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ON THE π-FUNCTION FOR NONCONVEX $l_{2,p}$-CIRCLE DISCS

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Abstract. For $0 < p < 1$, circle numbers $\pi(p)$ are defined to reflect the Euclidean area-content property $A_p(r) = \pi(p)r^2$ and circumference property $AL_{p,p^{**}}(r) = 2\pi(p)r$ of the $l_{2,p}$-circle discs with p-generalized radius r, where the arc-length measure $AL_{p,p^{**}}$ is based upon the nonconvex star-shaped set $S(p^{**}) = \{\frac{1}{|x|^{p^{**}}} + \frac{1}{|y|^{p^{**}}} \geqslant 1\}$ with $p^{**} > 0$ satisfying $-\frac{1}{p^{**}} + \frac{1}{p} = 1$. The resulting π-function extends the function $p \to \pi(p)$ recently defined in [2] from the case of convex discs, $p \geqslant 1,$ to the nonconvex case $0 < p < 1.$ This function is continuous, increasing, and taking values in $(0, 2)$.

The presented approach can be considered as reflecting a modified method of indivisibles in the sense that the indivisibles are the $l_{2,p}$-circles and that integrating their $S(p^{**})$-arc-lengths is equivalent to measuring the Euclidean area content.

Keywords: generalized circle numbers, generalized π, extended π-function, generalized perimeter, star-shaped arc-length measure, modified method of indivisibles, geometry of real numbers, p-generalized uniform distribution.

1 INTRODUCTION

The circle number π was recently generalized in [2] for $l_{2,p}$-circles $C_p(r)$ of p-generalized radius $r > 0$ and with $p \geqslant 1$ in the following sense. Let $A_p(r)$ be the Euclidean area content of the disc $K_p(r)$ inside $C_p(r)$, and let $AL_{p,p^*}(r)$ be the l_{2,p^*}-arc-length of $C_p(r)$ with 'dual' $p^* \geqslant 1$ satisfying $(p, p^*) = (1, \infty)$ or $\frac{1}{p^*} + \frac{1}{p} = 1$ for $p > 1$. The ratios $A_p(r)/r^2$ and $AL_{p,p^*}(r)/(2r)$ then do not depend on $r > 0$, their actual values are the same, this common number is denoted by $\pi(p)$, and the function $p \to \pi(p)$ is called the π-function.

Measuring the l_{2,p^*}-arc-length of $C_p(r)$ can be equivalently considered as being based upon the convex set $K_{p^*}(1)$. Replacing this set by a suitably chosen star-shaped one is the basic idea of this paper for extending the π-function to the case of nonconvex circle discs.

To be more specific, let us recall that if $p \geqslant 1$ and $q \geqslant 1$, then the $l_{2,q}$-arc-length of the $l_{2,p}$-circle $C_p(r)$ is given by

$$AL_{p,q}(r) = \int_0^{2\pi} \left(|x'(\varphi)|^q + |y'(\varphi)|^q\right)^{1/q} \mathrm{d}\varphi,$$

where $\varphi \to (x(\varphi), y(\varphi))$ is an arbitrary differentiable parameter representation for $C_p(r)$. If d_K denotes the Minkowski functional of a suitable star-shaped set K, then $AL_{p,q}(r)$ can also be written as

$$AL_{p,q}(r) = \int_0^{2\pi} d_{K_q(1)}\Big((x'(\varphi), y'(\varphi))\Big)\,\mathrm{d}\varphi. \tag{1.1}$$

This formula will be modified below for the nonconvex case $p \in (0,1)$ by replacing the convex disc $K_q(1)$ with a certain star-shaped set $S(q)$.

The special role the 'dual' p^* plays in the convex case for the definition of $\pi(p)$ is reflected by the equation

$$A_p(r) = \int_0^r AL_{p,p^*}(\varrho)\,\mathrm{d}\varrho, \quad p \geqslant 1,\ r > 0. \tag{1.2}$$

This equation allows a comprehensive interpretation. First of all, it reflects a generalization of the method of indivisibles of Cavalieri and Torricelli in the sense that the indivisibles are here the circles $C_p(\varrho), \varrho \in [0,r]$, and measuring the area content of the convex disc $K_p(r)$ means integrating just the l_{2,p^*}-arc-lengths $AL_{p,p^*}(\varrho)$ of the indivisibles. In comparison, the co-area formula of measure theory says that integrating the Euclidean or $l_{2,2}$-arc-lengths of the same indivisibles yields the area content of $K_p(r)$ if and only if $p = 2$.

