On $l_{2,p}$-circle numbers

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Abstract. Circle numbers are defined which reflect the Euclidean area-content- and a suitably defined non-Euclidean, if $p \neq 2$, circumference-properties of the $l_{2,p}$-circles, $p \in [1, \infty]$. The resulting function is continuous and monotonously increasing and takes all values from $[2, 4]$.
The actually chosen dual l_{2,p^*}-geometry for measuring the arc-length is closely connected with a generalization of the indivisiblen method of Cavalieri and Torricelli in the sense that integrating such arc-lengths means measuring area content. Moreover, this approach enables one to look in a new way onto the co-area formula of measure theory which says that integrating Euclidean arc-lengths does not yield area content unless for $p = 2$.
The new circle numbers play a natural role, e.g., as norming constants in geometric measure representation formulae for p-generalized uniform probability distributions on $l_{2,p}$-circles.
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1. Introduction People are dealing with the Archimedes or Ludolph number π more or less explicitly since millenniums. For an excellent and considerable complete introduction to the theme π we refer to [2, 3, 4, 6, 11]. Because of its long history, the idea of π is for many people that of unchangeableness. However, if we look nearer then we can see that things are nevertheless in motion. Without going into any details, we refer to [1, 7, 8, 10, 15, 16, 17, 19, 20, 22, 23] where several contributions were made on the geometric way to the generalization of π.
The famous name circle number is due to the following well known facts. Both the ratio of the circle's circumference to its diameter is the same for all circles and the ratio of the inside disc's area content to the square of the radius is the same for all circles. The corresponding constants coincide and their value is just π. What can we say about similar chosen ratios if we consider $l_{2,p}$-circles for arbitrary $p \in [1, \infty]$ and $r > 0$?
In the present paper, we let measuring the area content unchanged Euclidean and use, for each $p \in [1, \infty], p \neq 2$, that non-Euclidean geometry for measuring the circumference of the $l_{2,p}$-circle which is generated by the 'dual' l_{2,p^*}-circle. The notion of the diameter will be replaced by twice the p-generalized radius.
Recall that the co-area formula of measure theory (see, e.g., in [9]) says that integrating the $l_{2,2}$-arc-lengths of the $l_{2,p}$-circles with radii $r \in [0, R]$ does not yield the area content of $K_p(R)$ unless for $p = 2$. The present approach, however, is based upon a disintegration formula for the Lebesgue measure which shows that integrating the l_{2,p^*}-arc-lengths of the $l_{2,p}$-circles yields the area content of $K_p(R)$. This circumstance can also be interpreted as a generalization of the famous indivisiblen method of Cavalieri and Torricelli where the indivisibles are now the

circles $C_p(r)$. For the original probabilistic justification of this approach, we refer to [18] from where we see, e.g., that the notion of the uniform distribution on the unit circle $C_p(1)$ is closely connected with the notion of the p-generalized circle number presented here.
It turns out in the present paper that, just like in the Euclidean case, the area content of the unit disc is suitable for defining this p-generalized circle number for each $p \in [1, \infty]$. The resulting π-function takes therefore all values from $[2, 4]$.

2. Results The set $C_p(r) = \{(x, y) \in R^2 : |x|^p + |y|^p = r^p\}$ will be considered for $p \geq 1$ as the $l_{2,p}$-circle with p-generalized radius $r > 0$. The Euclidean area content of the disc $K_p(r)$ inside $C_p(r)$ will be denoted by $A_p(r)$. It is well known that $A_p(r) = A_p r^2$ where

$$A_p = \frac{2\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}. \qquad (1)$$

For defining the circumference of $C_p(r)$, the circle will be considered as a subset of a Minkowski plane $(R^2, d_q), q \geq 1$ where $d_q((u, v)) = (|u|^q + |v|^q)^{1/q}$ denotes the $l_{2,q}$-norm of the point (u, v) from R^2 and $q \geq 1$ will be chosen later.

Let $z_0, z_1, ..., z_n$ be an arbitrary successive partition of the circle $C_p(r)$ and define the d_q-arc-length $AL_{p,q}(r)$ of this circle as the supremum of $\sum_{j=1}^{n} d_q(z_j - z_{j-1})$ taken over all such partitions. We have then $AL_{p,q}(r) = rAL_{p,q}$ where $AL_{p,q}$ denotes the length of the unit circle.

