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Ellipses numbers and geometric measure representations

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Abstract. Ellipses will be considered as subsets of suitably defined Minkowski planes in such a way that, additionally to the well known area content property $A(r) = \pi_{(a,b)}r^2$, the number $\pi_{(a,b)} = ab\pi$ reflects a generalized circumference property $U_{(a,b)}(r) = 2\pi_{(a,b)}r$ of the ellipses $E_{(a,b)}(r)$ with main axes of lengths $2ra$ and $2rb$, respectively. In this sense, the number $\pi_{(a,b)}$ is an ellipse number w.r.t. the Minkowski functional r of the reference set $E_{(a,b)}(1)$. This approach is closely connected with a generalization of the method of indivisibles and avoids elliptical integrals. Further, several properties of both a generalized arc-length measure and the ellipses numbers will be discussed, e.g. disintegration of the Lebesgue measure and an elliptically contoured Gaussian measure indivisiblen representation, wherein the ellipses numbers occur in a natural way as norming constants.

Keywords. Ellipse number, generalized arc-length, generalized perimeter, generalized method of indivisibles, isoperimetric constant, Minkowski plane, geometric measure representation, intersection-percentage function, generalized uniform distribution on the ellipse, generalized trigonometric functions, generalized elliptical coordinates, disintegration of Lebesgue measure, elliptically contoured Gaussian measure representation, stochastic representation.

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1 Introduction

The Archimedes or Ludolf number π was recently generalized in [3] for $l_{2,p}$-circles C_p with $p \geq 1$ by certain p-circle numbers $\pi(p)$. The main principle behind the construction in [3] is a certain generalization of the method of indivisibles of Cavalieri and Torricelli which was developed in [2]. Thereby, the indivisibles are the concentric $l_{2,p}$-circles $r \cdot C_p, 0 \leq r \leq 1$ and measuring their arc-lengths is based upon the geometry induced by the 'dual' circle C_q with q satisfying the equation $\frac{1}{p} + \frac{1}{q} = 1$. Unless for $p = 2$, this approach differs from that of measuring the length of C_p as usual w.r.t. Euclidean geometry or based upon the geometry generated by C_p itself as it was favored by several authors when discussing the possibility of generalizing π.
It follows from the consideration in [3] that if one wants to generalize the circle

number π for $l_{2,p}$-circles with $p \geq 1$ then one has to replace at least two elements of the triple (area-content, circumference, diameter) with suitably defined quantities. The author's motivation to let the area content be unchanged the usual Euclidean one comes from probabilistic applications where certain probability distributions are absolutely continuous w.r.t. the Lebesgue measure or Euclidean area content. The process of generalizing the circle number π is accompanied in [3] by the following aspects.
-) As mentioned already, $\pi(p)$ has a suitably defined generalized circumference property and the usual, i.e. Euclidean, area-content property.
-) The p-circle C_p is the solution to the q-isoperimetric problem and will actually be measured w.r.t. the geometry generated by C_q.
-) The norming constants of certain density generating functions are identified to be generalized circle numbers.
-) The circle numbers $\pi(p)$ can also be defined for non-convex $l_{2,p}$-circles and may be even generalized for the multidimensional $l_{n,p}$-balls.

The notion of the generalized circumference
-) allows to derive a specific disintegration formula for the Lebesgue measure in two dimensions,
-) allows to define a generalization of the method of indivisibles,
-) allows to explain the p-generalized uniform distribution on C_p,
-) allows to formulate a so called thin-layers-property,
-) coincides with that of the mixed area content, i.e., it gives an additional explanation for the notion of mixed volume in two dimensions
and
-) can be considered from a local point of view as a derivative and from a global point of view as an integral.

In the present paper, we adopt a similar strategy for ellipses. An additional specific aspect here is that the notion of the generalized circumference makes the notion of an elliptical integral superfluous for our purposes.

We consider the ellipses $E_{(a,b)}(r) = rE_{(a,b)}, r > 0$ with

$$E_{(a,b)} = \{(x, y) \in \mathbb{R}^2 : ||(x, y)||_{(a,b)} = 1\}, \quad 0 < b \leq 1 \leq a$$

and the discs $K_{(a,b)}(r)$ inside $E_{(a,b)}(r)$, where

$$||(x, y)||_{(a,b)} = \sqrt{(\frac{x}{a})^2 + (\frac{y}{b})^2}.$$

Recognize that at any point (x, y) from $E_{(a,b)}(r)$ the value of the Minkowski functional w.r.t. the set $E_{(a,b)}$ equals r.

