Circle numbers for star discs

By W.-D. Richter

University of Rostock, Institute of Mathematics, Ulmenstraße 69, Haus 3, D-18057 Rostock, Germany

E-mail: wolf-dieter.richter@uni-rostock.de

Abstract The notion of a generalized circle number which has recently been discussed for $l_{2,p}$-circles and ellipses will be extended here for star bodies and a class of unbounded star discs.

Key words: Generalized radius; Minkowski functional; generalized circumference; distance based arc-length; positive directed arc-length; generalized method of indivisibles; generalized circle number; generalized trigonometric functions; generalized polar coordinates; norm; dual norm; anti-norm; semi-anti-norm; reverse triangle inequality; rotated gradient condition; star body; unbounded Orlicz disc.

2010 MSC: 51M25, 52A10, 52A30, 52A38, 53A35.

Introduction

Generalized circle numbers have been discussed for $l_{2,p}$-circles in Richter [2008a,b] and for ellipses in Richter [2011]. All these numbers correspond on the one hand to an area content property of the considered discs which is based upon the usual, i.e. Euclidean, area content measure and a suitably adopted radius variable. On the other hand, they reflect a circumference property of the considered generalized circles w.r.t. a non-Euclidean length measure which is generated by a suitably chosen non-Euclidean disc. Several basic and specific properties of the circumference measure have been discussed in Richter [2008a,b, 2011]. We refer here to only two of them which are closely connected with each other by the main theorem of calculus. The first one is that the generalized circumference of the generalized circle coincides with the derivative of the area content function w.r.t. the adopted radius variable. The second one is that, vice versa, the area content of the circle disc equals the integral of the generalized circumferences of the circles within the disc w.r.t. the adopted radius variable. Integrating this way may be considered as a generalization of Cavalieri's and Torricelli's method of indivisibles where the indivisibles are now the generalized circles within the given disc and measuring them is based upon a non-Euclidean geometry. The far reaching usefulness of this generalized method of indivisibles has been demonstrated in Richter [2007, 2008a,b, 2011] where several applications are dealt with and where it was also shown that integrating usual, i.e. Euclidean, lengths of the same indivisibles does not yield the area content, in general. In the present paper, we will prove that this method still applies when generalized circle numbers are derived for general star discs. In this sense, this paper deals with bounded and unbounded star discs. Notice that because we shall not assume symmetry of the unit disc, distances will depend on directions, in general.

To become more specific, let S be a star body in $\mathbb{R}^2$ and let its area content be defined as usual by its Lebesgue measure. Furthermore, let us call the boundary of ϱ times the star disc S the S-circle of S-radius ϱ, $\varrho > 0$ and denote it by $\mathfrak{C}_S(\varrho)$. If we define the perimeter of S by using different length measures then we can observe different perimeter-to-(two-times-S-radius) ratios and these ratios differ from the corresponding (area-content)-to-(squared-S-radius) ratio, in general. If we choose, however, the length measure in a certain specific way then the first ratio coincides with the second one for all $\varrho > 0$. In the most famous case when S is the Euclidean disc and measuring circumference is based upon Euclidean arc-length, the common constant value of the two ratios is the well known circle number π.

If S is the symmetric and convex $l_{2,p}$-circle, centered at the origin and thus defining a norm then, according to Richter [2008a], the suitable arc-length measure is based upon the dual norm, i.e. the norm which is generated by the l_{2,p^*}-circle $\{(x,y) : |x|^{p^*} + |y|^{p^*} = 1\}$ with $p^* \geq 1$ satisfying the equation $\frac{1}{p} + \frac{1}{p^*} = 1$.

Similarly, if S is an $l_{2,p}$-circle, $p \in (0,1)$, corresponding to an antinorm (see Moszyńska and Richter [2011]) then, according to Richter [2008b], the suitable arc-length measure is based upon the star disc $S(p^{**}) = \{(x,y) \in \mathbb{R}^2 : |x|^{p^{**}} + |y|^{p^{**}} \geq 1\}$ with $p^{**} < 0$ satisfying $\frac{1}{p} + \frac{1}{p^{**}} = 1$. The star disc $S(p^{**})$ corresponds to a specific semi-antinorm w.r.t. the canonical fan (see Moszyńska and Richter [2011]).

The situation for ellipses has been discussed in Richter [2011]. If $S = D_{a,b} = \{(x,y) \in \mathbb{R}^2 : (\frac{x}{a})^2 + (\frac{y}{b})^2 \leq 1\}$ is an elliptically contoured disc and $E_{(a,b)} = \partial S$ its boundary then the suitable arc-length measure on the Borel σ-field of subsets of the ellipse $E_{(a,b)}$ is based upon the disc

$D_{(\frac{1}{b},\frac{1}{a})}$. Note that again $D_{(a,b)}$ and $D_{(\frac{1}{b},\frac{1}{a})}$ correspond to dual norms.

