On the ball number function

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Summary The ball number function was recently defined in [6] for $l_{n,p}$-balls. It was shown there that ball numbers occur naturally in factorizations of the normalizing constants of density generating functions if such functions depend on the $l_{n,p}$-norm. For an analogous situation in the case of ellipsoids we refer to [7] and [8]. Here, we discuss some additional properties of the ball number function and state the problem of extending it to sets which are generated by arbitrary norms or anti-norms (for definition, see [4]), and to even more general balls. As an application, we present a method for deriving new specific representation formulae for values of the Beta function.
Keywords Generalized method of indivisibles, non-Euclidean surface content, disintegration of Lebesgue measure, co-area-formula, ball number asymptotics, generalization of the circle number π, thin layers property, geometric measure representation, intersection percentage function, Beta function, anti-norm, semi-anti-norm.
MSC 2010 28A50, 28A75, 33B15, 51F99, 53A35, 60D05, 60E05.

1 Introduction

The circle number π reflects in a well known sense the area content property and the circumference property of Euclidean circles. The Euclidean ball number function extends these properties to arbitrary dimension $n \geq 2$ and reflects the volume property and surface content property of Euclidean balls. If $V(r)$ and $O(r)$ denote volume and surface content of the n-dimensional Euclidean ball $B(r)$ with radius $r > 0$ then the ratios $V(r)/r^n$ and $O(r)/(nr^{n-1})$ do not depend on the radius of the ball and their constant values coincide with each other. Hence, it is natural to think of the actual value of this constant as a ball number. If this will be denoted by π_n then

$$\frac{V(r)}{r^n} = \pi_n = \frac{O(r)}{nr^{n-1}},\ \forall r > 0$$

and

$$\pi_n = V(1) = \frac{O(1)}{n} = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})}.$$

It may be surprising that in fact many books on geometry and analysis although giving explicit expressions for $V(r)$ and $O(r)$ do not mention these relations which connect volume and surface content with each other. In what follows, let λ and $\mathfrak{O}$ denote the Lebesgue measure and the Euclidean surface measure in $\mathbb{R}^n$, respectively. Further, let $S(r)$ be the Euclidean sphere of radius r. The obvious conclusions from the above equations,

$$V'(r) = O(r) \text{ and } V(r) = \int_0^r O(\varrho)d\varrho,$$

are closely related to the method of indivisibles of Cavalieri and its extension by Torricelli which is nowadays basically reflected by the $S(r)$-adapted disintegration formulae for the Lebesgue measure,

$$\lambda(A) = \int_0^\infty \mathfrak{O}(A \cap S(r))dr = n\pi_n \int_0^\infty r^{n-1}\mathfrak{F}(A,r)dr \text{ for } A \in \mathfrak{B}^n \text{ satisfying } \lambda(A) < \infty.$$

Here, $\mathfrak{B}^n$ denotes the Borel σ-algebra in $\mathbb{R}^n$ and

$$\mathfrak{F}(A,r) := \frac{\mathfrak{O}(\frac{1}{r}A \cap S(1))}{O(1)}$$

is the Euclidean intersection percentage function of the set A. These conclusions, which refer to the main theorem of calculus, are furthermore closely related to the thin layers property of the Lebesgue measure which is expressed by the asymptotic equivalence relation

$$\lambda(r \leq ||x|| \leq r+\varepsilon) \sim n\pi_n r^{n-1}\varepsilon,\ \varepsilon \to +0.$$

Let us describe the upper half-sphere of radius r by

$$y(x) := (r^2 - \sum_1^{n-1} x_i^2)^{1/2},\ i=1,...,n-1,\ \sum_1^{n-1} x_i^2 < r^2.$$

The normal vector to this surface at the point $(x, y(x))$ with $\sum_1^{n-1} x_i^2 < r^2$ is

$$N(x) = \sum_1^{n-1} [\frac{\partial}{\partial x_i} y(x)] e_i - e_n$$

where $e_1, ..., e_n$ are standard unit vectors in $\mathbb{R}^n$. The surface content of a half-sphere may be represented as

$$\int\limits_{\sum_1^{n-1} x_i^2 < r^2} ||N(x)||dx = \frac{1}{2}O(r)$$

where $||.||$ denotes the Euclidean norm in $\mathbb{R}^n$. The resulting equation

$$2 \int\limits_{\sum_1^{n-1} x_i^2 < r^2} ||N(x)||dx = V'(r)$$

may therefore be thought of as reflecting both the global look and the local look at the notion of surface content which reflect integrating the norm of the normal vector and taking the derivative of the volume function, respectively.

