\mathbf{y} - \boldsymbol{\eta}(\gamma_0)\| =: \nu > 0,$$

from the definition of $\mathcal{K}_{f,i}(t_{n-1,1-\alpha_i})$, it follows that $\phi_i(\mathbf{y}) := \angle(\mathbf{y} - \boldsymbol{\eta}(\gamma_0), \mathrm{e}(\gamma_i(\mathbf{y},\gamma_0)))$ satisfies the inequality

$$\cos\phi_i(\mathbf{y}) > \frac{t_{n-1,1-\alpha_i}}{\sqrt{(n-1)+t^2_{n-1,1-\alpha_i}}}, \quad i = 1,2. \tag{13}$$

Because of $\mathbf{y} \in S_n(\boldsymbol{\eta}(\gamma_0), \nu)$, we have $\gamma_i(\mathbf{y},\gamma_0) = \gamma_i(\mathbf{z}(\nu),\gamma_0)$, $i = 1,2$, and consequently

$$\angle(\mathbf{e}(\gamma_1(\mathbf{y},\gamma_0)), \mathbf{e}(\gamma_2(\mathbf{y},\gamma_0))) = \angle(\mathbf{e}(\gamma_1(\mathbf{z}(\nu),\gamma_0)), \mathbf{e}(\gamma_2(\mathbf{z}(\nu),\gamma_0))) = \zeta(\nu)$$

for all $\nu > 0$. If $\mathbf{y} \in L(\mathbf{e}(\gamma_1(\mathbf{y},\gamma_0)), \mathbf{e}(\gamma_2(\mathbf{y},\gamma_0)))$, then $\zeta(\nu) = \phi_1(\mathbf{y}) + \phi_2(\mathbf{y})$ and

$$\cos\zeta(\nu) = \cos(\phi_1(\mathbf{y}) + \phi_2(\mathbf{y})) = \cos\phi_1(\mathbf{y})\cos\phi_2(\mathbf{y}) - \sin\phi_1(\mathbf{y})\sin\phi_2(\mathbf{y}).$$

From equation (13) it follows that

$$\cos\zeta(\nu) > \frac{t_{n-1,1-\alpha_1}t_{n-1,1-\alpha_2} - (n-1)}{\sqrt{(n-1)+t^2_{n-1,1-\alpha_1}}\sqrt{(n-1)+t^2_{n-1,1-\alpha_2}}},$$

which is in contradiction to the assumption (12).

If $\mathbf{y} \notin L(\mathbf{e}(\gamma_1(\mathbf{y},\gamma_0)), \mathbf{e}(\gamma_2(\mathbf{y},\gamma_0)))$, then consider the three vectors $\mathbf{y}$, $\mathbf{e}(\gamma_1(\mathbf{y},\gamma_0))$, and $\mathbf{e}(\gamma_2(\mathbf{y},\gamma_0))$. It holds

$$\angle(\mathbf{e}(\gamma_1(\mathbf{y},\gamma_0)), \mathbf{e}(\gamma_2(\mathbf{y},\gamma_0))) < \angle(\mathbf{y}, \mathbf{e}(\gamma_1(\mathbf{y},\gamma_0))) + \angle(\mathbf{y}, \mathbf{e}(\gamma_2(\mathbf{y},\gamma_0)))$$

and consequently

$$\zeta(\nu) < \phi_1(\mathbf{y}) + \phi_2(\mathbf{y}).$$

From this it follows with $\zeta(\nu) \in [0,\pi]$ and Eq. (13) that

$$\cos\zeta(\nu) > \cos(\phi_1(\mathbf{y}) + \phi_2(\mathbf{y})) > \frac{t_{n-1,1-\alpha_1}t_{n-1,1-\alpha_2} - (n-1)}{\sqrt{(n-1)+t^2_{n-1,1-\alpha_1}}\sqrt{(n-1)+t^2_{n-1,1-\alpha_2}}},$$

being again in contradiction to equation (12). ■

THEOREM 3.2 *If the regression function f and the experimental design point* $\mathbf{x}$ *satisfy the condition* (12), *then the test*

$$\Psi_3(\mathbf{y}) = \begin{cases} 1, & \mathbf{y} \in \mathcal{K}_{f,1}(t_{n-1,1-\alpha_1}) \cup \mathcal{K}_{f,2}(t_{n-1,1-\alpha_2}), \\ 0 & \textit{otherwise} \end{cases}$$

is a size-α test for the two-sided problem (11) *with $\alpha = \alpha_1 + \alpha_2$.*

Proof The assertion follows from Lemma 3.2 and Theorem 3.1. ■

COROLLARY 3.2 *The two-sided test Ψ_3 is a level-α test with $\alpha = \alpha_1 + \alpha_2$ and*

$$\{\gamma_0 \in \Gamma\colon T_1(\mathbf{y}) < t_{n-1,1-\alpha_1} \wedge T_2(\mathbf{y}) < t_{n-1,1-\alpha_2}\}$$

is a confidence region for the parameter γ with a confidence level $1 - \alpha$.

If condition (12) *is satisfied for all $\gamma_0 \in \Gamma$, then the size of this confidence region is exactly by equal to $1 - \alpha$.*

3.3 *Modified model assumptions*

There are several regression functions that do not satisfy the assumption (RA31) or (RA32) in the sense that their γ_0-related expected curve points distance function ν may increase strongly monotonously as γ approaches b or a, but are not unbounded. In such case, we shall construct tests for proving the hypothesis H_0: $\gamma = \gamma_0$ versus the one-sided alternatives H_{A_1} and H_{A_2} or the two-sided alternative H_{A} by the help of a suitable continuation of the curve $\mathfrak{M}$. To be more concrete, we shall make use of one of the following modified model assumptions or their combination, respectively:

(RA31m) ν increases strongly monotonously as $\gamma > \gamma_0$ approaches b and is bounded by the finite limit

$$\lim_{\gamma \to b} \|\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)\| =: b_1.$$

(RA32m) ν increases strongly monotonously as $\gamma < \gamma_0$ approaches a and is bounded by

$$b_2 := \lim_{\gamma \to a} \|\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_0)\|.$$