Lemma (a) If p^* satisfies the equation $\frac{1}{p} + \frac{1}{p^*} = 1$ for $p > 1$ and $p^* = \infty$ for $p = 1$ then the following equation is valid for each $p \geq 1$:

$$AL_{p,p^*} = 2A_p.$$

(b) The value of p^* is unique in the sense that $AL_{p,q} > AL_{p,p^*}$ if $q < p^*$ and $AL_{p,q} < AL_{p,p^*}$ if $q > p^*$.

Proof (a) It is well known that

$$AL_{p,q} = 4\int_0^1 (1 + |y'(x)|^q)^{1/q} dx, \quad y(x) = (1 - x^p)^{1/p}.$$

On substituting $u = x^p$, we get

$$AL_{p,q} = \frac{4}{p}\int_0^1 (u^{(1-p)q/p} + (1 - u)^{(1-p)q/p})^{1/q} du.$$

Under the assumptions of the lemma, it follows

$$AL_{p,p^*} = \frac{4}{p}\int_0^1 (u^{-1} + (1-u)^{-1})^{(p-1)/p} du = \frac{4}{p}\int_0^1 u^{\frac{1}{p}-1}(1-u)^{\frac{1}{p}-1} du = \frac{4}{p}B(\frac{1}{p}, \frac{1}{p})$$

where $B(.,.)$ denotes the Beta function. Hence, $AL_{p,p^*} = 2A_p$ for all $p \geq 1$.
(b) Let the positive quadrant part of $C_p(1)$ be described by

$$x = \cos\varphi/N_p(\varphi),\ y = \sin\varphi/N_p(\varphi)$$

where $N_p(\varphi) = (|\sin\varphi|^p + |\cos\varphi|^p)^{1/p}$ and $0 \le \varphi < \pi/2$. The arc-length $AL_{p,q}$ can be written as

$$AL_{p,q} = 8 \int_0^{\pi/4} (|x'(\varphi)|^q + |y'(\varphi)|^q)^{1/q} d\varphi$$

with

$$x'(\varphi) = -\frac{(\sin\varphi)^{p-1}}{(N_p(\varphi))^{p+1}}, \quad y'(\varphi) = \frac{(\cos\varphi)^{p-1}}{(N_p(\varphi))^{p+1}}.$$

Hence,

$$AL_{p,q} = 8 \int_0^{\pi/4} ((\frac{\sin\varphi}{N_p(\varphi)})^{(p-1)q} + (\frac{\cos\varphi}{N_p(\varphi)})^{(p-1)q})^{1/q} \frac{d\varphi}{N_p^2(\varphi)}$$

$$= 8 \int_0^{\pi/4} \frac{(1 + (\tan\varphi)^{q(p-1)})^{1/q}}{(1 + (\tan\varphi)^p)^{1-1/p}} \frac{d\varphi}{N_p^2(\varphi)}.$$

Because of $0 \le \tan\varphi \le 1$ for $0 \le \varphi \le \pi/4$ and

$$q(p-1) = (>)(<)p \Leftrightarrow q = (>)(<)p^* \Leftrightarrow 1 - 1/p = (>)(<)1/q$$

it follows that $\frac{(1+(\tan\varphi)^{q(p-1)})^{1/q}}{(1+(\tan\varphi)^p)^{1-1/p}} = (<)(>)1$ iff $q = (>)(<)p^*$, respectively $\Box$

Summarizing what we know so far, we have, for all $r > 0$,

$$\frac{A_p(r)}{r^2} = \frac{AL_{p,p^*}(r)}{2r} = A_p \qquad (2)$$

and

$$AL_{p,p^*}(r) = A_p'(r). \qquad (3)$$

In the case $p = 2$, the equations (2) reflect the above mentioned well known properties of the Euclidean circle which we shall refer to as its Euclidean area-content- and its Euclidean circumference-properties. And, obviously, there holds $A_2 = \pi$.

Equation (3) allows a comprehensive interpretation. To start with, notice that it may be rewritten as

$$A_p(r) = \int_0^r AL_{p,p^*}(\varrho) d\varrho. \qquad (4)$$

This representation formula for $A_p(r)$ may be considered as reflecting a generalization of the indivisiblen method of Cavalieri and Torricelli in the sense that the indivisibles are the circles $C_p(\varrho), \varrho \in [0, r]$ and measuring the area content means to integrate the d_{p^*}-arc-lengths of the indivisibles.