2 The $l_{n,p}$-ball number function

While all ingredients for defining the ball number function $n \mapsto \pi_n$ for Euclidean balls in the last section were known from the literature already for a long time the situation is quite different in the case of arbitrary $l_{n,p}$-balls. We start this section with explaining a background problem for this case coming from measure theory. A conclusion which follows from the co-area-formula (see, e.g. [3]) for the function

$$f(x) := |x|_p,\ |x|_p := (\sum_1^n |x_i|^p)^{1/p}, x \in \mathbb{R}^n, p > 0, p \neq 1$$

is that for $A \in \mathfrak{B}^n$ satisfying $\lambda(A) < \infty$

$$\int_0^\infty \mathfrak{O}(A \cap f^{-1}(r))dr = \int_A J(f)(x)dx =: \nu(A).$$

Here, $f^{-1}(r) = \{x \in \mathbb{R}^n : (\sum\limits_1^n |x_i|^p)^{1/p} = r\} =: S_{n,p}(r)$ denotes the $l_{n,p}$-sphere of 'p-radius' r and $J(f)(x) = (\frac{|x|_{2p-2}}{|x|_p})^{p-1}$ is the Jacobian of f. The measure ν satisfies the relations

$$\nu(A) \begin{cases} < \\ = \\ > \end{cases} \lambda(A) \quad \text{if and only if} \quad p \begin{cases} > \\ = \\ < \end{cases} 2.$$

Hence, there is no extension of the disintegration formula for the Lebesgue measure if the indivisibles are $l_{n,p}$-spheres and measuring them will be done using the Euclidean surface content. If one measures, however, the $l_{n,p}$-spheres $S_{n,p}(r)$ by another surface content than the Euclidean one then it may become possible to extend the $S(r)$-adapted method of indivisibles to the case that the indivisibles are $l_{n,p}$-spheres.

It has been proved in [5] that this actually holds if one defines the surface measure as follows. Let the upper half of the $l_{n,p}$-sphere $S_{n,p}(r)$ be described by

$$y(x) := (r^p - \sum_1^{n-1} |x_i|^p)^{1/p}, \ \sum_1^{n-1} |x_i|^p < r^p, \ p > 0.$$

Making use of the normal vector to this surface, $N(x) = \sum_{i=1}^{n-1} [\frac{\partial}{\partial x_i} y(x)] e_i - e_n$, we define now a surface measure for Borel subsets D from the upper half sphere by

$$\mathfrak{O}_p(D) := \int\limits_{G(D)} |N(x)|_q dx, \ G(D) := \{x = (x_1, ..., x_{n-1}) : \sum_1^{n-1} |x_i|^p < r^p \text{ and } (x_1, ..., x_n) \in D\}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, i.e., \ q = \frac{p}{p-1} \in \begin{cases} [1, \infty] & \text{if } p \in [1, \infty] \\ (-\infty, 0) & \text{if } p \in (0, 1). \end{cases}$$

If $V_p(r)$ and $O_p(r)$ denote volume and $\mathfrak{O}_p$-surface content of the $l_{n,p}$-ball with p-radius $r > 0$ then, according to [6], the global look and the local look at the generalized surface content are reflected by the equation

$$2 \int\limits_{\sum\limits_1^{n-1} |x_i|^p < r^p} |N(x)|_q dx = V_p'(r).$$

By combining this with the well known explicit formula for $V_p(r)$, we obtain

$$\frac{V_p(r)}{r^n} = \pi_n(p) = \frac{O_p(r)}{nr^{n-1}}, \ \forall r > 0$$

where

$$\pi_n(p) := V_p(1) = \frac{O_p(1)}{n} = \frac{2^n\Gamma^n(\frac{1}{p})}{np^{n-1}\Gamma(\frac{n}{p})}, p > 0.$$