Put

$$\mathbf{B}_1 := \lim_{\gamma \to b} \boldsymbol{\eta}(\gamma), \quad \mathbf{B}_2 := \lim_{\gamma \to a} \boldsymbol{\eta}(\gamma) \quad \text{and} \quad \mathbf{e}_i := \frac{\mathbf{B}_i - \boldsymbol{\eta}(\gamma_0)}{\| \mathbf{B}_i - \boldsymbol{\eta}(\gamma_0)\|}, \quad i \in \{1, 2\}$$

and let the critical regions of the one-sided tests for the modified model assumptions be defined as

$$\widetilde{\mathcal{K}}_{f,i} = \widetilde{\mathcal{K}}_{f,i}(t_{n-1,1-\alpha}) := \{\mathbf{y} \in \mathbb{R}^n\colon \widetilde{T}_i(\mathbf{y}) \geq t_{n-1,1-\alpha}\}, \quad i \in \{1, 2\},$$

where $\widetilde{T}_i(\mathbf{y}) = T_i(\mathbf{y})$ if $\|\mathbf{y} - \boldsymbol{\eta}(\gamma_0)\| < b_i$ and

$$\widetilde{T}_i(\mathbf{y}) = \sqrt{n-1}\,\frac{\|\Pi_{\mathbf{e}_i}(\mathbf{y} - \boldsymbol{\eta}(\gamma_0))\| \operatorname{sign}(\langle \mathbf{y} - \boldsymbol{\eta}(\gamma_0), \mathbf{e}_i\rangle)}{\|\mathbf{y} - \boldsymbol{\eta}(\gamma_0) - \Pi_{\mathbf{e}_i}(\mathbf{y} - \boldsymbol{\eta}(\gamma_0))\|}$$

otherwise. An example for such a critical region is given in figure 3.

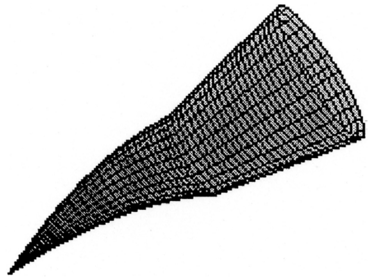

Figure 3. $\tilde{\mathcal{K}}_{f,1}$ for $f(x,\gamma) = 130 - 30\exp(-\gamma x)$, $n = 3$, $x_i = 10i$, $i = 1, 2, 3$, $\gamma_0 = 0.03$.

COROLLARY 3.3 *The one-sided tests* $\widetilde{\Psi}_i \mid \mathbb{R}^n \to \{0, 1\}$, $i \in \{1, 2\}$, *defined by*

$$\widetilde{\Psi}_i(\mathbf{y}) = \begin{cases} 1 & \textit{if } \mathbf{y} \in \widetilde{\mathcal{K}}_{f,i}(t_{n-1,1-\alpha}), \\ 0 & \textit{otherwise}, \end{cases}$$

are exact size-α *tests.*

Example 3.1 Let the regression function be $f(x,\gamma) = a - b\exp(-\gamma x)$, $x > 0$, $\gamma \in \Gamma = (0,\infty)$, and assume that the positive parameters a and b are known. Any experimental design point $\mathbf{x}$ with $\xi_j \in (0,\infty)$, $j = 1, \ldots, m$, satisfies the conditions (RA31m) and (RA32m) for every $\gamma_0 \in \Gamma$.

Analogously, the critical region of the two-sided test for the modified model assumptions is defined as

$$\widetilde{\mathcal{K}}_{f,3} = \widetilde{\mathcal{K}}_{f,1}(t_{n-1,1-\alpha_1}) \cup \widetilde{\mathcal{K}}_{f,2}(t_{n-1,1-\alpha_2})$$

for some α_1, α_2 with $0 < \alpha_1 + \alpha_2 < 1$. Assuming $\widetilde{\mathcal{K}}_{f,1}(t_{n-1,1-\alpha_1}) \cap \widetilde{\mathcal{K}}_{f,2}(t_{n-1,1-\alpha_2}) = \emptyset$, the test $\widetilde{\Psi}_3 \mid \mathbb{R}^n \to \{0, 1\}$ defined by $\widetilde{\Psi}_3(\mathbf{y}) := I_{\widetilde{\mathcal{K}}_{f,3}}(\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^n$, is also a size-$\alpha$ test with $\alpha = \alpha_1 + \alpha_2$.

4. Lower bounds for the power functions in the case of unknown σ^2

The power functions p_{Ψ_i} of the tests Ψ_i, $i = 1, 2, 3$, describe the probabilities of rejecting the null hypothesis H_0: $\gamma = \gamma_0$ if the true parameter is some $\gamma \neq \gamma_0$. In this case, it holds $\mathbf{Y} \sim EC_n(\boldsymbol{\eta}(\gamma), \sigma^2 I_n, g)$ with $\gamma \in \Gamma_1$, $\gamma \in \Gamma_2$, or $\gamma \in \Gamma_1 \cup \Gamma_2 =: \Gamma_3$, respectively. Note that

$$p_{\Psi_i}(\gamma) := \Phi_{\boldsymbol{\eta}(\gamma),\sigma^2 I_n,g}(\mathcal{K}_{f,i}), \quad \gamma \in \Gamma_i, \ \ i \in \{1, 2, 3\}. \tag{14}$$

An exact evaluation of these power functions by the help of the geometric measure representation formula (3) seems to be very complicated. It turns out to be much more easier to derive lower bounds for the power functions by measuring suitable subsets of the critical regions $\mathcal{K}_{f,i}$ with the elliptically symmetric measure $\Phi_{\boldsymbol{\eta}(\gamma),\sigma^2 I_n,g}$. The construction and measuring of a single ball or a union of several balls inside the critical regions will be favored here.

In this article, we shall consider such possibilities for estimating the power function only for the one-sided test Ψ_1, because the approximations for the power functions of the tests Ψ_2 and Ψ_3 may be carried out in a similar way.