For a further interpretation of equation (4), let us recall from [9] what the co-area formula says in this connection:

$$\int_0^r AL_{p,2}(\varrho)d\varrho = \int_{K_p(r)} (\frac{||x||_{2p-2}}{||x||_p})^{p-1}dx. \quad (5)$$

The integral on the right side of equation (5), however, is larger or smaller than $A_p(r)$ iff $1 \leq p < 2$ or $p > 2$, respectively. Hence, integrating the Euclidean arc-lengths of the indivisibles $C_p(r)$ as on the left side of equation (5) does not coincide with measuring the area content, unless for $p = 2$.

Let us come back again to equation (3). Busemann [5] studied the arc-lengths $AL_{p,q}$ within a more general problem than considered here. He proved that $AL_{p,q}$ may be understood as two times a mixed area (volume), say $V(K_p(r), I)$, for a certain set I. On combining this result with equation (3), it follows that $A'_p(r)$ may be considered as two times this mixed area, and vice versa. This circumstance may be considered both as an additional reason for considering here just the special arc-length $AL_{p,p^*}(r)$ and as a characterization of the mixed area $V(K_p(r), I)$ in terms of the d_{p^*}-arc-length.
Notice that the derivative of the area content function, $A'_p(r)$, in formula (3) should not be confused with a congenial quantity which is frequently used in convexity considerations and defined, e.g., in [24, p. 295] as, say, $f'_p(0)$. Here, $f_p(\lambda) = v_2((K_p(r))_\lambda)$ denotes the two-dimensional Lebesgue measure v_2 of the Minkowski sum $K_p(r) + \lambda \cdot K_2(1)$. It is known that $f'_p(0)$ equals two times a mixed area which is nothing else than the Euclidean or $l_{2,2}$-arc-length of $C_p(r)$. Hence, $A'_p(r) = f'_p(0)$ if and only if $p = 2$. For the origin of the theory of mixed volumes, we refer to Minkowski [14].

All what was said in the discussion of equation (3) means that the non-Euclidean, if $p \neq 2$, geometry of the Minkowski plan (R^2, d_{p^*}) is a suitable geometry for defining the circumference of an $l_{2,p}$-circle. Therefore, and according to the notation in [18], the d_{p^*}-arc-length of the $l_{2,p}$-circle $C_{2,p}(r)$, $AL_{p,p^*}(r)$, will be called its $l_{2,p}$-generalized circumference. Additional justification for this approach comes from the applications presented below and in [18].

The equations (2) may be interpreted now for arbitrary $p \geq 1$ as reflecting Euclidean area-content- and the $l_{2,p}$-generalized circumference-properties of the $l_{2,p}$-circle $C_p(r)$. The following definition is therefore well motivated.

Definition For arbitrary $p \in [1, \infty]$, the quantity A_p will be called the $l_{2,p}$- or p-generalized circle number and denoted by $\pi(p)$.

As because $A_p = A_p(1)$ is the area content of the unit disc, the function $p \to \pi(p), p \in [1, \infty]$ is continuous and monotonously increasing and satisfies the relations $2 = \pi(1) \leq \pi(p) < \lim_{p\to\infty} \pi(p) =: \pi(\infty) = 4$. Hence, all numbers from $[2, 4]$ are geometrically interpretable as $l_{2,p}$-circle numbers.

Let us give some numerical values of these numbers.

Table 1 Values of selected $l_{2,p}$-circle numbers.

p	1	12/11	1,482	2	3	4,245	5	10	255
$\pi(p)$	2	2,172	e	π	3,533	3,736	3,801	3,9429	3,999

Theorem The $l_{2,p}$-circle numbers satisfy the representation formulae

$$\pi(p) = 4\int_0^1 (1-y^p)^{\frac{1}{p}}dy = 4\int_0^\infty (1+\tau^p)^{-\frac{2}{p}-1}d\tau = 2\int_0^1 (1-\mu^p)^{\frac{1-p}{p}}d\mu. \quad (6)$$