The arc-length measure used for defining the p-generalized circle number allows both for $p \geq 1$ and for $0 < p < 1$ the same additional interpretation in terms of the derivative of the area content function w.r.t. the p-radius variable. This way the notion of the p-generalized circumference of the p-circle was introduced first in Richter [2007] under more general circumstances and motivated there by several of its applications. Several geometric interpretations of this notion in cases of special norms and anti-norms have been discussed so far. As just to refer to a few of them let us recall that this notion is a basic one for the generalized method of indivisibles, that it allows to prove the so called thin layers property of the Lebesgue measure and to think of a certain mixed area content in a new way and that it is closely connected with the solution to a certain isoperimetric problem. From a technical point of view, a basic difference between the two situations is that in the convex case one uses triangle inequality for showing convergence of a sequence of suitably defined integral-sums and that one makes use of the reverse triangle inequality from Moszyńska and Richter [2011] for proving such convergence if the p-generalized circle corresponds to an anti-norm.

The arc-length measure used for defining ellipses numbers has also an interpretation in terms of the derivative of the area content function but w.r.t. a generalized radius variable corresponding to $E_{(a,b)}$. The common notion behind the different definitions of a generalized radius variable discussed so far in the literatur is that of the Minkowski functional (or gauge function) of a star body, but looking onto the motivating applications, e.g. from probability theory and mathematical statistics, let it become clear that further generalizations are desirable in future work.

Here, we start our consideration with the definition of the S-generalized circumference of an S-circle corresponding to a star body and shall discuss in Section 2 both its general geometric meaning and its specific interpretation either when S is generated by an arbitrary norm or by an anti-norm of special type.

It should be mentioned here that the Minkowski functional of a star body generates a distance which is not symmetric in general, i.e., it does not assign a length in the usual sense to a generalized circle but a certain directed length.

In this sense, Section 2 deals mainly with generalized circle numbers for star bodies while Section 3 is devoted to a certain class of unbounded star discs.

Definition 1. *Let λ_2 be the Lebesgue measure in $\mathbb{R}^2, S$ a star body and $A_S(\varrho) = \lambda_2(\varrho S)$ the corresponding area content function. Then*

$$\frac{d}{d\varrho} A_S(\varrho) =: \mathfrak{U}_S(\varrho), \varrho > 0 \tag{1}$$

will be called the S-generalized circumference of ϱS or the S-generalized arc-length of the boundary $\mathfrak{C}_S(\varrho)$ of ϱS, $\varrho S = \{(\varrho x, \varrho y) \in \mathbb{R}^2 : (x, y) \in S\}$.

It follows from the properties of the Lebesgue measure that

$$\frac{A_S(\varrho)}{\varrho^2} = \frac{\mathfrak{U}_S(\varrho)}{2\varrho} = A_S(1), \, \forall \varrho > 0. \tag{2}$$

The representation

$$A_S(\varrho) = \int_0^{\varrho} \mathfrak{U}_S(r) dr$$

may be understood as a generalized method of indivisibles for the Lebesgue measure where the indivisibles are multiples of the boundary of S and measuring their circumferences is based upon $\mathfrak{U}_S$. The equations (2) may suggest on the one hand to call $A_S(1)$ the S-generalized circle number. On the other hand, one may consider at this stage of consideration a method of introducing generalized circle numbers which follows basically the idea of the main theorem of calculus being rather elementary if not even trivial. However, the papers Richter [2007, 2008a,b, 2011] which are closely connected with this approach allow a new look onto a class of geometric measure representations or, similarly, onto a class of stochastic representations which are quite fruitful for many applications. Several of these applications, especially in probability theory and mathematical statistics, are discussed therein.

Clearly, there is always a necessity to give a geometric or otherwise mathematical interpretation of the circumference $\mathfrak{U}_S(\varrho)$. In other words, one naturally looks for a geometry such that the arc-length of $\mathfrak{C}_S(\varrho)$ w.r.t. this geometry coincides with the S-generalized circumference $\mathfrak{U}_S(\varrho)$. The non-Euclidean geometries being identified in this way may be considered as geometries "being close to the Euclidean one" as those were discussed in Hilbert [1900] in connection with his fourth problem. If we can uniquely identify a geometry such that the arc-length measure of S, $AL_{S,S^*}(\varrho)$, which is based upon the geometry's unit ball S^*, satisfies the equation

$$AL_{S,S^*}(\varrho) = \mathfrak{U}_S(\varrho), \tag{3}$$

then we can observe already the non-trivial situation that

$$\frac{A_S(\varrho)}{\varrho^2} = \frac{AL_{S,S^*}}{2\varrho} = A_S(1),\ \forall \varrho > 0. \tag{4}$$

At such stage of investigation, it will then be already much more motivated that the area content of the unit star, $A_S(1)$, is called the S-generalized circle number, $\pi(S)$.

In this sense, the considerations in Richter [2008a,b, 2011] deal with restrictions of the function $S \to \pi(S)$ to $l_{2,p}$-balls, $p > 0$ and to axes aligned ellipses.

Star bodies

A subset S from $\mathbb{R}^2$ is called a star body if it is star-shaped with respect to the origin and compact and has the origin in its interior. A set of this type has the property that for every $z \in \mathbb{R}^2$ there exists a uniquely determined $\varrho > 0$ such that $z/\varrho \in \partial S$ where ∂S denotes the boundary of the set S. This ϱ equals the value of the Minkowski functional w.r.t. the reference set S,

$$h_S(x,y) = \inf\{\lambda > 0 : (x,y) \in \lambda S\},$$

at any point $(x,y) \in \partial S$. The function h_S is often called the gauge function of S (see, e.g., in Webster [1994]) and coincides, for $(x,y) \neq (0,0)$, with the reciprocal of the radial function (see, e.g., in Thompson [1996] and Moszyńska [2006])

$$\varrho_S((x,y)) = \sup\{\lambda \geq 0 : \lambda(x,y) \in S\}.$$

The special cases that h_S is a norm or an anti-norm are of particular interest and will be separately dealt with in examples 14 and 15.