In this paper, we replace the convex disc $K_q(1)$ with a star-shaped set in such a way that a formula analogous to (1.2) can be also proved in the nonconvex case $p \in (0,1)$.

To this end, the arc-length measure for defining the circumference of the $l_{2,p}$-circle in the case $p \in (0,1)$ will be based upon the star-shaped set $\{(x,y) \in R^2\colon\ \frac{1}{|x|^{p^{**}}} + \frac{1}{|y|^{p^{**}}} \geqslant 1\}$ with $p^{**} > 0$ satisfying $-\frac{1}{p^{**}} + \frac{1}{p} = 1$. Using this notion, p-generalized circle numbers will be defined in the case $p \in (0,1)$ analogously to those for convex $l_{2,p}$-circle discs. The resulting extension of the π-function from [2] is continuous and increasing. The described approach can be considered as reflecting a certain modification of the method of indivisibles in the sense that the indivisibles are now the boundaries of nonconvex $l_{2,p}$-circle discs and integrating their suitably chosen star-shaped arc-lengths means measuring the Euclidean area content of the disc.

When studying further properties of the present extension of the π-function, special emphasis will be on its asymptotic behavior as p tends to zero.

Just like in the convex case $p \geqslant 1$, it turns out that the area content of the unit disc is suitable for defining the p-generalized circle numbers for each $p \in (0,1)$. In this case, the resulting π-function therefore takes all values from $(0,2)$.

2 RESULTS

We consider the set $C_p(r) = \{(x,y) \in R^2\colon\ |x|^p + |y|^p = r^p\}$ for $0 < p < 1$ as the $l_{2,p}$-circle with p-generalized radius $r > 0$. The Euclidean area content of the disc $K_p(r)$ inside $C_p(r)$ is denoted by $A_p(r)$. It is known that $A_p(r) = A_p r^2$, where

$$A_p = \frac{2\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}. \tag{2.1}$$

For determining this constant using p-generalized polar coordinates, we refer to [1].

The length measure in a Minkowski plane is generated by a norm or, equivalently, by the corresponding convex body. In this paper, we leave the Minkowski plane and equip R^2 instead with the star-shaped set

$$S(q) = \Big\{(x,y) \in R^2\colon\ \frac{1}{|x|^q} + \frac{1}{|y|^q} \geqslant 1\Big\}, \quad q > 0,$$

for generating an arc-length measure, where q will be chosen later. To this end, let the Minkowski functional of a star-shaped set S be defined as

$$d_S(z) = \inf\{\lambda > 0\colon\ z \in \lambda S\}, \quad z \in R^2,$$

where $\lambda S = \{(\lambda x, \lambda y) \in R^2\colon\ (x,y) \in S\}$. Note that

$$d_{S(q)}((x,y)) = 0 \quad \text{if } x = 0 \text{ or } y = 0$$

and

$$d_{S(q)}((x,y)) = \frac{1}{(\frac{1}{|x|^q} + \frac{1}{|y|^q})^{1/q}} \quad \text{for all other } (x,y) \in R^2.$$

The function $d_{S(q)}$ is symmetric w.r.t. the origin,

$$d_{S(q)}((x,y)) = d_{S(q)}((|x|,|y|)), \quad (x,y) \in R^2,$$

nonnegative and homogeneous,

$$d_{S(q)}(\lambda z) = |\lambda| d_{S(q)}(z), \quad z \in R^2,\ \lambda \in R.$$

For defining now $AL_{p,q}$, let an arbitrary differentiable parameter representation of the circle $C_p(r)$ be given by

$$C_p(r) = \Big\{\big(x(\varphi), y(\varphi)\big)^T, 0 \leqslant \varphi < 2\pi\Big\}.$$

DEFINITION. The $S(q)$-based arc-length of the $l_{2,p}$-circle $C_p(r)$ is defined for $0 < p < 1$ and $0 < q < \infty$ as

$$AL_{p,q}(r) = \int_0^{2\pi} d_{S(q)}(x'(\varphi), y'(\varphi))\, \mathrm{d}\varphi.$$

Note that $AL_{p,q}(r) = r AL_{p,q}$, where $AL_{p,q}$ means the length of the unit circle.