In the next section, we shall study some properties of this ball number function $(n,p) \mapsto \pi_n(p)$ which was defined in [6]. Before doing this, let us emphasize the remarkable circumstance that two generalizations are going here hand in hand. Changing the Euclidean surface measure with $\mathfrak{O}_p$, on the one hand the ball number function $(n,p) \mapsto \pi_n(p)$ extends the function $n \mapsto \pi_n$ from the set of Euclidean balls to the set of $l_{n,p}$-balls and on the other hand the $l_{n,p}$-adapted disintegration formula for the Lebesgue measure is a generalization of the $S(r)$-adapted one. This can be seen from the following result in [5] and has been observed in an analogous way in [7] and [8] for ellipses and ellipsoids, respectively.

If a Borel measurable set A satisfies the assumption $\lambda(A) < \infty$ then the Lebesgue measure satisfies the following $l_{n,p}$-adapted disintegration formulae

$$\lambda(A) = \int_0^\infty \mathfrak{O}_p(A \cap S_{n,p}(r))dr = n\pi_n(p)\int_0^\infty r^n\mathfrak{F}_p(A,r)dr.$$

Here,

$$\mathfrak{F}_p(A,r) := \frac{\mathfrak{O}_p(\frac{1}{r}A \cap S(1))}{O_p(1)}$$

is the (non-Euclidean) p-generalized intersection-percentage-function of the set A.

Finally, let us remark that our choice of the geometry for measuring the surface content is closely connected with that one for solving the isoperimetric problem for Minkowski area in [2]. It turns out that certain types of ”parallel sets” discussed there are similar to those which occur when one is taking the derivative of the volume function $V_p(r)$ w.r.t. the p-radius r. For more details, we refer to [5].

3 Properties of the $l_{n,p}$-ball number function

Theorem 1 *For fixed dimension n, the ball number function $p \mapsto \pi_n(p)$ is continuous and increasing and satisfies the following relations*

$$0 = \lim_{p\to 0}\pi_n(p) < \pi_n(p) < \lim_{p\to\infty}\pi_n(p) =: \pi(\infty) = 2^n.$$

Proof :
All properties of the $l_{n,p}$-ball number function mentioned in the theorem follow immediately when thinking of $\pi_n(p)$ as the volume of the $l_{n,p}$-unit ball □

Theorem 2 *For fixed dimension n, the ball number function $p \mapsto \pi_n(p)$ satisfies the following asymptotic relations*

$$\pi_n(p) = \frac{2^{\frac{3n-1}{2}}\pi^{\frac{n-1}{2}}}{p^{\frac{n-1}{2}}n^{\frac{n}{p}+\frac{1}{2}}}(1 + \frac{n^2-1}{12n}p + O(p^2)),\ p \to 0$$

and

$$\pi_n(p) = 2^n(1 - \frac{\pi^2 n(n-1)}{12p^2} + O(\frac{1}{p^3})),\ p \to \infty.$$

Proof :
Proof of the first assertion follows essentially the line of the corresponding proof in the two-dimensional case and may therefore be omitted here. We start now from the representation for the ball number $\pi_n(p) = 2^n\Gamma^n(\frac{1}{p})/(np^{n-1}\Gamma(\frac{n}{p})), p > 0$. Combining this with the asymptotic relation

$$\Gamma(\frac{k}{p}) = p(1 - \frac{\gamma k}{p} + \frac{\delta k^2}{p^2})/k + O(\frac{1}{p^3}),\ p \to \infty$$

where $\gamma = 0,577...$ is the Euler constant and δ is a suitably chosen other constant, (see [1]), we obtain the second assertion □

It is an immediate conclusion from Theorem 1 that every positive number is a ball number. Moreover, there are infinitely many possibilities to represent a positive number as a ball number.

The second assertion in Theorem 2 motivates to consider normalized $l_{n,p}$-ball numbers

$$\pi_n^*(p) := \pi_n(p)/2^n,$$

see Figure 1.