With a star body S, the pair $(\mathbb{R}^2, h_S)$ may be considered as a generalized Minkowski plane. The star disc and the star circle of S-radius ϱ will be defined then by $K_S(\varrho) = \{r \cdot s, s \in S, 0 \leq r \leq \varrho\}$ and

$\mathfrak{C}_S(\varrho) = \partial K_S(\varrho)$, respectively. The set S will be called the unit star in this plane.

Let T be another star disc in $\mathbb{R}^2$ which will be specified later. Whenever possible, we may define the T-arc-length of the curve $\mathfrak{C}_S(\varrho)$ as follows.

Definition 2. *If $\mathfrak{Z}_n = \{z_0, z_1, ..., z_n = z_0\}$ denotes a successive and positive (anticlockwise) oriented partition of $\mathfrak{C}_S(\varrho)$ then the positive directed T-arc-length of $\mathfrak{C}_S(\varrho)$ is defined by*

$$AL_{S,T}(\varrho) := \lim_{n\to\infty} \sum_{j=1}^{n} h_T(z_j - z_{j-1})$$

if the limit exists for and is independent of all described partitions of $\mathfrak{C}_S(\varrho)$ with $F(\mathfrak{Z}_n) = \max\limits_{1\le j\le n} h_T(z_j - z_{j-1})$ tending to zero as $n \to \infty$.

Using triangle inequality or its reverse, one can show that if h_S is a norm or anti-norm then taking the limit may be changed with taking the supremum or the infimum, respectively. Notice that because h_T is in general not a symmetric function, the orientation in the partition may have essential influence onto the value of $AL_{S,T}(\varrho)$ and is therefore assumed here always to be positive.

For studying $AL_{S,T}(\varrho)$, let a parameter representation of the unit-S-circle $\mathfrak{C}_S(1)$ be given by $\mathfrak{C}_S(1) = \{R_S(\varphi)(\cos\varphi, \sin\varphi)^T, 0 \le \varphi < 2\pi\}$. Later on we shall assume that R_S is a.e. differentiable. From the relation

$$h_S((x,y)^T) = 1, \forall (x,y)^T \in \mathfrak{C}_S(1)$$

it follows that

$$h_S((\cos\varphi, \sin\varphi)^T) = 1/R_S(\varphi), 0 \le \varphi < 2\pi.$$

In other words, with the notation $M_S(\varphi) = h_S((\cos\varphi, \sin\varphi)^T)$, we have

$$\mathfrak{C}_S(1) = \{(\cos\varphi/M_S(\varphi), \sin\varphi/M_S(\varphi))^T, 0 \le \varphi < 2\pi\}.$$

This motivates the following definition which generalizes more particular notions from earlier considerations.

Definition 3. *For an arbitrary star body S, the S-generalized sine and cosine functions are*

$$\sin_S(\varphi) = \sin\varphi / M_S(\varphi) \ \ \textit{and} \ \ \cos_S(\varphi) = \cos\varphi / M_S(\varphi),\ \varphi \in [0, 2\pi),$$

respectively.

Notice that there is an elementary geometric interpretation of these generalized trigonometric functions when one considers a right-angled triangle $Tr = ((0,0)^T, (x,0)^T, (x,y)^T)$ with $x > 0$ and $y > 0$ as follows. The S-generalized sine and cosine of the angle $\varphi \in [0, 2\pi)$ between the directions of the positive x-axes and the line through the points $(0,0)^T$ and $(x,y)^T$ are

$$\sin_S(\varphi) = y / h_S((x,y)^T) \quad \textit{and} \quad \cos_S(\varphi) = x / h_S((x,y)^T),$$

respectively. These functions satisfy the equation

$$h_S(\cos_S(\varphi), \sin_S(\varphi)) = 1$$

generalizing the well known formula $\cos^2\varphi + \sin^2\varphi = 1$.

Definition 4. *The S-generalized polar coordinate transformation*

$$Pol_S : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$$

is defined by

$$x = r\cos_S(\varphi), y = r\sin_S(\varphi), 0 \leq \varphi < 2\pi, 0 \leq r < \infty.$$

Let us denote the quadrants in $\mathbb{R}^2$ in the usual anticlockwise ordering by Q_1 up to Q_4.