To construct a suitable ball inside a curved transformed cone-type set, we consider the critical region $\mathcal{K}_{f,1}(t_{n-1,1-\alpha})$ of the test Ψ_1 and a fixed alternative parameter value $\overline{\gamma} \in \Gamma_1$, and try to construct a ball

$$K_n(\boldsymbol{\eta}(\overline{\gamma}), R(\overline{\gamma})) := \{\mathbf{z} \in \mathbb{R}^n \colon \|\mathbf{z} - \boldsymbol{\eta}(\overline{\gamma})\| \le R(\overline{\gamma})\}$$

with center in the point $\boldsymbol{\eta}(\overline{\gamma}) \in \mathfrak{M}$ and possibly maximal radius $R(\overline{\gamma})$ chosen in such a way that

$$K_n(\boldsymbol{\eta}(\overline{\gamma}), R(\overline{\gamma})) \subset \mathcal{K}_{f,1}(t_{n-1,1-\alpha}). \tag{15}$$

This problem is equivalent to determining the minimal Euclidean distance between the point $\boldsymbol{\eta}(\overline{\gamma})$ and $\partial\mathcal{K}_{f,1}(t_{n-1,1-\alpha})$, *i.e.*, $R(\overline{\gamma})$ is the solution of the minimization problem

$$R(\overline{\gamma})^2 = \min_{\mathbf{y} \in \partial\mathcal{K}_{f,1}} \|\mathbf{y} - \boldsymbol{\eta}(\overline{\gamma})\|^2. \tag{16}$$

Recognize that there exists a solution of this minimization problem. It can be determined by finding the solutions of a system of $m - 1$ nonlinear equations. In sections 4.1–4.3, we shall already make use of the solution of the problem (16). The respective technical details are given in refs. [24, 26].

4.1 *Measuring a suitable ball: a first lower bound for the power function p_{Ψ_1}*

The first lower bound for the power function of the one-sided test Ψ_1 at the point $\gamma_1 \in \Gamma_1$ will be simply the probability measure of a possibly large ball inside the critical region $\mathcal{K}_{f,1}(t_{n-1,1-\alpha})$ having its center in the point $\boldsymbol{\eta}(\gamma_1)$. Let $R(\gamma_1)$ denote the solution of the minimization problem (16) for $\overline{\gamma} = \gamma_1$ at a fixed $\gamma_1 \in \Gamma_1$.

THEOREM 4.1 *The power function of Ψ_1 satisfies the inequality*

$$p_{\Psi_1}(\gamma_1) > \Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)))$$

with the lower bound for $p_{\Psi_1}(\gamma_1)$ being representable as

$$\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1))) = I_{n,g}^{-1} \int_0^{R(\gamma_1)/\sigma} r^{n-1} g(r^2)\, \mathrm{d}r.$$

Proof Because the radius $R(\gamma_1)$ is a solution of the minimization problem (16) for $\overline{\gamma} = \gamma_1$, it holds

$$K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)) \subset \mathcal{K}_{f,1}(t_{n-1,1-\alpha}).$$

Consequently,

$$\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1))) < \Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(\mathcal{K}_{f,1}(t_{n-1,1-\alpha})) = p_{\Psi_1}(\gamma_1).$$

For evaluating the probability measure of the ball $K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1))$, we can now use the geometric measure representation formula (3).

Note that for the transformation $V_1 \mid \mathbb{R}^n \to \mathbb{R}^n$, $V_1(\mathbf{y}) = \sigma^{-1}(\mathbf{y} - \boldsymbol{\eta}(\gamma_1))$, $\mathbf{y} \in \mathbb{R}^n$, it holds

$$\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1))) = \Phi_{\mathbf{0}_n,I_n,g}(V_1(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1))))$$

and

$$V_1(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1))) = K_n\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right).$$

The spherical measure of the ball $K_n(\mathbf{0}_n, R(\gamma_1)/\sigma)$ is equal to the value of the distribution function of the central g-generalized χ^2-distribution at the point $(R(\gamma_1)/\sigma)^2$. The corresponding geometric measure representation formula for this distribution was given in ref. [18] and exploited in ref. [21]. The intersection percentage function is

$$\mathcal{F}\left(K_n\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right), r\right) = \begin{cases} 1, & 0 \le r \le \dfrac{R(\gamma_1)}{\sigma}, \\ 0, & \dfrac{R(\gamma_1)}{\sigma} < r. \end{cases}$$

Hence, we got

$$\begin{aligned}\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1))) &= \Phi_{\mathbf{0}_n,\sigma^2 I_n,g}\left(K_n\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right)\right) \\ &= I_{n,g}^{-1}\int_0^{R(\gamma_1)/\sigma} r^{n-1} g(r^2)\,\mathrm{d}r.\end{aligned}$$

■

A comparison of the first lower bound values with simulated values of the power function p_{Ψ_1} shows that this first approximation is already quite good for values $\gamma_1 \in \Gamma_1$ of the alternative, far away from γ_0. However, this first approximation fails for values γ_1 close to γ_0. That is why we shall improve this first approximation by constructing and measuring additional suitable balls inside the critical region. We consider two different possibilities in the forthcoming sections. We construct only one additional ball with center at a certain distance from the point $\boldsymbol{\eta}(\gamma_1)$ in section 4.2 and a sequence of additional balls in section 4.3.

4.2 *Measuring two balls: a first improvement of the lower bound for the power function p_{Ψ_1}*

Let a value $\gamma_1 \in \Gamma$ be fixed and let us define a function $\nu_1 \mid \Gamma \to [0, \infty)$ by

$$\nu_1(\gamma) := |\boldsymbol{\eta}(\gamma) - \boldsymbol{\eta}(\gamma_1)|, \quad \gamma \in \Gamma_1.$$

Put $\widetilde{\Gamma}_1 := \{\gamma \in \Gamma_1 : \gamma > \gamma_1\}$. Assume that the function ν_1 satisfies the assumption

(RA41) $\nu_1(\cdot)$ is monotonously and unbounded increasing in the strong sense as $\gamma \to b$.

We shall choose a second ball $K_n(\boldsymbol{\eta}(\widetilde{\gamma}), R(\widetilde{\gamma}))$ inside the critical region $\mathcal{K}_{f,1}(t_{n-1,1-\alpha})$ in such a way that the elliptically symmetric measure of this ball is as large as possible in some sense. To this end, the image $V_1(K_n(\boldsymbol{\eta}(\widetilde{\gamma}), R(\widetilde{\gamma})))$ of the second ball will be situated in such a part of the sample space $\mathbb{R}^n$ where the weighting function

$$w_g(r) := r^{n-1} g(r^2)$$

from the geometric measure representation formula attains its maximum value. It is possible to evaluate the radius r_g,

$$r_g := \underset{r\in(0,\infty)}{\operatorname{argmax}}\; w_g(r)$$

for different density generators g.

Example 4.1 (a) In the case of a Gaussian density generator g_G it holds $r_{g_G} = \sqrt{n-1}$.
(b) If the density generator is of Pearson-VII-type then

$$r_{g_P} = \sqrt{\frac{m^*(n-1)}{2M-(n-1)}}.$$

(c) The Kotz-type density generator g_K with $M = 1$ and $n > 1$ yields

$$r_{g_K} = \left[\frac{(n-1)}{2st}\right]^{1/(2s)}.$$

We choose the point $\boldsymbol{\eta}(\widetilde{\gamma}) \in \mathfrak{M}, \widetilde{\gamma} \in \widetilde{\Gamma}_1$, as the centre of the second ball, which has Euclidean distance σr_g from the point $\boldsymbol{\eta}(\gamma_1)$.