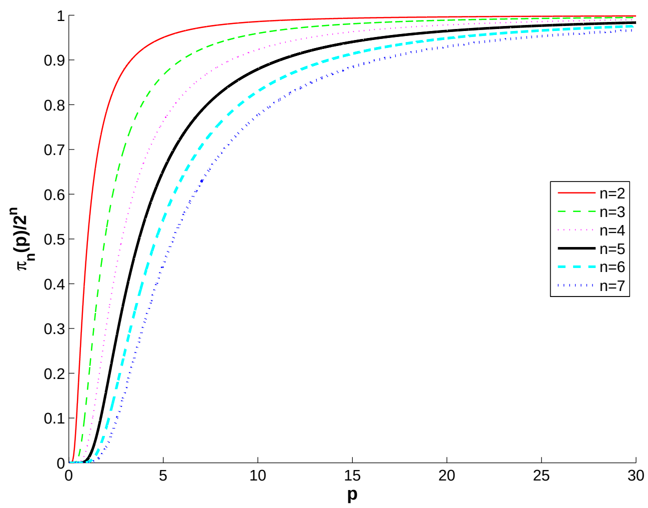

Figure 1: The normalized ball number function $(n,p) \mapsto \pi_n^*(p) = \pi_n(p)/2^n$

The following Figure 2 presents a more detailed look at the normalized $l_{n,p}$-ball number function for small values of p.

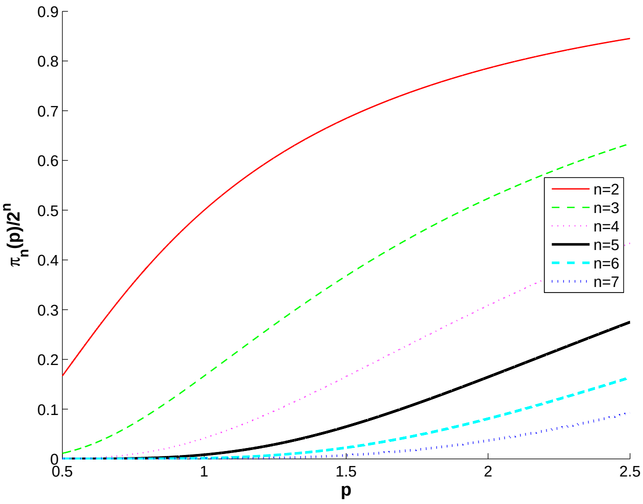

Figure 2: $\pi_n^*(p) = \pi_n(p)/2^n$ for small values of p

The original, i.e. non-normalized, $l_{n,p}$-ball number function for small values of p can be seen in Figure 3.

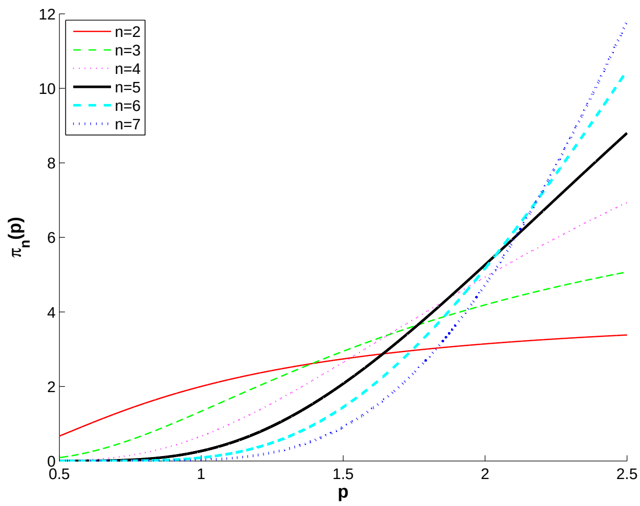

Figure 3: The ball number function $(n, p) \mapsto \pi_n(p)$ for small values of p

Theorem 3 *The Lebesgue measure has the $l_{n,p}$-thin layers property*

$$\lambda(r \leq |x|_p \leq r + \varepsilon) \,\sim\, n\pi_n(p)r^{n-1}\varepsilon,\ \varepsilon \to +0.$$