Theorem 5. *The map Pol_S is almost one-to-one, for $x \neq 0$, its inverse Pol_S^{-1} is given by*

$$r = h_S(x,y) \ \textit{and} \ \arctan(|\frac{y}{x}|) = \varphi \ \textit{in} \ Q_1, \pi - \varphi \ \textit{in} \ Q_2, \varphi - \pi \ \textit{in} \ Q_3, 2\pi - \varphi \ \textit{in} \ Q_4$$

and its Jacobian satisfies

$$J(r, \varphi) = \frac{D(x,y)}{D(r,\varphi)} = \frac{r}{M_S^2(\varphi)}.$$

Proof :
The proof follows that of Theorem 8 in Richter [2011] and makes essentially use of the fact that the derivatives of the S-generalized trigonometric functions $\sin_S$ and $\cos_S$ allow the representations

$$\sin'_S(\varphi) = \frac{1}{M_S^2(\varphi)}[\cos\varphi M_S(\varphi) - \sin\varphi M'_S(\varphi)]$$

and

$$\cos'_S(\varphi) = \frac{1}{M_S^2(\varphi)}[-\sin\varphi M_S(\varphi) - \cos\varphi M'_S(\varphi)].$$

□

Using S-generalized polar coordinates, we can write

$$\mathfrak{C}_S(\varrho) = \{(\varrho\cos_s(\varphi), \varrho\sin_S(\varphi))^T, 0 \le \varphi < 2\pi, \varrho > 0\}.$$

We assume from now on that h_T is positively homogeneous, put

$$h_T(z_j - z_{j-1}) = h_T((\Delta_j x, \Delta_j y)^T)$$

and consider

$$h_T(z_j - z_{j-1}) = h_T((\Delta_j x(\varphi)/\Delta_j\varphi, \Delta_j y(\varphi)/\Delta_j\varphi)^T)\Delta_j\varphi$$

for sufficiently thin partition $\mathfrak{Z}_n$ and $\Delta_j\varphi > 0$. We get in the limit, which was assumed in Definition 2 to be uniquely determined,

$$AL_{S,T}(\varrho) = \int\limits_0^{2\pi} h_T(x'(\varphi), y'(\varphi))d\varphi = \varrho\int\limits_0^{2\pi} h_T(x'(\varphi)/\varrho, y'(\varphi)/\varrho)d\varphi$$

$$= \varrho\int\limits_0^{2\pi} h_T((cos'_S(\varphi), \sin'_S(\varphi))^T)d\varphi = \varrho AL_{S,T}(1).$$

It follows from the proof of Theorem 5 that

$$AL_{S,T}(\varrho) = \varrho\int\limits_0^{2\pi} R_S^2(\varphi)h_T(M_S(\varphi)\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} + M'_S(\varphi)\begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix})d\varphi,$$

i.e.,

$$AL_{S,T}(\varrho) = \varrho\int\limits_0^{2\pi} R_S^2(\varphi)h_T(O(\varphi)\mathfrak{x}_S(\varphi))d\varphi \tag{5}$$

with

$$O(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \ \mathfrak{x}_S(\varphi) = \begin{pmatrix} R'_S(\varphi)/R^2_S(\varphi) \\ 1/R_S(\varphi) \end{pmatrix}.$$

The following lemmas and corollaries represent certain steps towards a reformulation of formula (5).

Lemma 6. *In the case of their existence, the partial derivatives $h_{S,x}$ and $h_{S,y}$ of the function $(x,y) \to h_S(x,y)$ satisfy the representation*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi)) \\ h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi)) \end{pmatrix} = \mathfrak{O}(\varphi)\mathfrak{x}_S(\varphi).$$

Proof. It follows from the relation

$$h_S(r\cos_S(\varphi), r\sin_S(\varphi)) = r$$

that the partial derivatives $h_{S,x}$ and $h_{S,y}$ satisfy the equation system

$$\frac{\partial}{\partial\varphi} h_S(\cos_S(\varphi), \sin_S(\varphi)) = 0, \frac{\partial}{\partial r} h_S(r\cos_S(\varphi), r\sin_S(\varphi)) = 1.$$

Solving this differential equation system we get

$$h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi)) = \frac{1}{R^2_S(\varphi)}(R_S(\varphi)\cos(\varphi) + R'_S(\varphi))\sin(\varphi)$$

and

$$h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi)) = \frac{1}{R^2_S(\varphi)}(R_S(\varphi)\sin(\varphi) - R'_S(\varphi))\cos(\varphi).$$

Hence,

$$\begin{pmatrix} h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi)) \\ h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi)) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathfrak{O}(\varphi)\mathfrak{x}_S(\varphi).$$

□

Let B be a 2×2-matrix and $BT = \{B(x,y)^T : (x,y)^T \in T\}$. Clearly, multiplying the set T by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ causes an anticlockwise rotation of T through the angle $\pi/2$. Hence, if T is a star disc then BT is a star disc, too.

Corollary 7. *For positively homogeneous h_T, differentiable h_S, formula* (5) *may be rewritten as*

$$AL_{S,T}(\varrho) = \varrho \int_0^{2\pi} R_S^2(\varphi) h_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T} (\nabla h_S(x,y)|_{(x,y)=Pol_S(r,\varphi)}) d\varphi.$$

Proof :
Based upon Lemma 6 formula (5) may be reformulated as

$$AL_{S,T}(\varrho) = \varrho \int_0^{2\pi} R_S^2(\varphi) h_T(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_{S,x}(r\cos_s(\varphi), r\sin_s(\varphi)) \\ h_{S,y}(r\cos_s(\varphi), r\sin_s(\varphi)) \end{pmatrix}) d\varphi.$$

Because of

$$h_T(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}) = \inf\{\lambda > 0 : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \lambda T\}$$

$$= \inf\{\lambda > 0 : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T\} = h_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T} (\begin{pmatrix} \xi \\ \eta \end{pmatrix})$$

it follows the assertion.