Lemma. (a) *If $p^{**} > 0$ satisfies $-\frac{1}{p^{**}} + \frac{1}{p} = 1$, then, for all $p \in (0,1)$,*

$$AL_{p,p^{**}} = 2A_p.$$

(b) *The value of p^{**} is unique in the sense that*

$$AL_{p,q} > AL_{p,p^{**}} \quad \text{if } q > p^{**} \text{ and } AL_{p,q} < AL_{p,p^{**}} \text{ if } q < p^{**}.$$

Proof. (a) Making use of the p-generalized trigonometric functions defined in [1], a concrete parameter representation of $C_p(1)$ is given by $x(\varphi) = \cos_p(\varphi)$ and $y(\varphi) = \sin_p(\varphi), 0 \leqslant \varphi < 2\pi$. Hence,

$$AL_{p,q} = \int_0^{2\pi} d_{S(q)}\Big(\big(\cos_p'(\varphi), \sin_p'(\varphi)\big)^T\Big)\, \mathrm{d}\varphi.$$

According to [1], it follows that

$$AL_{p,q} = \int_0^{2\pi} d_{S(q)}\bigg(\frac{-\sin\varphi|\sin\varphi|^{p-2}}{N_p(\varphi)^{p-1}}, \frac{\cos\varphi|\cos\varphi|^{p-2}}{N_p(\varphi)^{p-1}}\bigg)\frac{\mathrm{d}\varphi}{N_p^2(\varphi)}$$

with $N_p(\varphi) = (|\sin\varphi|^p + |\cos\varphi|^p)^{1/p}$. Since $N_\delta(\varphi) = d_{K_\delta(1)}(\cos\varphi, \sin\varphi), \delta > 0$, we can rewrite $AL_{p,q}$ as

$$AL_{p,q} = \int_0^{2\pi} \left(\frac{d_{K_p(1)}(\cos\varphi, \sin\varphi)}{d_{K_{(1-p)q}(1)}(\cos\varphi, \sin\varphi)} \right)^{1-p} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)}. \tag{2.2}$$

Under the assumption of the lemma, we have $(1-p)p^{**} = p$ and

$$AL_{p,p^{**}} = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)}.$$

Further, the area content of the unit disc can be written as $A_p = \int_{K_p(1)} \mathrm{d}(x,y)$. Changing Cartesian with p-generalized polar coordinates, we have

$$A_p = \int_{\varrho=0}^{1} \int_{\varphi=0}^{2\pi} \frac{\varrho}{N_p^2(\varphi)} \,\mathrm{d}(\varrho,\varphi) = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)}.$$

(b) The ratio under the integral sign in (2.2) satisfies

$$\frac{d_{K_p(1)}(\cos\varphi, \sin\varphi)}{d_{K_{(1-p)q}(1)}(\cos\varphi, \sin\varphi)} > (<) 1 \quad (\forall \varphi \in [0, 2\pi)) \ \text{ if } \ q > (<) p^{**}. \qquad \square$$

Summarizing what is already known, we have, for all $r > 0$,

$$\frac{A_p(r)}{r^2} = \frac{AL_{p,p^{**}}(r)}{2r} = A_p \tag{2.3}$$

and

$$AL_{p,p^{**}}(r) = \frac{d}{dr} A_p(r). \tag{2.4}$$

Equation (2.4) can be rewritten in the sense of (1.2) and may therefore be considered as reflecting a modification of the method of indivisibles of Cavalieri and Torricelli which says that the indivisibles are the circles $C_p(\varrho), \varrho \in [0, r]$ and measuring the area content of the nonconvex disc $K_p(r)$ means integrating the $d_{S(p^{**})}$-based arc-lengths of the indivisibles.

Hence, the geometry of the plane $(R^2, d_{S(p^{**})})$ is suitable for defining the circumference of an $l_{2,p}$-circle if $p \in (0,1)$. Therefore, and in accordance with the notation in [1], the special star-shaped or $d_{S(p^{**})}$-based arc-length $AL_{p,p^{**}}(r) = A_p'(r)$ of the $l_{2,p}$-circle $C_p(r)$ will be called its $l_{2,p}$-generalized circumference.