Proof :
We start from the $l_{n,p}$-adapted disintegration formula for the Lebesgue measure and observe the corresponding intersection percentage function of the set $A = \{x \in \mathbb{R}^n : r \leq |x|_p \leq r + \varepsilon\}$,

$$\mathfrak{F}_p(A, \rho) = I_{(r,r+\varepsilon)}(\rho), \rho > 0.$$

Hence,

$$\lambda(A) = n\pi_n(p) \int_0^\infty \rho^{n-1} I_{(r,r+\varepsilon)}(\rho) d\rho.$$

Finally, $\lambda(A) = \pi_n(p)[(r+\varepsilon)^n - r^n]$ □

Studying additional recurrence properties of the $l_{n,p}$-ball number function, which are of interest for their own, we prove the following specific formula for values of the Beta function of which the author has not been aware.

Theorem 4 *For all $x > 0$ and $n \in \{3, 4, ...\}$, the Beta function satisfies the representation*

$$xB(x,(n-1)x) = \int_0^{\pi/2} \frac{(\sin\varphi)^{n-2} d\varphi}{((\sin\varphi)^{1/x} + (\cos\varphi)^{1/x})^{nx}}.$$

Proof :
It follows immediately from the explicit formula for $\pi_n(p)$ that the $l_{n,p}$-ball number function satisfies the (first) recurrence formula

$$(*)\ \pi_n(p) = \frac{2}{p}(\frac{n-1}{n})B(\frac{1}{p}, \frac{n-1}{p})\pi_{n-1}(p),\ n = 3, 4, ...$$

where the starting value $\pi_2(p)$ allows several known integral representations. Change of Cartesian coordinates into p-generalized spherical coordinates from [5] in the integral

$$\int\limits_{|x|_p \le 1} dx = V_p(1) = \pi_n(p)$$

yields

$$\pi_n(p) = \int_0^1 r^{n-1} dr \int_0^{\pi} ... \int_0^{\pi} \int_0^{2\pi} \frac{\sin^{n-2}\phi_1 \cdot ... \cdot \sin\phi_{n-2}}{N_p^n(\phi_1) \cdot ... \cdot N_p^3(\phi_{n-2}) N_p^2(\phi_{n-1})} d\phi_{n-1} d\phi_{n-2} ... d\phi_1$$

with

$$N_p(\varphi) = (|\sin\varphi|^p + |\cos\varphi|^p)^{1/p}.$$

Hence, the $l_{n,p}$-ball number function satisfies also the (second) recurrence formula

$$(**)\ \pi_n(p) = (\frac{n-1}{n}) \int_0^{\pi} (\frac{\sin\varphi}{N_p(\varphi)})^{n-2} \frac{d\varphi}{N_p^2(\varphi)} \pi_{n-1}(p), n = 3, 4, ...,$$

$$\pi_2(p) = \int_0^{\pi} \frac{d\varphi}{N_p^2(\varphi)}.$$

Comparing (*) with (**), we obtain

$$\frac{1}{p}B(\frac{1}{p}, \frac{n-1}{p}) = \int_0^{\pi/2} \frac{(\sin\varphi)^{n-2} d\varphi}{((\sin\varphi)^p + (\cos\varphi)^p)^{n/p}}.$$

Finally, put $x = \frac{1}{p}$ □

As a consequence of Theorem 4 for $n = 3$ we obtain the following:

$$\frac{x\Gamma(x)\Gamma(2x)}{\Gamma(3x)} = \int_0^{\pi/2} \frac{\sin\varphi d\varphi}{((\sin\varphi)^{1/x} + (\cos\varphi)^{1/x})^{3x}}, x > 0.$$

Combining, e.g., this formula with the analogous one derived from Theorem 4 for $n = 4$ we infer that

$$\frac{\Gamma^2(3x)}{\Gamma(2x)\Gamma(4x)} = \int_0^{\pi/2} \frac{\sin^2\varphi d\varphi}{((\sin\varphi)^{1/x} + (\cos\varphi)^{1/x})^{4x}} / \int_0^{\pi/2} \frac{\sin\varphi d\varphi}{((\sin\varphi)^{1/x} + (\cos\varphi)^{1/x})^{3x}}, x > 0.$$

An additional representation formula for the $l_{n,p}$-ball number function is given in the next theorem.