□

Remark 8. *The plug-in version $\nabla h_S(x,y)|_{(x,y)=Pol_S(r,\varphi)}$ of the gradient $\nabla h_S(x,y)$ coincides with the image of the gradient $\nabla h_S(x,y)$ after changing Cartesian with S-generalized polar coordinates, $Pol_S(\nabla h_S(x,y))(r,\varphi)$.*

Proof. Changing Cartesian coordinates (x,y) with S-generalized polar coordinates (r,φ), we have $r = h_S(x,y)$ and $\varphi = \arctan \frac{y}{x}$. Starting from the equation

$$\frac{\partial}{\partial x} h_S(x,y) = \frac{\partial}{\partial r} h_S(x,y) \frac{\partial}{\partial x} r + \frac{\partial}{\partial \varphi} h_S(x,y) \frac{\partial}{\partial x} \varphi,$$

and the analogous one for $\frac{\partial}{\partial y} h_S(x,y)$, and taking into account that

$$\frac{\partial}{\partial r} h_S(x,y)|_{(x,y)=Pol_S(r,\varphi)} = \frac{\partial}{\partial r} h_S(r\cos_S(\varphi), r\sin_S(\varphi)) = 1$$

and

$$\frac{\partial}{\partial \varphi} h_S(x,y)|_{(x,y)=Pol_S(r,\varphi)} = \frac{\partial}{\partial \varphi} h_S(r \cos_S(\varphi), r \sin_S(\varphi)) = 0,$$

it follows

$$\frac{\partial}{\partial x} h_S(x,y) = \frac{\partial r}{\partial x}|_{(x,y)=Pol_S(r,\varphi)} = h_{S,x}(r \cos_S(\varphi), r \sin_S(\varphi))$$

and

$$\frac{\partial}{\partial y} h_S(x,y) = \frac{\partial r}{\partial y}|_{(x,y)=Pol_S(r,\varphi)} = h_{S,y}(r \cos_S(\varphi), r \sin_S(\varphi)).$$

□

Remark 9. *For positively homogeneous h_T and differentiable Minkowski functional h_S of the star disc S, formula* (5) *may be rewritten as*

$$AL_{S,T}(\varrho) = \varrho \int_0^{2\pi} R_S^2(\varphi) h_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T} (Pol_S(\nabla h_S(x,y))(r,\varphi)) d\varphi. \quad (6)$$

Definition 10. *A star body S and a star disc T satisfy the rotated gradient condition if*

$$h_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T} (\nabla h_S(x,y)|_{(x,y)=Pol_S(r,\varphi)}) = 1, a.e. \quad (7)$$

Let us notice that, at the point (x,y) from $\mathfrak{C}_S(\varrho)$, the gradient $\nabla h_S(x,y)$ is normal to the level set $\mathfrak{C}_S(\varrho), \varrho > 0$ of h_S . The following lemma is a consequence of the above consideration.

Lemma 11. *For a star body S and a star disc S^* satisfying the rotated gradient condition* (7)*, the positive directed S^*-arc-length of $\mathfrak{C}_S(\varrho)$ allows the representation*

$$AL_{S,S^*}(\varrho) = \varrho \int_0^{2\pi} R_s^2(\varphi) d\varphi.$$

We consider now the area function

$$A_S(\varrho) = \varrho^2 A_S(1) = \frac{\varrho^2}{2} \int\limits_0^{2\pi} R_S^2(\varphi) d\varphi$$

where $A_s(1)$ denotes the area content of the unit disc $K_S(1)$. The derivative of the area function satisfies obviously the equation

$$\frac{d}{d\varrho} A_S(\varrho) = \varrho \int\limits_0^{2\pi} R_S^2(\varphi) d\varphi.$$

The following theorem has thus been proved.

Theorem 12. *If the star body S and the star disc S^* satisfy the rotated gradient condition* (7) *then*

$$\mathfrak{U}_S(\varrho) = AL_{S,S^*}(\varrho), \tag{8}$$

i.e., the S-generalized circumference of S coincides with the positive directed S^*-circumference of S.

If relation (8) holds then

$$AL_{S,S^*}(1) = 2A_S(1). \tag{9}$$

Consequently, the ratios $A_S(\varrho)/\varrho^2$ and $AL_{S,S^*}(\varrho)/2\varrho$ satisfy the relations

$$\frac{A_S(\varrho)}{\varrho^2} \overset{(a)}{=} A_S(1) \overset{(c)}{=} \frac{AL_{S,S^*}(\varrho)}{2\varrho}, \forall \varrho > 0. \tag{10}$$

In this sense, the geometry and the arc-length measure generated by S^* fulfill our expectations. The following definition is thus well motivated if a star body S and a star disc S^* are chosen in such a way that the limit in Definition 2 is uniquely determined, h_T is positively homogeneous, h_S is a.e. differentiable and the rotated gradient condition (7) is satisfied.

Definition 13. *(a) The properties of the star bodies $K_S(\varrho), \varrho > 0$, which are expressed by the equations* (a) *and* (c) *in (10) are called the area-content and the S-generalized circumference properties of the discs, respectively.*
(b) The quantity $A_S(1) =: \pi(S)$ is called the S-generalized circle number of the star bodies $K_S(\varrho), \varrho > 0$.