Equations (2.3) may be interpreted now for arbitrary $p \in (0,1)$ as reflecting the Euclidean area-content- and the $l_{2,p}$-generalized circumference-properties of the $l_{2,p}$-circle $C_p(r)$. The following definition is therefore well motivated.

DEFINITION. For $p \in (0,1)$, the Euclidean area content A_p of the $l_{2,p}$-unit disc $K_p(1)$ is called the $l_{2,p}$-circle number and henceforth denoted by $\pi(p)$.

From this definition it follows immediately that the function $p \to \pi(p), p \in (0,1)$, is continuous and increasing and satisfies the relations $0 = \lim_{p\to 0+} \pi(p) < \pi(p) < \lim_{p \to 1-} \pi(p) = 2$. Hence, the π-function which was recently defined in [2] for the convex case has been extended here to $l_{2,p}$-circles with $0 < p < 1$. All numbers from $(0,2)$ are therefore geometrically interpretable as such $l_{2,p}$-circle numbers for nonconvex

Table 1. Values of selected $l_{2,p}$-circle numbers

p	0,113	1/4	0,3	1/3	1/2	0,607	0,783	0,99995
$\pi(p)$	0,0001	2/35	0,1323	1/5	2/3	0,9992	1,5	1,9999

discs. Note further that $\lim_{p\to 1-}\pi(p)$ coincides with the circle number $\pi(1)=2$ which was introduced in [2]. In Table 1, the numerical values of some p-generalized circle numbers are given.

It can be proved word by word in the same way as Theorem in [2] that the $l_{2,p}$-circle numbers $\pi(p)$, $p\in(0,1)$, satisfy the representation formulae

$$\pi(p)=4\int_0^1(1-y^p)^{\frac{1}{p}}\,\mathrm{d}y=4\int_0^\infty(1+\tau^p)^{-\frac{2}{p}-1}\,\mathrm{d}\tau=2\int_0^1(1-\mu^p)^{\frac{1-p}{p}}\,\mathrm{d}\mu. \tag{2.5}$$

Let us now consider a few specific properties of the π-function for arguments from $(0,1)$.

Remark. From (2.1) and from the definition of the Gamma function it follows immediately that the $l_{2,1/k}$-circle numbers satisfy the representation formula

$$\pi(\frac{1}{k})=\frac{(2k!)^2}{(2k)!},\quad k=2,3,\dots$$

These numbers tend to zero as k tends to infinity at least as fast as their upper bounds $4\cdot 2^{-k}$. The following theorem describes this asymptotic behavior more precisely.

Theorem. *The π-function satisfies the asymptotic relation*

$$\pi(p)=\frac{4\pi^{1/2}}{p^{1/2}2^{2/p}}\Big(1+\frac{p}{8}+\mathrm{O}(p^2)\Big),\quad p\to+0.$$

Proof. Starting from (2.1) and using the asymptotic representation formula of the Gamma function

$$\Gamma(x+1)=x^xe^{-x}\sqrt{2\pi x}\bigg(1+\frac{1}{12x}+\mathrm{O}\Big(\frac{1}{x^2}\Big)\bigg),\quad x\to\infty,$$

it follows that

$$\pi(p)=\frac{2((-1+1/p)^{1/p-1/2}e^{1-1/p}\sqrt{2\pi}(1+\frac{p}{12(1-p)}+\mathrm{O}(p^2)))^2}{p(-1+2/p)^{2/p-1/2}e^{1-2/p}\sqrt{2\pi}(1+\frac{p}{12(2-p)}+\mathrm{O}(p^2))},\quad p\to 0.$$

Hence,

$$\pi(p)=\frac{4\sqrt{\pi}}{2^{2/p}\sqrt{p}}\,\frac{1+\frac{p}{6}+\mathrm{O}(p^2)}{1+\frac{p}{24}+\mathrm{O}(p^2)}\quad\text{as}\quad p\to 0,$$

which immediately completes the proof. □

According to this theorem, $\pi(p)$ tends to zero as p tends to zero. The asymptotic behavior of $AL_{p,p^{**}}(r(p))$ therefore strongly depends on the behavior of $r(p)$ as $p\to 0$. Clearly, $r(p)$ should tend to infinity sufficiently fast for $AL_{p,p^{**}}(r(p))$ not tending to zero. The following corollary makes this more precise.