Theorem 5 *The $l_{n,p}$-ball number function satisfies the equation*

$$\pi_n(p) = \frac{2^{n-1}\pi_2(p)}{n} \prod_{i=1}^{n-2} \int_0^{\pi/2} \frac{d\varphi}{(1 + (\cot\varphi)^p)^{i/p} N_p^2(\varphi)}.$$

Proof :
It can be seen from the second recurrence formula in the proof of Theorem 4 that

$$\pi_n(p) = \frac{1}{n} \prod_{i=1}^{n-2} \int_0^{\pi} (\frac{\sin\phi}{N_p(\phi)})^i \frac{d\phi}{N_p^2(\phi)} \cdot 2\pi_2(p).$$

The assertion of the theorem follows now immediately □

Finally, let $g \geq 0$ be a density generating function satisfying the assumption $I_{n,g,p} < \infty$ where $I_{n,g,p} := \int_0^\infty r^{n-1} g(r^p) dr$. We recall from [6] that the normalizing constant of the corresponding $l_{n,p}$-symmetric density

$$\varphi_{p,g}(x) := C_p(n, g) g(|x|_p^p)$$

allows a factorization involving the geometric quantity $\pi_n(p)$ and the analytical one, $I_{n,g,p}$,

$$C_p(n, g) = \frac{1}{n\,\pi_n(p)\, I_{n,g,p}}.$$

Examples of density generating functions are the p-generalized Gaussian one, $g_G(r) := e^{-r/p} I_{(0,\infty)}$, the Kotz-type one, $g_K(r) := r^{M-1} e^{-\beta r^\gamma} I_{(0,\infty)}$, and the Pearson-VII-type one, $g_P(r) := \frac{1}{(1+\frac{r}{m})^M} I_{(0,\infty)}$.

4 Outlook: more general balls

Let us recall that in Sections 1 and 2 we considered the ball numbers π_n and $\pi_n(p)$. Moreover, the ellipsoid number function $(n, a) \mapsto \pi_n^E(a)$ is considered in [8]. It assigns a number $\pi_n^E(a)$ to each axes aligned ellipsoid with half-axes of lengths $r \cdot a_1, ..., r \cdot a_n$. Here, $a = (a_1, ..., a_n)$, and

r is the "a-radius" of the ellipsoid which can be interpreted in terms of a Minkowski functional. In this sense, one may think of the ellipsoid also as a generalized ball. This gives rise to search for a more general approach to defining just one general ball number function. We shall do this in such a way that the set of all specific ball (or ellipsoid) number functions studied so far may be understood as a restriction of the new one. To this end, we consider a function

$$(n, B) \mapsto \pi_n^{\mathfrak{S}}(B),\ B \in \mathfrak{S}$$

where $\mathfrak{S}$ denotes the set of all star bodies (with respect to 0) in $\mathbb{R}^n$. The restriction of $\pi_n^{\mathfrak{S}}$ to any subset $\mathfrak{T}$ of $\mathfrak{S}$ will be denoted by $\pi_n^{\mathfrak{T}}$. In this sense, the ball number function $(n, p) \mapsto \pi_n(p)$ is the restriction of the function $\pi_n^{\mathfrak{S}}$ to the set $\mathfrak{T}_1$ of all $l_{n,p}$-balls and the ellipsoid number function $(n, a) \mapsto \pi_n^E(a)$ is the restriction of $\pi_n^{\mathfrak{S}}$ to the set $\mathfrak{T}_2$ of all exes aligned ellipsoids.
We recall that the notion of the $\mathfrak{O}_p$-surface content of an $l_{n,p}$-ball of radius r is defined by

$$O_p(r) = 2\int_{|x|_p<r} |N(x)|_q dx$$

where in the case $p \geq 1$,

$$\begin{cases} |.|_p & \text{is a norm} \\ \text{and} & \\ |.|_q & \text{is the norm dual to } |.|_p : q = \frac{p}{p-1} \in [1, \infty]. \end{cases}$$