We may write now the equations in (10) as

$$AL_{S,S^*}(\varrho) = 2\pi(S)\varrho \qquad \textit{and} \qquad A_S(\varrho) = \pi(S)\varrho^2. \tag{11}$$

Notice that the circle number function $S \to \pi(S)$ assigns a generalized circle number to any star body $K_S(\varrho)$ satisfying assumption (7) . The restrictions of this function to $l_{2,p}$-balls or axes aligned ellipses were considered in Richter [2008a,b, 2011].

Example 14. *Here we consider a first, rather general case where the rotated gradient condition* (7) *is satisfied. Let* $||.||_{(p)}$ *and* $||.||_{(d)}$ *denote a (primary)* C^1*-norm in* $\mathbb{R}^n$ *and the corresponding dual one, respectively. It is proved in Yang [1991] that*

$$||\,\nabla\,||\,\mathfrak{x}\,||_{(p)}\,||_{(d)} = 1, \forall\, \mathfrak{x} \in \mathbb{R}^n.$$

Hence, if S *is a convex body, i.e.* $h_S(\mathfrak{x}) = ||\,\mathfrak{x}\,||_{(p)}$ *is a (primary) norm, and if*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T = \{\mathfrak{x} \in \mathbb{R}^n : ||\,\mathfrak{x}\,||_{(d)} \leq 1\} = S^*$$

is the unit ball w.r.t. the corresponding dual norm then the condition (7) *is satisfied.*

For determining the actual value of a generalized circle number $\pi(S) = A_S(1)$ *we may refer, e.g., to Pisier [1999] where volumes of convex bodies are dealt with. Alternatively, one may use* S*-generalized polar coordinates for making the respective calculations in given cases. The particular results for* $l_{2,p}$*-circles with* $p \geq 1$ *and of axes aligned ellipses as well as the corresponding generalized circle numbers have been dealt with in Richter [2008a, 2011].*

Example 15. *We consider now the non-convex* $l_{2,p}$*-circles* $C_p = \{(x, y) \in \mathbb{R}^2 : |x|^p + |y|^p = 1\}$ *with* $0 < p < 1$*. Such generalized circles correspond to anti-norms. A suitable arc-length measure for measuring* C_p *is based upon the star disc* $S(p^{**}) = \{(x, y) \in \mathbb{R}^2 : |x|^{p^{**}} + |y|^{p^{**}} \geq 1\}$ *with* $p^{**} < 0$ *satisfying* $\frac{1}{p} + \frac{1}{p^{**}} = 1$*. The star discs* $S(p^{**})$ *are closely related to a specific semi-anti-norms w.r.t. the canonical fan. The corresponding generalized circle numbers have been determined in Richter [2008b]. As because this was done without referring explicitely to* (7)*, we may state here the following problem.*

Problem 16. *Give a general description of sets T satisfying condition* (7) *for sets S being generated by anti-norms.*

As was indicated in Richter [2008b], p-generalized circle numbers for $0 < p < 1$ may occur, e.g., within certain combinatorial formulae. Notice further that the reciprocal values of the coefficients of the binomial series expansion

$$\frac{1}{4\sqrt{1-4u}} = \sum_{n=0}^{\infty} \frac{1}{\pi(\frac{1}{n})} \cdot u^n,\ u \in (0, \frac{1}{4}),$$

are just the generalized circle numbers corresponding to the non-convex $l_{2,1/n}$-circles. One could also ask for a (possibly elementary geometric ?) explanation of this fact.

Unbounded star discs

In this section, we consider a class of (truncated) unbounded Orlicz discs. More generally than in the preceding section, a star-shaped subset of $\mathbb{R}^2$ is called a star disc if all its intersections with balls centered at the origin are star bodies. The boundary of a star disc is called a star circle. Notice that a star circle is not necessarily bounded. Special sets of this type will be studied in this section. To be more specific, let us consider, for arbitrary $p < 0$, the function

$$(x, y) \to |(x, y)|_p = (|x|^p + |y|^p)^{1/p}, x \neq 0, y \neq 0$$

which denotes a semi-anti norm, and the p-generalized circle

$$C_p = \{(x, y) \in \mathbb{R}^2 : |(x, y)|_p = 1\}.$$

The pairs of straight lines $|y| = 1$ and $|x| = 1$ represent asymptotes for the circle C_p as $|x| \to \infty$ or $|y| \to \infty$, respectively. The intersection point of the p-circle C_p with the line $y = x$ is for positive coordinates $(x_0, y_0) = 2^{-1/p} \cdot (1, 1)$.

Let further $C_p(r) = r \cdot C_p, r > 0$ denote the p-generalized circle of p-generalized radius $r > 0$. It is the boundary of the unbounded p-generalized disc of p-generalized radius r,

$$K_p(r) = \{(x, y) \in R^2 : |(x, y)|_p \leq r\} = rK_p, \quad K_p = K_p(1).$$

As because

$$|(x, y)|_p \leq r \iff |x|^p + |y|^p \geq r^p$$

$$\iff f(|x|) + f(|y|) \geq f(r) \text{ for } f(\lambda) = \lambda^p, \lambda \geq 0,$$