Proof The first equation in (6) reflects the definition of the area content integral and the second one follows on substituting $1-y^p=(1+\tau^p)^{-1}$. The proof of the third representation formula reflects the geometric nature of the circle number $\pi(p)$ in an additional way. To this end, let us consider the area integral $A_p(R) = \int\limits_{K_p(R)} dxdy, R>0$. Changing Cartesian with p-generalized standard triangle coordinates from [18] separately for nonnegative and negative y's, i.e., $x=r\mu, y=+(-)r(1-|\mu|^p)^{1/p}$, we get

$$A_p(R) = 2\int_0^R\int_{-1}^1 r(1-|\mu|^p)^{(1-p)/p}d\mu dr$$

and therefore

$$AL_{p,p^*}(R) = 4R\int_0^1 (1-\mu^p)^{(1-p)/p}d\mu \qquad \square$$

A further justification for considering the geometry of the Minkowski plane (R^2, d_{p^*}) when measuring the lengths of the $l_{2,p}$-circles $C_p(r)$ and when defining the circle numbers $\pi(p)$ is given by the following probabilistic application from [18]. Let the random vector (X,Y) be uniformly distributed on $K_p(1)$ or distributed according to the p-generalized normal distribution. The normalized vector $(\xi,\eta)=(X,Y)/(|X|^p+|Y|^p)^{1/p}$ is then p-generalized uniformly distributed on the $l_{2,p}$-unit circle in the sense that for arbitrary Borel subset A of $C_p(1)$ it holds

$$P((\xi,\eta)\in A) = \frac{l_{2,p^*}\text{-arc-length of } A}{l_{2,p^*}\text{-arc-length of } C_p(1)} = \int\limits_{POL_p^{*-1}(A)} \frac{d\varphi}{N_p^2(\varphi)} / \int_0^{2\pi} \frac{d\varphi}{N_p^2(\varphi)}.$$

Here, $POL_p^{*-1}(A)$ denotes the inverse image of the set A under the transformation $POL_p^*(\varphi) = POL_p(1,\varphi)$ where the p-generalized polar coordinate transformation $POL_p|[0,\infty)\times[0,2\pi)\to R^2$ is defined in [18] by

$$x = r\cos\varphi/N_p(\varphi),\ y = r\sin\varphi/N_p(\varphi).$$

Because of

$$\frac{1}{2}AL_{p,p^*} = \frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{N_p^2(\varphi)} = \pi(p), \quad (7)$$

it follows

$$P((\xi,\eta)\in A) = \frac{l_{2,p^*}\text{-arc-length of } A}{2\pi(p)} = \frac{1}{2\pi(p)}\int\limits_{POL_p^{*-1}(A)}\frac{d\varphi}{N_p^2(\varphi)}. \quad (8)$$

According to the equations (2), formula (8) may also be interpreted in the spirit of Kepler's second law as follows. Imagine a point is moving on $C_p(1)$. Then, equal areas $\int\limits_{POL_p^{*-1}(A_1)} \frac{d\varphi}{N_p^2(\varphi)}/2$ and $\int\limits_{POL_p^{*-1}(A_2)} \frac{d\phi}{N_p^2(\phi)}/2$ are swept out (from the origin) in equal "times" $T_1 = P((\xi,\eta) \in A_1)$ and $T_2 = P((\xi,\eta) \in A_2)$, respectively. For related considerations in the case $p = 1$, we refer to [23].

For another application of the generalized circle numbers, let us continue the example from [18]. Imagine a machine tool which moves along an $l_{2,p}$-circle line of p-generalized radius $r + \varepsilon$, creating thereby a thin protective coat on a workpiece of p-generalized radius r by applying a special material to it's surface. The consumed material has then approximately, i.e. for small ε, the area content $2\pi(p)r\varepsilon$.

For another interpretation of the $l_{2,p}$-circle numbers, notice that there is a broad analytical research area dealing with constructing periodic functions which are generalizations of the classical trigonometric functions and have a period differently from 2π. This research direction was started with some work of Gauss (unpublished, see, e.g., in [25, § 1.14.17]) in 1796 and Lundberg [13] in 1879 and continued then by many authors. Following this way, and in the spirit of Landau, Shelupsky [21] and Lindqvist and Peetre [12] defined for $p \in \{1, 2, ...\}$ numbers π_p as $\pi_p/2$ being the first positive zero point of a certain p-generalized cosine function. For $p \in \{1, 2, ...\}$, the value of the $l_{2,p}$-circle number $\pi(p)$ defined here coincides with that of π_p. The third integral representation formula in (6) is just what was proved in [21] for the quantities π_p to hold and the second one corresponds to a representation formula in [13].
The present paper, however, does not rely on this analytical approach to generalizing π.

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