Let a parameter representation of the ellipse $E_{(a,b)}(r)$ be given by

$$x = ar\cos\varphi, \quad y = br\sin\varphi, \quad 0 \le \varphi < 2\pi. \tag{PD1}$$

The Euclidean circumference of $E_{(a,b)}(r)$ can be written then as

$$U(r) = 4\int_0^{\pi/2} \sqrt{x'^2(\varphi) + y'^2(\varphi)}d\varphi = 4ar\int_0^{\pi/2} \sqrt{1-\varepsilon^2\cos^2(\varphi)}d\varphi,$$
$$\varepsilon^2 = 1 - (\frac{b}{a})^2 \in [0,1).$$

With $\cos\varphi = \sin(\frac{\pi}{2} - \varphi), \psi = \frac{\pi}{2} - \varphi$, it follows that $U(r) = 4arE(\varepsilon)$, where

$$E(\varepsilon) = \int_0^{\pi/2} \sqrt{1-\varepsilon^2\sin^2\psi}\, d\psi$$

denotes the elliptical integral of second type which cannot be evaluated explicitly.

The area content of the disc $K_{(a,b)}(r)$ is

$$A(r) = ab\pi r^2.$$

Hence, integrating Euclidean circumferences yields area content,

$$A(r) = \int_0^r U(r')dr'\,, \tag{$\circ$}$$

if and only if

$$2E(\varepsilon) = b\pi. \tag{$\times$}$$

If $a = b = 1$ then $\varepsilon = 0$ and with $E(0) = \frac{\pi}{2}$ it follows ($\times$).
If $a = 1$ and $b \to +0$ then the quantity on the right hand side of ($\times$) tends to 0 and that on the left hand side tends to $2E(1) = 2$. Hence, relation ($\circ$) does not hold in general and no method of indivisibles applies in the sense that the indivisibles are the ellipses $E_{(a,b)}(r'), 0 \le r' < r$ and integrating their Euclidean lengths gives the area content of the disc $K_{(a,b)}(r)$.
Therefore, the principle of constructing ellipses numbers applied in the present paper will essentially make use of a generalized method of indivisibles. This method will be based upon a suitably defined arc-length measure which will be introduced

in Section 2 using integration. In Section 3, we introduce an arc-length measure in quite another way by taking the derivative of the area content of an ellipse-sector w.r.t. the Minkowski functional r. This local approach will be shown then to lead to the same result as the global one from Section 2. Finally, Section 4 deals with geometric measure representations making use of the arc-length measure considered in Sections 2 and 3.

2 The ellipses numbers and their basic properties

We consider the Minkowski space $(\mathbb{R}^2, d_{(1/b,1/a)})$ where the metric $d_{(1/b,1/a)}|\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$is given by

$$d_{(1/b,1/a)}((x,y),(s,t)) = (b^2(x-s)^2 + a^2(y-t)^2)^{1/2}.$$

The corresponding norm in $\mathbb{R}^2$ is $||(x,y)||_{(1/b,1/a)} = d_{(1/b,1/a)}((x,y),(0,0))$ and the unit ball w.r.t. this norm is the disc $K_{(1/b,1/a)}(1) = \{(x,y) : b^2x^2 + a^2y^2 \leq 1\}$. Applying Brunn-Minkowski Theory and results for mixed areas, Buseman [1] proved a theorem from which it follows that the ellipse $E_{(a,b)}$ solves the $d_{(1/b,1/a)}$-isoperimetric problem, i.e. among all curves having $d_{(1/b,1/a)}$-length L, $L = 2\pi ab$, the ellipse $E_{(a,b)}$ includes the area having maximum content, πab. We consider now the ellipse $E_{(a,b)}(r)$ as a subset of the Minkowski plane $(\mathbb{R}^2, ||.||_{(\frac{1}{b},\frac{1}{a})})$ and define a generalized arc-length in the following way. Let $(x_i, y_i), i = 0, 1, 2, ..., n$ be an arbitrary successive partition of the ellipse and put $(x_0, y_0) = (1,0)$, $(x_n, y_n) = (1,0)$, $F(\mathfrak{Z}) = \sup\limits_i ||(x_i,y_i) - (x_{i-1},y_{i-1})||_{(\frac{1}{b},\frac{1}{a})}$ and

$$S(\mathfrak{Z}) = \sum_{i=1}^{n} ||(x_i,y_i) - (x_{i-1},y_{i-1})||_{(\frac{1}{b},\frac{1}{a})}.$$

Because of the triangle inequality

$$||(x_{i+2},y_{i+2}) - (x_i,y_i)||_{(\frac{1}{b},\frac{1}{a})} \leq$$
$$||(x_{i+2},y_{i+2}) - (x_{i+1},y_{i+1})||_{(\frac{1}{b},\frac{1}{a})} + ||(x_{i+1},y_{i+1}) - (x_i,y_i)||_{(\frac{1}{b},\frac{1}{a})},$$

the sequence of sums $(S(\mathfrak{Z}_n))_{n=1,2,...}$ is increasing as a successive sequence of partitions $(\mathfrak{Z}_n)_{n=1,2,...}$ satisfies $\lim\limits_{n\to\infty} F(\mathfrak{Z}_n) = 0$. The sequence $(S(\mathfrak{Z}_n))_{n=1,2,...}$ is bounded above and hence convergent. The limit does neither depend on the parameter representation of $E_{(a,b)}(r)$ nor on the sequence $(\mathfrak{Z}_n)_{n=1,2,...}$.