one may call $K_p(r)$ a two-dimensional Orlicz-anti-ball corresponding to the Young function f. The disc K_p is a star-shaped but non-compact set and therefore not a star body. For any $(x, y) \in C_p(r)$ one may think of r as the value of the Minkowski-functional w.r.t. the reference set K_p. The area content and the Euclidean circumference of the unit p-circle are obviously unbounded. That's why we consider from now on truncated p-circles. To this end, let us introduce truncation cones

$$C(x_1) := \{(x, y) \in \mathbb{R}^2 : \frac{||(x, y) - \Pi_1(x, y)||}{||\Pi_1(x, y)||} < \frac{||(x_1, y_1) - \Pi_1(x_1, y_1)||}{||\Pi_1(x_1, y_1)||}\}$$

$$= \{(x, y) \in \mathbb{R}^2 : \frac{|x - y|}{|x + y|} < \frac{|x_1 - y_1|}{|x_1 + y_1|}\}$$

where $||.||$ denotes Euclidean norm, $1 = (1, 1)$, x_1 is chosen according to $x_1 > x_0 = 2^{-\frac{1}{p}}$ and $|y_1| = (1 - |x_1|^p)^{1/p} < 1$.
The question of interest is now whether we may define in a reasonable way circle numbers for the truncated p-discs $K_p^{x_1}(r) = rK_p^{x_1}$ the boundaries of which are the p-circles $C_p^{x_1}(r) = rC_p^{x_1}$ of p-radius r and where

$$K_p^{x_1} := K_p \cap C(x_1) \text{ and } C_p^{x_1} := C_p \cap C(x_1).$$

To this end, let $\mathfrak{Z}_n = (z_0, z_1, ..., z_n)$ be an arbitrary successive anticlockwise oriented partition of the truncated circle $C_p^{x_1}$ satisfying $z_0 = (x_0, (1 - x_0^p)^{1/p})$ and $z_n = (x_1, (1 - x_1^p)^{1/p})$. We consider the sum

$$S(\mathfrak{Z}_n) = \sum_{j=1}^{n} |z_j - z_{j-1}|_q = \sum_{j=1}^{n} (|x_j - x_{j-1}|^q + |y_j - y_{j-1}|^q)^{1/q}, q \in (0, 1)$$

and observe that due to the reverse triangle inequality it decreases monotonously as

$$F(\mathfrak{Z}_n) := \sup_{1 \leq j \leq n} |z_j - z_{j-1}|_q \to 0.$$

According to the symmetry of $C_p^{x_1}$, the following remark is justified.

Remark 17. *For $q \in (0, 1)$, the $l_{2,q}$-arc-length of the truncated circle $C_p^{x_1}$ is defined as*

$$AL_{p,q}^{x_1} = 8 \lim_{F(\mathfrak{Z}_n) \to 0} S(\mathfrak{Z}_n).$$

If $x \to y(x)$ denotes an arbitrary parameter representation of the truncated circle $C_p^{x_1}$ then

$$\frac{1}{8}AL_{p,q}^{x_1} = \lim_{F(\mathfrak{Z}_n)\to 0} \sum_{j=1}^{n} (1 + |\frac{\Delta y_j}{\Delta x_j}|^q)^{1/q} \Delta x_j = \int_{x_0}^{x_1} (1 + |y'(x)|^q)^{1/q} dx.$$

Let us denote the usual Euclidean area content of the truncated circle disc $K_p^{x_1}$ by $A_p^{x_1}$.

Lemma 18. *Let for arbitrary $p < 0$ the number $p^* \in (0,1)$ be uniquely defined by the equation $\frac{1}{p} + \frac{1}{p^*} = 1$. Then*

$$A_p^{x_1} = \frac{1}{2} AL_{p,p^*}^{x_1} \,. \tag{12}$$

Proof :
With

$$y(x) = (1 - |x|^p)^{1/p} = (1 - x^p)^{1/p}, y'(x) = -(1 - x^p)^{1/p-1} x^{p-1},$$

it follows from the above formulae that

$$AL_{p,q}^{x_1} = 8 \int_{1/2^{1/p}}^{x_1} (1 + (1 - x^p)^{(1/p-1)q} x^{(p-1)q})^{1/q} dx.$$

Changing variables $u = x^p, dx = du/(pu^{(p-1)/p})$ causes a change of the limits of integration:

$$AL_{p,q}^{x_1} = \frac{8}{-p} \int_{x_1^p}^{1/2} (1+(1-u)^{\frac{1-p}{p}q} u^{\frac{p-1}{p}q})^{\frac{1}{q}} \frac{du}{u^{\frac{p-1}{p}}} = \frac{8}{|p|} \int_{x_1^p}^{1/2} (u^{\frac{1-p}{p}q} + (1-u)^{\frac{1-p}{p}q})^{\frac{1}{q}} du.$$

Assuming now $\frac{1}{p} + \frac{1}{q} = 1$, or equivalently $q = \frac{p}{p-1} =: p^*$, it follows

$$AL_{p,p^*}^{x_1} = \frac{8}{|p|} \int_{x_1^p}^{1/2} (u^{-1} + (1-u)^{-1})^{\frac{p-1}{p}} du = \frac{8}{|p|} \int_{x_1^p}^{1/2} (\frac{1-u+u}{u(1-u)})^{1-\frac{1}{p}} du.$$

Hence,

$$AL_{p,p^*}^{x_1} = \frac{8}{|p|} \int_{x_1^p}^{1/2} u^{\frac{1}{p}-1} (1-u)^{\frac{1}{p}-1} du. \quad (o)$$