Corollary. (a) If, for some constant $C > 0$,

$$r(p) \sim \frac{C}{8\sqrt{\pi}} \cdot \sqrt{p} \cdot 2^{2/p} \quad \text{as} \quad p \to 0,$$

then

$$AL_{p,p^{**}}(r(p)) \to C \quad \text{as} \quad p \to 0.$$

(b) If $r(p) = \mathrm{o}(\sqrt{p} \cdot 2^{2/p})$ as $p \to 0$, then

$$AL_{p,p^{**}}(r(p)) \to 0 \quad \text{as } p \to 0.$$

(c) If $r(p)/(\sqrt{p} \cdot 2^{2/p}) \to \infty$ as $p \to 0$, then

$$AL_{p,p^{**}}(r(p)) \to \infty \quad \text{as } p \to 0.$$

Proof. Since of $AL_{p,p^{**}}(\varrho) = 2\varrho\pi(p)$, by Theorem we have

$$AL_{p,p^{**}}(r(p)) \sim r(p)\frac{8}{2^{2/p}}\sqrt{\frac{\pi}{p}} \quad \text{as } p \to 0.$$

Replacing here $r(p)$ with one of the assumptions made on the asymptotic behavior $r(p)$ in a), b), or c), we immediately get the asymptotic relation stated in a), b), or c), respectively. □

A further justification for considering the geometry of the plane $(R^2, d_{S(p^{**})})$ when measuring the lengths of the $l_{2,p}$-circles $C_p(r)$ and when defining the circle numbers $\pi(p)$ is given by the following probabilistic application from [1]. Let a random vector (X, Y) be uniformly distributed on $K_p(1)$ or distributed according to the p-generalized normal distribution. The normalized vector $(\xi, \eta) = (X, Y)/(|X|^p + |Y|^p)^{1/p}$ is then p-generalized uniformly distributed on the $l_{2,p}$-unit circle in the sense that, for arbitrary Borel subset A of $C_p(1)$,

$$P((\xi, \eta) \in A) = \frac{S(p^{**})\text{-arc-length of } A}{S(p^{**})\text{-arc-length of } C_p(1)} = \int_{POL_p^{*-1}(A)} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)} \Big/ \int_0^{2\pi} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)}.$$

Here $POL_p^{*-1}(A)$ denotes the inverse image of the set A under the transformation $POL_p^*(\varphi) = POL_p(1, \varphi)$, where the p-generalized polar coordinate transformation $POL_p|[0, \infty) \times [0, 2\pi) \to R^2$ is defined by $x = r\cos\varphi/N_p(\varphi)$ and $y = r\sin\varphi/N_p(\varphi)$. Since

$$\frac{1}{2}AL_{p,p^*} = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)} = \pi(p), \tag{2.6}$$

it follows that

$$P((\xi, \eta) \in A) = \frac{S(p^{**})\text{-arc-length of } A}{2\pi(p)} = \frac{1}{2\pi(p)} \int_{POL_p^{*-1}(A)} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)}. \tag{2.7}$$

According to Eqs. (2.3), formula (2.7) may also be interpreted as follows. Imagine that a point is moving on $C_p(1)$. Then the equal areas

$$\int_{POL_p^{*-1}(A_1)} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)}/2 \quad \text{and} \quad \int_{POL_p^{*-1}(A_2)} \frac{\mathrm{d}\varphi}{N_p^2(\varphi)}/2$$

are swept out (from the origin) in equal “times” $T_1 = P((\xi,\eta) \in A_1)$ and $T_2 = P((\xi,\eta) \in A_2)$, respectively.

For another application of the generalized circle numbers, let us continue the example from [1]. Imagine that a machine tool moves along an $l_{2,p}$-circle line of p-generalized radius $r+\varepsilon$, creating thereby a thin protective coat on a workpiece of p-generalized radius r by applying a special material to its surface. The consumed material then approximately (i.e., for small ε) has the area content $2\pi(p)r\varepsilon$.

Let us finally give an elementary probabilistic interpretation of the circle numbers $\pi(\frac{1}{k})$ for $k \in \{2,3,...\}$. To this end, let m_{2k} be the total number of successes in a Bernoulli trial of length $2k$ with the probability one half of a single success. It then immediately follows from Eq. (2.1) that

$$\frac{\pi(1/k)}{4} = \frac{1}{4^k \cdot P(m_{2k} = k)} \quad \text{for} \quad k = 2,3,\ldots. \tag{2.8}$$

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