THEOREM 4.2 *Let a certain $\gamma_1 \in \Gamma_1$ be fixed and assume that the condition* (RA41) *is fulfilled. Put $r_g = \operatorname{argmax}_{r\in(0,\infty)} w_g(r)$ and let $\widetilde{\gamma} \in \Gamma_1$ denote the uniquely determined solution of the equation*

$$\nu_1(\gamma) = \sigma r_g.$$

Denote the solutions of the minimization problem (16) *for $\overline{\gamma} = \gamma_1$ and $\overline{\gamma} = \widetilde{\gamma}$ by $R(\gamma_1)$ and $R(\widetilde{\gamma})$, respectively, and let*

$$M := K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)) \cup K_n(\boldsymbol{\eta}(\widetilde{\gamma}), R(\widetilde{\gamma})).$$

(a) *The power function p_{Ψ_1} of the one-sided test Ψ_1 satisfies the inequality*

$$p_{\Psi_1}(\gamma_1) > \Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(M).$$

(b) *Let $\boldsymbol{\mu} := \sigma^{-1}(\boldsymbol{\eta}(\widetilde{\gamma}) - \boldsymbol{\eta}(\gamma_1))$. The lower bound for $p_{\Psi_1}(\gamma_1)$ satisfies the representation formula*

$$\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(M) = I_{n,g}^{-1} \int_0^\infty \mathcal{F}(\widetilde{M}, r) r^{n-1} g(r^2)\,\mathrm{d}r,$$

wherein $\mathcal{F}(\widetilde{M}, \cdot)$ is the intersection percentage function (i.p.f.) of a well-defined set $\widetilde{M}$. This i.p.f. can be written in terms of the i.p.f. of the ball $K_n(\boldsymbol{\mu}, R(\widetilde{\gamma})/\sigma)$ as follows:

$$\mathcal{F}(\tilde{M}, r) = I_{[0,R(\gamma_1)/\sigma]}(r) + I_{[r_g - R(\tilde{\gamma})/\sigma, r_g + R(\tilde{\gamma})/\sigma]}(r) \cdot \mathcal{F}\left(K_n\left(\boldsymbol{\mu}, \frac{R(\tilde{\gamma})}{\sigma}\right), r\right)$$

for $r_g > (R(\gamma_1) + R(\tilde{\gamma}))/\sigma$ and

$$\mathcal{F}(\tilde{M}, r) = I_{[0,R(\gamma_1)/\sigma]}(r) + I_{(R(\gamma_1)/\sigma, r_g + R(\tilde{\gamma})/\sigma]} \cdot \mathcal{F}\left(K_n\left(\boldsymbol{\mu}, \frac{R(\tilde{\gamma})}{\sigma}\right), r\right)$$

otherwise. For the i.p.f. of the ball $K_n(\boldsymbol{\mu}, R(\tilde{\gamma})/\sigma)$, we refer to ref. [21].

Proof Recognize that $\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(M) = \Phi_{\mathbf{0}_n,I_n,g}(\tilde{M})$, where

$$\tilde{M} = K_n\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right) \cup K_n\left(\boldsymbol{\mu}, \frac{R(\tilde{\gamma})}{\sigma}\right).$$

According to assumption (RA41), the value $\tilde{\gamma}$ is uniquely determined. As a solution of the minimization problem (16), $R(\tilde{\gamma})$ satisfies

$$K_n(\boldsymbol{\eta}(\tilde{\gamma}), R(\tilde{\gamma})) \subset \mathcal{K}_{f,1}(t_{n-1,1-\alpha})$$

and thus we have for $M := K_n(\boldsymbol{\eta}(\gamma_1), R(\widetilde{\gamma}_1)) \cup K_n(\boldsymbol{\eta}(\tilde{\gamma}), R(\widetilde{\gamma}))$ the relation

$$M \subset \mathcal{K}_{f,1}(t_{n-1,1-\alpha}).$$

Consequently,

$$\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(M) < \Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(\mathcal{K}_{f,1}(t_{n-1,1-\alpha})) = p_{\Psi_1}(\gamma_1).$$

The elliptically symmetric measure of the set M is equal to the spherical measure of the set $V_1(M) = \tilde{M}$. The set $\tilde{M}$ can be decomposed as follows:

$$\tilde{M} = K_n\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right) \cup \left[K_n\left(\boldsymbol{\mu}, \frac{R(\tilde{\gamma})}{\sigma}\right) \cap K_n\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right)^C\right].$$

By construction, it holds $\|\boldsymbol{\mu}\| = r_g$ and

$$\mathcal{F}(\tilde{M}, r) = 1 \quad \text{for } r \in \left[0, \frac{R(\gamma_1)}{\sigma}\right].$$

Taking into account the cases that $K_n(\mathbf{0}_n, R(\gamma_1)/\sigma) \cap K_n(\boldsymbol{\mu}, R(\widetilde{\gamma})/\sigma)$ may be empty or not and applying the i.p.f. for balls in ref. [21], we obtain assertion (b). ■

4.3 *Measuring several balls: improved lower bounds for p_{Ψ_1}*

Different improvements of the first lower bound for the power function p_{Ψ_1} at a fixed point $\gamma_1 \in \Gamma_1$ can be obtained by constructing and measuring several balls inside the critical region $\mathcal{K}_{f,1}(t_{n-1,1-\alpha})$.

Let the function ν_1 be defined as in section 4.2 and assume that the condition (RA41) is still satisfied. Fix $N \in \mathbb{N}$, $N \geq 2$, and choose $N-1$ parameter values $\gamma_k \in \widetilde{\Gamma}_1$, $k = 2, \ldots, N$, in such a way that $\gamma_1 < \gamma_2 < \cdots < \gamma_N$ with $\nu_1(\gamma_3) > R(\gamma_1)$ if $\nu_1(\gamma_2) \leq R(\gamma_1)$. An example for a reasonable concrete choice of the values γ_k will be given in section 4.4.