Now, what about the area content of the truncated circle $C_p^{x_1}$? The $l_{2,p}$-generalized standard triangle coordinate transformation Tr from Richter [2007] is defined by

$$Tr_p(r,\mu) = (x,y) \text{ with } x = r\mu, y = +(-)r(1-|\mu|^p)^{1/p}.$$

Because of

$$\{(x,y) : |x|^p + |y|^p = 1, x_0 \le x \le x_1\} = Tr_p(\{1\} \times [x_0, x_1])$$
$$r\{(x,y) : |x|^p + |y|^p = 1, x_0 \le x \le x_1\} =$$
$$\{(rx, ry) : |x|^p + |y|^p = 1, x_0 \le x \le x_1\}$$
$$= \{(\xi,\eta) : |\frac{\xi}{r}|^p + |\frac{\eta}{r}|^p = 1, x_0 \le \frac{\xi}{r} \le x_1\}$$
$$= \{(x,y) : |x|^p + |y|^p = r^p, x_0 \le \frac{x}{r}(=: \mu) \le x_1\}$$
$$= Tr_p(\{r\} \times [x_0, x_1])$$

it follows

$$\bigcup_{0 \le r \le 1} r\{(x,y) : |x|^p + |y|^p = 1, x_0 \le x \le x_1\}$$
$$= Tr_p([0,1] \times [x_0, x_1]) = K_p^{x_1},$$

i.e.,

$$K_p^{x_1} = Tr_p([0,1] \times [x_0, x_1]). \qquad (*)$$

Changing Cartesian with standard triangle coordinates in the integral

$$A_p^{x_1} = \int_{K_p^{x_1}} d(x,y),$$

we get

$$A_p^{x_1} = 8 \int_{r=0}^{1} (\int_{\mu=x_0}^{x_1} r(1-\mu^p)^{\frac{1-p}{p}} d\mu) dr = \frac{8}{2} \int_{2^{-1/p}}^{x_1} (1-\mu^p)^{1/p-1} d\mu.$$

Substituting $y = \mu^p, \frac{dy}{d\mu} = py^{\frac{p-1}{p}}$, it follows

$$A_p^{x_1} = \frac{4}{p} \int_{1/2}^{x_1^p} (1-y)^{1/p-1} y^{1/p-1} dy = \frac{4}{|p|} \int_{x_1^p}^{1/2} y^{1/p-1} (1-y)^{1/p-1} dy.$$

Hence, the lemma is proved. □

Remark 19. *Because of $p < 0$,*

$$A_p^{x_1} \to \infty \text{ as } x_1 \to \infty.$$

The following corollary and definition are now quite obvious and well motivated.

Corollary 20. *For arbitrary $x_1 > x_0 = 2^{-1/p}$, the truncated star discs $K_p^{x_1}$ have the area content and p-generalized circumference properties (a^*) and (c^*), respectively,*

$$\frac{A_p^{x_1}(r)}{r^2} \overset{(a^*)}{=} A_p^{x_1} \overset{(c^*)}{=} \frac{AL_{p,p*}^{x_1}(r)}{2r}, \tag{13}$$

from which it follows immediately

$$\frac{d}{dr} A_p^{x_1}(r) = AL_{p,p^*}^{x_1}(r). \tag{14}$$

Remark 21. *One may think of equation (14) as reflecting a generalized method of indivisibles for each $x_1 > x_0$ where the truncated circles $C_p^{x_1}$ are the indivisibles and measuring them is based upon the geometry generated by the disc K_{p^*}.*

Definition 22. *For arbitrary $x_1 > 2^{-1/p}$, the quantity $A_p^{x_1} =: \pi^{x_1}(p)$ will be called the circle number of the truncated circle $C_p^{x_1}$.*

Literaturverzeichnis

Hilbert,D.(1900), Mathematische Probleme. Internationaler Mathematikerkongress, Paris.

Moszyńska,M., (2006), Selected topics in convex geometry. Birkhäuser.

Moszyńska, M. and W.-D. Richter (2011). Reverse triangle inequality. Norms and Anti-norms. Submitted.

Pisier, G., (1999), On the volume of convex bodies and Banach space geometry. Cambridge tracts in mathematics; 94.

Richter,W.-D. (2007), *Generalized spherical and simplicial coordinates*, J.Math.Anal.Appl., 336, 1187-1202. doi:10.1016/j.jmaa.2007.03.047

Richter,W.-D. (2008a). *On $l_{2,p}$-circle numbers.* Lith. Math. J., 48, 2, 228-234. doi.org./10.1007/s10986-008-9002-Z

Richter,W.-D. (2008b). *On the π-function for non-convex $l_{2,p}$-circle discs.*Lith. Math. J., 48, 3, 332-338. doi.org./10.1007/s10986-008-9016-6

Richter, W.-D. (2011), *Ellipses numbers and geometric measure representations*, Journal of Applied Analysis, vol. 17, in print.

Thompson, A.C., (1996), Minkowski geometry. Cambridge University Press.

Webster, R., (1994), Convexity. Oxford University Press.

Yang, Wei H. (1991) *On generalized Hölder inequality*, Nonlinear Analysis, Theory, Methods Appl., Vol. 16, No.5, 489-498.