Instead of assigning a ball number to each convex $l_{n,p}$-ball, we could think of $\pi_n^{\mathfrak{S}}$ as a function which assigns in this case a ball number to each $l_{n,p}$-norm.
It could be then a reasonably restricted problem to consider ball numbers for all norms.
Let us recall the well known fact there is a biunique correspondence between the norms in $\mathbb{R}^n$ and the convex bodies symmetric w.r.t. 0. This correspondence is extended in [4] over functionals of another general type.
It follows as a special result from [4] that in the case $p \in (0, 1)$,

$$\begin{cases} |.|_p & \text{is an anti-norm} \\ \text{and} & \\ |.|_q & \text{is a semi-anti-norm } : q = \frac{p}{p-1} \in (-\infty, 0). \end{cases}$$

Instead of assigning a ball number to each non-convex $l_{n,p}$-ball, we could think of $\pi_n^{\mathfrak{S}}$ as a function which, in the present case, assigns a ball number to each $l_{n,p}$-anti-norm. Hence, it could be then another reasonably restricted problem to consider ball numbers for all anti-norms.

For suggesting a very first idea of the notions of an anti-norm and a semi-anti-norm, Figure 4 shows the $l_{2,q}$-unit circles (and balls) of norms, $q \in [1, \infty]$, anti-norms, $q \in (0, 1)$, and semi-anti-norms, $q \in [-\infty, 0)$.
For the convenience of the reader, we summarize shortly the basic notions needed to define those of anti-norm and semi-anti-norm which were introduced in [4].
Any closed convex cone $C \subset \mathbb{R}^n$ containing no half-space, with vertex 0 and non-empty interior, will be called a sector of $\mathbb{R}^n$. A finite collection $\mathfrak{C}$ of sectors in $\mathbb{R}^n$ will be called a fan if its members have pairwise disjoint interiors and their union is $\mathbb{R}^n$. A function $g : \mathbb{R}^n \to R_+$ is positively homogeneous iff for every $t \in \mathbb{R}$ and $x \in \mathbb{R}$: $g(tx) = |t|g(x)$. In what follows, the

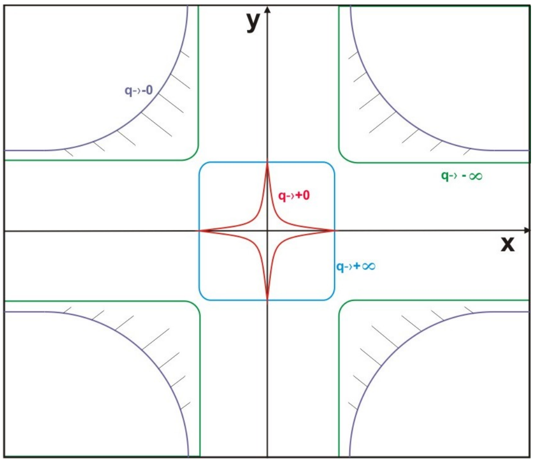

Figure 4: $l_{2,q}$-unit circles

generalized ball $B_g := \{x \in \mathbb{R}^n : g(x) \leq 1\}$ is assumed to be star-shaped, with positive and continuous radial function, and symmetric w.r.t. 0. We call B_g radially concave w.r.t. a sector C whenever $C \setminus B_g$ is convex. Further, B_g is called radially concave w.r.t. a fan if it is radially concave w.r.t. every sector in this fan. This property of a generalized ball is closely connected with the following property of the function g generating this ball. A function $g : \mathbb{R}^n \to \mathbb{R}_+$ is superadditive in a sector C if $g(x+y) \geq g(x)+g(y)$ for every $x, y \in C$. The function g is superadditive in a fan $\mathfrak{C}$ if it is superadditive in every sector of the fan $\mathfrak{C}$. A function $g : \mathbb{R}^n \to R_+$ will be called a semi-anti-norm provided that g is continuous, positively homogeneous and superadditive in some fan. A function $g : \mathbb{R}^n \to R_+$ is an anti-norm if g is a semi-anti-norm which is non-degenerate.

Acknowledgement The author would like to express his gratitude to the Referee for giving several constructive hints the realization of which made the paper more pleasant to read.

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