Let $R(\gamma_k)$ denote the solutions of the minimization problem (16) corresponding to $\overline{\gamma} = \gamma_k$, then

$$\bigcup_{k=1}^{N} K_n(\boldsymbol{\eta}(\gamma_k), R(\gamma_k)) \subset \mathcal{K}_{f,1}(t_{n-1,\alpha}).$$

We construct a disjoint decomposition of this union and derive the corresponding representation formula to compute this improved lower bound for $p_{\Psi_1}(\gamma_1)$. To this end, let

$$KS_n(\boldsymbol{\eta}(\gamma), r_1, r_2) := \{\mathbf{z} \in \mathbb{R}^n\colon r_1 < \|\mathbf{z} - \boldsymbol{\eta}(\gamma)\| \leq r_2\}, \quad 0 < r_1 < r_2 < \infty$$

denote a spherical shell with center $\boldsymbol{\eta}(\gamma)$, $\gamma \in \Gamma$.

Put $S_1 := R(\gamma_1)$, $S_2 := [\max(R(\gamma_1), \nu_1(\gamma_2)) + \nu_1(\gamma_3)]/2$, $S_k := [\nu_1(\gamma_k) + \nu_1(\gamma_{k+1})]/2$, $k = 3, \ldots, N-1$, and $S_N := \nu_1(\gamma_N) + R(\gamma_N)$.

Let a set M_N be defined by

$$M_N := \bigcup_{k=2}^{N} [K_n(\boldsymbol{\eta}(\gamma_k), R(\gamma_k)) \cap KS_n(\boldsymbol{\eta}(\gamma_1), S_{k-1}, S_k)]. \tag{17}$$

THEOREM 4.3 *Let $\gamma_1 \in \Gamma_1$ be fixed and assume that condition* (RA31) *is fulfilled.*

(a) *The power function p_{Ψ_1} of the one-sided test Ψ_1 satisfies the inequality*

$$p_{\Psi_1}(\gamma_1) > \Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)) \cup M_N),$$

with $R(\gamma_1)$ being the solution of the minimization problem (16) *for $\overline{\gamma} = \gamma_1$ and M_N being given in equation* (17).

(b) *Let* $\boldsymbol{\mu}_k := \sigma^{-1}(\boldsymbol{\eta}(\gamma_k) - \boldsymbol{\eta}(\gamma_1))$, $k = 2, \ldots, N$. *The lower bound for* $p_{\Psi_1}(\gamma_1)$ *satisfies the representation formula*

$$\begin{aligned}&\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)) \cup M_N)\\ &\quad = I_{n,g}^{-1}\left(\int_0^{R(\gamma_1)/\sigma} r^{n-1} g(r^2)\,\mathrm{d}r + \sum_{k=2}^{N}\int_{S_{k-1}/\sigma}^{S_k/\sigma} \mathcal{F}\left(K_n\left(\boldsymbol{\mu}_k, \frac{R(\gamma_k)}{\sigma}\right), r\right) r^{n-1} g(r^2)\,\mathrm{d}r\right).\end{aligned}$$

For the i.p.f. of the ball $K_n(\boldsymbol{\mu}_k, R(\gamma_k)/\sigma)$, *we refer to the representation formula in refs.* [21, 24].

Proof The radii $R(\gamma_k)$ satisfy

$$K_n(\boldsymbol{\eta}(\gamma_k), R(\gamma_k)) \subset \mathcal{K}_{f,1}(t_{n-1,1-\alpha}), \quad k = 1, \ldots, N,$$

and

$$K_n(\boldsymbol{\eta}(\gamma_k), R(\gamma_k)) \cap KS_n(\boldsymbol{\eta}(\gamma_1), S_{k-1}, S_k) \subset \mathcal{K}_{f,1}(t_{n-1,1-\alpha}), \quad k = 2, \ldots, N.$$

Hence, $K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)) \cup M_N \subset \mathcal{K}_{f,1}(t_{n-1,1-\alpha})$ and it holds

$$\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)) \cup M_N) < \Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(\mathcal{K}_{f,1}(t_{n-1,1-\alpha})).$$

Recall that

Q2

$$\Phi_{\boldsymbol{\eta}(\gamma_1),\sigma^2 I_n,g}(K_n(\boldsymbol{\eta}(\gamma_1), R(\gamma_1)) \cup M_N) = \Phi_{\mathbf{0}_n,I_n,g}\left(K_n\left(B_f\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right)\right) \cup \tilde{M}_N\right)$$

with

$$\tilde{M}_N := V_1(M_N) = \bigcup_{k=2}^{N}\left[K_n\left(\boldsymbol{\mu}_k, \frac{R(\gamma_k)}{\sigma}\right) \cap KS_n\left(\mathbf{0}_n, \frac{S_{k-1}}{\sigma}, \frac{S_k}{\sigma}\right)\right].$$

Q2 The set $K_n(B_f(\mathbf{0}_n, R(\gamma_1)/\sigma)) \cup \tilde{M}_N$ is a union of N disjoint subsets, consequently

Q2

$$\begin{aligned}&\Phi_{\mathbf{0}_n,I_n,g}\left(K_n\left(B_f\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right)\right) \cup \tilde{M}_N\right)\\ &\quad = \Phi_{\mathbf{0}_n,I_n,g}\left(K_n\left(\mathbf{0}_n, \frac{R(\gamma_1)}{\sigma}\right)\right) + \sum_{k=2}^{N}\Phi_{\mathbf{0}_n,I_n,g}\left(K_n\left(\boldsymbol{\mu}_k, \frac{R(\gamma_k)}{\sigma}\right) \cap KS_n\left(\mathbf{0}_n, \frac{S_{k-1}}{\sigma}, \frac{S_k}{\sigma}\right)\right).\end{aligned}$$

Assertion (b) follows directly with

$$\begin{aligned}&\Phi_{\mathbf{0}_n,I_n,g}\left(K_n\left(\boldsymbol{\mu}_k, \frac{R(\gamma_k)}{\sigma}\right) \cap KS_n\left(\mathbf{0}_n, \frac{S_{k-1}}{\sigma}, \frac{S_k}{\sigma}\right)\right)\\ &\quad = I_{n,g}^{-1}\int_{S_{k-1}/\sigma}^{S_k/\sigma} \mathcal{F}\left(K_n\left(\boldsymbol{\mu}_k, \frac{R(\gamma_k)}{\sigma}\right), r\right) r^{n-1} g(r^2)\,\mathrm{d}r.\end{aligned}$$

■

Notice that these considerations include the case of constructing $N-2$ balls in addition to the two balls considered already in section 4.2. Namely, we choose $\gamma_l = \tilde{\gamma}$ for a suitable $l \in \{2, \ldots, N\}$ in dependence on the relation between the quantities $\nu_1(\tilde{\gamma})$ and $R(\gamma_1)$. If $\nu_1(\tilde{\gamma}) \leq R(\gamma_1)$ put $\gamma_2 = \tilde{\gamma}$. According to the concrete difference $\nu_1(\tilde{\gamma}) - R(\gamma_1)$ and the density generating function g, in the remaining case $\nu_1(\tilde{\gamma}) > R(\gamma_1)$ there are several reasonable possibilities for choosing the index l.

4.4 *Numerical results*

We consider the regression function $f(x, \gamma) = x^\gamma$, $\gamma \in \Gamma = (-\infty, \infty)$, and the experimental design point $\mathbf{x} \in \mathbb{R}^4$ with $\xi_1 = 2$, $\xi_2 = 3$, $n_1 = 3$, $n_2 = 1$.

For comparing the different lower bounds for the power functions of the one-sided test Ψ_1 for $\gamma_0 = 2$, $\sigma^2 = 1$, and $\alpha = 0.05$, we choose three density generating functions g_{K}, g_{G}, and g_{P_1} with the parameters $M = t = s = 1$ for the Kotz-type generator g_{K} and with $m^* = 3$ and $M = (n + m^*)/2 = 7/2$ for the Pearson-VII-type generator g_{P_1}.

Notice that in the case of unknown parameter σ, the existence of the moments is also not necessary for the construction of the tests and the evaluation of the different lower bounds for the power functions. To illustrate this, we choose an additional density generator $g = g_{\mathrm{P}_2}$ of Pearson-VII-type with parameters $m^* = 1$ and $M = (n + 1)/2 = 5/2$, *i.e.*, $\mathbf{Y}$ follows a multivariate Cauchy distribution.

Recognize that the Kotz-type density generator g_{K} defines lighter tails and the Pearson-VII-type density generators g_{P_1} and g_{P_2} define heavier tails than the Gaussian density generator g_{G}. For a comparison of the functions $I_{n,g}^{-1} w_g(r) = I_{n,g}^{-1} r^{n-1} g(r^2)$ for the four density generators, see figure 4.

We simulate the power functions of the test Ψ_1 for all four density generating functions with 100.000 repetitions. The results are given in figure 5.

Now, we evaluate the first lower bounds and the first improved lower bounds for the power function p_{Ψ_1} according to Theorems 4.1 and 4.2. The integrations are performed by Simpson's rule with 100.000 steps.

Besides this, we use the result of section 4.3 with $N = 4$ and $N = 8$ for a further improvement of the approximations especially for values $\gamma_1 \in \Gamma_1$ close to γ_0. We choose different parameter values $\gamma_{k,g}$, $k = 2, \ldots, N$, for all four density generating functions. To determine these parameter values $\gamma_{k,g}$ for $N = 4$, we choose first three radii $r_g(0.25)$, $r_g(0.5)$, and $r_g(0.75)$ in such a way that

$$\Phi_{\mathbf{0}_4, I_4, g}\left(K_4\left(\mathbf{0}_4, r_g\left(\frac{k}{4}\right)\right)\right) = \frac{k}{4}, \quad k = 1, 2, 3,$$

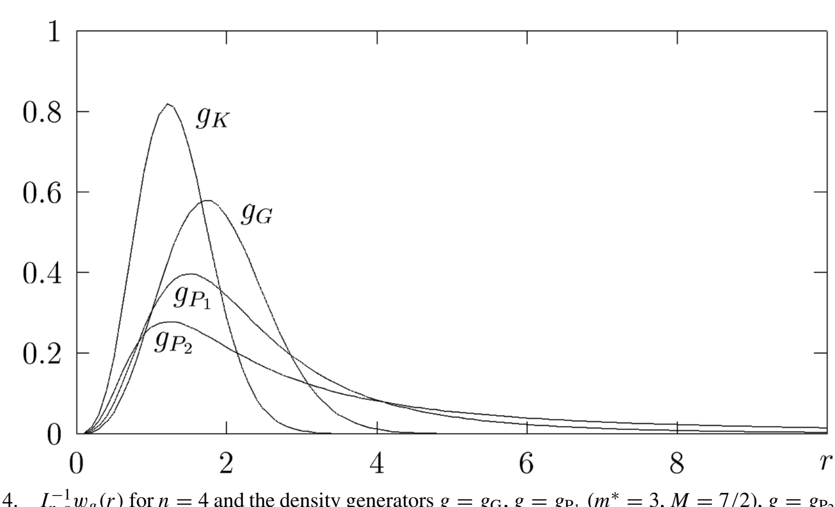

Figure 4. $I_{n,g}^{-1} w_g(r)$ for $n = 4$ and the density generators $g = g_{\mathrm{G}}$, $g = g_{\mathrm{P}_1}$ ($m^* = 3$, $M = 7/2$), $g = g_{\mathrm{P}_2}$ ($m^* = 1$, $M = 5/2$), and $g = g_{\mathrm{K}}$ ($M = t = s = 1$).

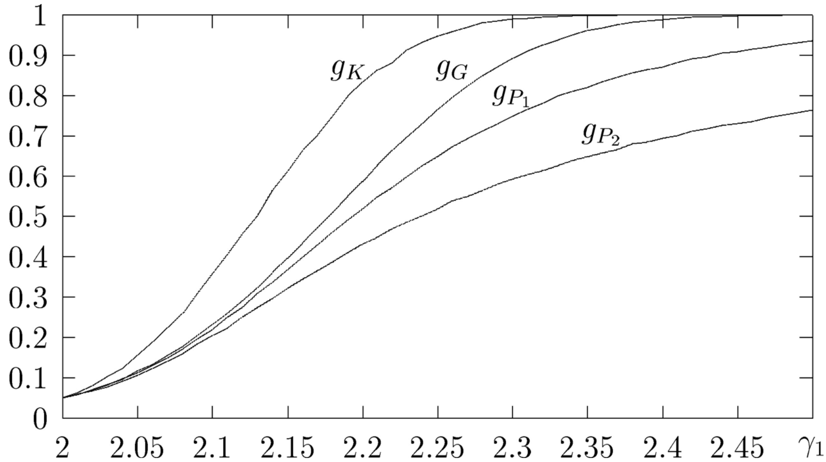

Figure 5. Simulated power functions for the density generators g_{K}, g_{G}, g_{P_1}, and g_{P_2}.

i.e., the radius $r_g(k/4)$ is the square root of the $(k/4)$-quantile of the central g-generalized χ^2-distribution with four degrees of freedom.

In the case $g = g_{\mathrm{G}}$, we use the quantiles of the usual χ^2-distribution with four degrees of freedom given in ref. [27], namely, $r_{g\mathrm{G}}(0.25) = \sqrt{1.923}$, $r_{g\mathrm{G}}(0.5) = \sqrt{3.357}$, $r_{g\mathrm{G}}(0.75) = \sqrt{5.385}$.

If $g = g_{\mathrm{P}_1}$, $g = g_{\mathrm{P}_2}$, or $g = g_{\mathrm{K}}$, we evaluate the radii numerically by the help of the bisection method and the geometric measure representation formula for the distribution function $\mathrm{CQ}(n; g)(\cdot)$ of the central g-generalized χ^2-distribution, in ref. [18], is given by

$$\mathrm{CQ}(n; g)(R^2) = \Phi_{\mathbf{0}_n, I_n, g}(K_n(\mathbf{0}_n, R)) = I_{n,g}^{-1} \int_0^R r^{n-1} g(r^2)\,\mathrm{d}r.$$

The integration is performed again by Simpson's rule with 100.000 steps and the iteration algorithm stops if the approximation R for $r_{g_{\mathrm{P}_i}}(k/4)$, $i = 1, 2$, or $r_{g_{\mathrm{K}}}(k/4)$ satisfies $|CQ(4; g)(R^2) - k/4| < 10^{-10}$. In this way, we get

$$\begin{aligned} &r_{g_{\mathrm{P}_1}}(0.25) = 1.397997357, \quad r_{g_{\mathrm{P}_1}}(0.5) = 2.062256723, \quad r_{g_{\mathrm{P}_1}}(0.75) = 3.091995344, \\ &r_{g_{\mathrm{P}_2}}(0.25) = 1.487655189, \quad r_{g_{\mathrm{P}_2}}(0.5) = 2.700159136, \quad r_{g_{\mathrm{P}_2}}(0.75) = 5.858649731 \end{aligned}$$

and

$$r_{g_{\mathrm{K}}}(0.25) = 0.980448246, \quad r_{g_{\mathrm{K}}}(0.5) = 1.295510320, \quad r_{g_{\mathrm{K}}}(0.75) = 1.640924900.$$

Put $\gamma_{1,g} := \gamma_1$. The parameter values $\gamma_{k,g}$ in section 4.3 are the solutions of the equations

$$\nu_1(\gamma_{k,g}) = r_g\left(\frac{k-1}{4}\right), \quad k = 2, 3, 4.$$

For the value γ_1 far enough from γ_0, the radius $R(\gamma_1)$ of the first ball exceeds the value $r_g(0.5)$. In this case, we evaluate only $\gamma_{3,g}$ and $\gamma_{4,g}$. If $R(\gamma_1)$ also exceeds the value $r_g(0.75)$, we compute only the parameter value $\gamma_{4,g}$. Hence, we use only for γ_1-values of the alternative close to γ_0 the elliptically symmetric measure of the union of the four balls $K_4(\boldsymbol{\eta}(\gamma_{k,g}), R(\gamma_{k,g}))$, $k = 1, \ldots, 4$, for the evaluation of the lower bound of the power function.

To check the influence of additional balls, we consider also the case $N = 8$ and compute, in addition to $r_g(0.25)$, $r_g(0.5)$, and $r_g(0.75)$, the four radii $r_g(0.125)$, $r_g(0.375)$, $r_g(0.625)$, and $r_g(0.875)$ by the same procedure. The parameter values $\gamma_{k,g}$, $k = 2, \ldots, 8$ are the solutions of the equation

$$\nu_1(\gamma_{k,g}) = r_g\left(\frac{k-1}{8}\right).$$

If the radius $R(\gamma_1)$ exceeds the values $r_g((k-1)/8)$, $k = 3, \ldots, 8$, we compute only the appropriate parameter values $\gamma_{k,g}$.

We evaluate these two additional lower bounds for the power function p_{Ψ_1} according to the result of section 4.3 and use Simpson's rule with 100.000 steps for each integration.

The results of our numerical study are given in figures 6, 7, 8 and 9. For the considered density generators, the first lower bound b_1 could be substantially improved by the computation of the first improved lower bound b_{imp} according to Theorem 4.2.

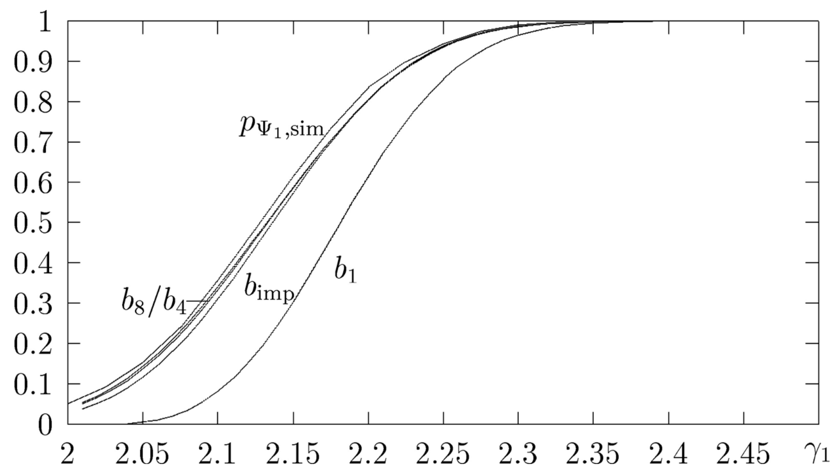

Figure 6. Kotz-type density generator g_{K}.

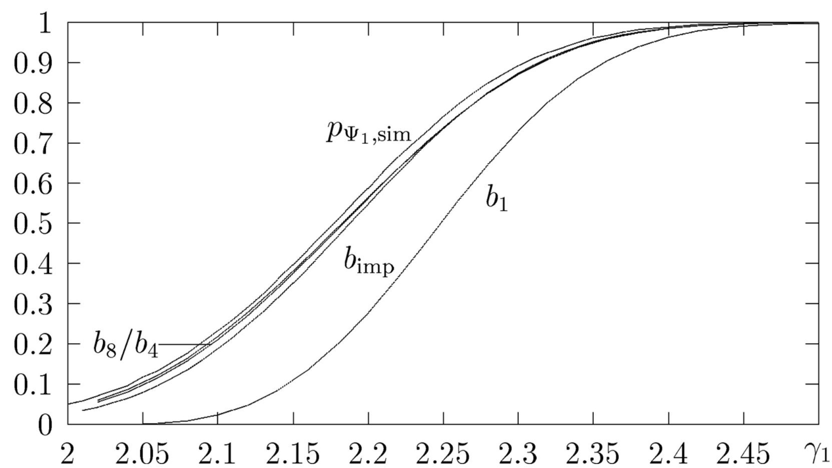

Figure 7. Gaussian density generator g_{G}.

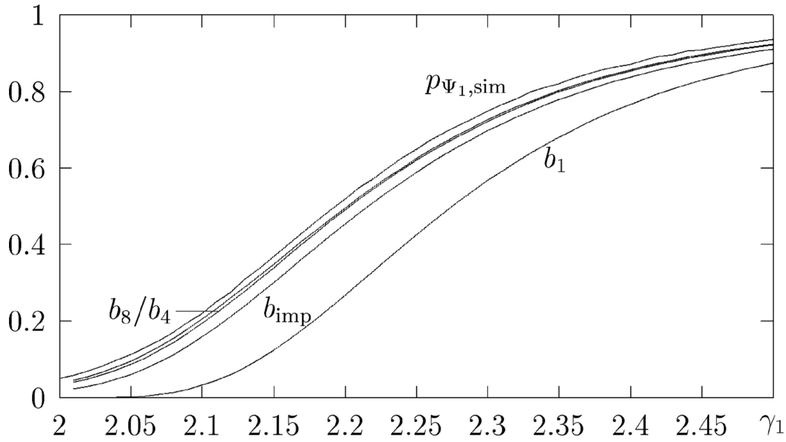

Figure 8. Pearson-VII-type density generator g_{P_1}.

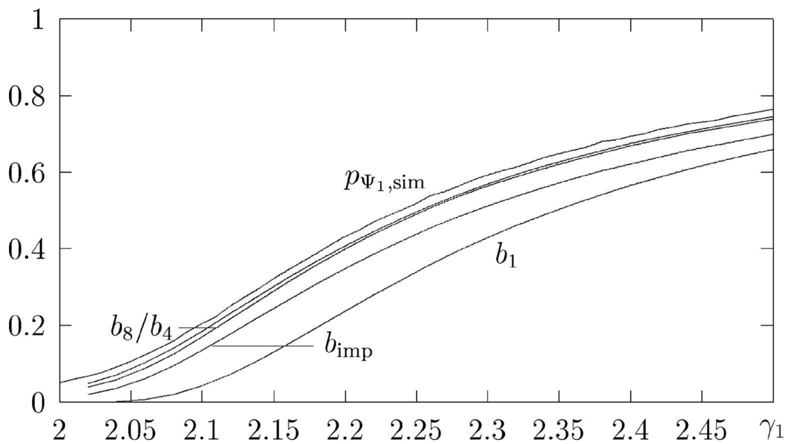

Figure 9. Pearson-VII-type density generator g_{P_2}.

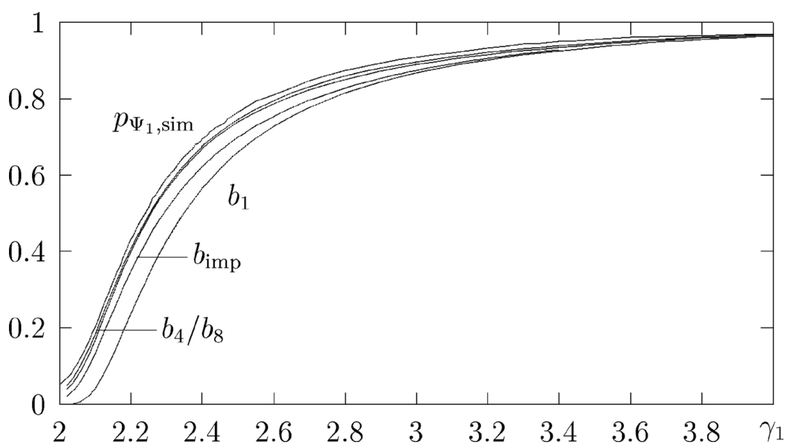

Figure 10. Pearson-VII-type density generator g_{P_2}.

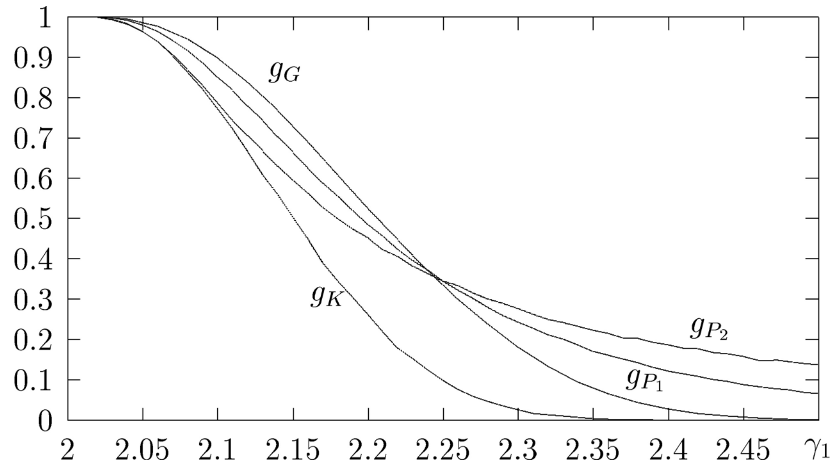

Figure 11. Relative error of the first lower bound.

A further slight improvement for values γ_1 close to γ_0 could be observed for the lower bounds b_N according to the result of section 4.3 for $N = 4$ and $N = 8$. The differences between the lower bounds for $N = 4$ and $N = 8$ are very small.

If $g = g_{\mathrm{P}_2}$, we evaluate the different lower bounds for a wider range of values $\gamma_1 \in \Gamma_1$. The result is given in figure 10.

Figure 11 allows a certain joint interpretation of the effects reflected in figures 4 and 5. Note that the relative error between the first lower bound and the simulated value of the power function for the density generators g_{P_1} and g_{P_2} are smaller than the corresponding relative error for the density generator g_{G} for values γ_1 close to γ_0, because $I_{n,g_{\mathrm{P}_i}} w_{g_{\mathrm{P}_i}}(r) > I_{n,g_{\mathrm{G}}} w_{g_{\mathrm{G}}}(r)$, $i = 1, 2$, for small values of r.

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