Definition 2.1. (a) The limit of the sequence $(S(\mathfrak{Z}_n))_{n=1,2,...}$ will be called the $||.||_{(\frac{1}{b},\frac{1}{a})}$-arc-length of $E_{(a,b)}(r)$ and denoted by $U_{(a,b)}(r)$:

$$U_{(a,b)}(r) = \lim_{F(\mathfrak{Z}_n)\to+0} \sum_{i=1}^{n} ||(x_i, y_i) - (x_{i-1}, y_{i-1})||_{(\frac{1}{b},\frac{1}{a})}.$$

(b) For $\{(x(\varphi), y(\varphi)), \varphi \in [0, 2\pi)\}$ being an arbitrary parameter representation of $E_{(a,b)}$, we write

$$U_{(a,b)}(r) = \int_0^{2\pi} ||(x'(\varphi), y'(\varphi))||_{(\frac{1}{b},\frac{1}{a})} d\varphi, \qquad (*)$$

and using the representation $x \to y(x)$ of the upper half-ellipse

$$E^+_{(a,b)}(r) = \{(x,\ (r^2 - (\frac{x}{a})^2)^{1/2} b\ ), -r \le x \le r\},$$

we write alternatively

$$U_{(a,b)}(r) = 2\int_{-r}^{r} ||(1, y'(x))||_{(\frac{1}{b},\frac{1}{a})} dx. \qquad (**)$$

Notice that the notations $(*)$ and $(**)$ are well motivated by the relations

$$U_{(a,b)}(r) = \lim_{F(\mathfrak{Z}_n)\to+0} \sum_{i=1}^{n} \frac{(|b\Delta x_i|^2 + |a\Delta y_i|^2)^{1/2}}{\Delta\varphi_i} \Delta\varphi_i$$

and

$$U_{(a,b)}(r) = \lim_{F(\mathfrak{Z}_n)\to+0} \sum_{i=1}^{n} (|b|^2 + \frac{|a\Delta y_i|^2}{|\Delta x_i|^2})^{1/2} \Delta x_i.$$

On the one hand side, the following lemma can be understood as an immediate consequence of solving the q-isoperimetric problem. For an easier understanding of which technical role plays the chosen metric, however, we present this lemma together with a short proof.

Lemma 2.2. *For all* $r > 0$, $U_{(a,b)}(r) = 2ab\pi r$.

Proof. By definition and symmetry,

$$U_{(a,b)}(r) = 4\int_0^{\pi/2} (b^2 x'^2(\varphi) + a^2 y'^2(\varphi))^{1/2} d\varphi.$$

On choosing (PD1) as a parameter representation of $E_{(a,b)}(r)$, we get the result

$$U_{(a,b)}(r) = 4\int\limits_0^{\pi/2} (b^2a^2r^2(\sin^2\varphi + \cos^2\varphi))^{1/2}d\varphi = 4abr\frac{\pi}{2}$$

which in fact does not depend on the actually chosen parameter representation. □

Corollary 2.3. *The Lebesgue measure λ satisfies the following (first) disintegration formula:*

$$\int\limits_0^r U_{(a,b)}(r')dr' = \lambda(K_{(a,b)}(r)), \quad r > 0.$$

This formula may be considered as reflecting a generalization of the method of indivisibles of Cavalieri and Torricelli in the sense that the indivisibles of the disc $K_{(a,b)}(r)$ are the ellipses $E_{(a,b)}(r'), 0 < r' < r$ and measuring the latter is based upon the geometry of the Minkowski plane $(\mathbb{R}^2, ||.||_{(\frac{1}{b},\frac{1}{a})})$.
Summarizing what is already known, we have, for all $r > 0$,

$$\frac{A(r)}{r^2} \overset{(1)}{=} a\,b\,\pi \overset{(2)}{=} \frac{U_{(a,b)}(r)}{2\,r}, \quad r > 0.$$

Definition 2.4. (a) The properties of the ellipses $E_{(a,b)}(r)$ which are expressed by the relations (1) and (2) are called the area-content and generalized circumference-properties of the ellipses.
(b) For fixed a, b, the quantity $a\,b\,\pi =: \pi_{(a,b)}$ will be called ellipse number.

3 Properties of the generalized arc-length measure

While we have considered yet a generalized arc-length from a certain global point of view as an integral, we will consider it now from a certain local point of view as a derivative. It follows immediately from the consideration in Section 2 that

$$\frac{d}{d\varrho}A(\varrho) = 2\pi_{(a,b)}\varrho = U_{(a,b)}(\varrho), \quad \varrho > 0. \qquad (\heartsuit)$$

The following definition is therefore well motivated. To this end, let $\mathfrak{A}_{(a,b)}$ denote the σ-field consisting of all measurable subsets of $E_{(a,b)}$. Further, for arbitrary $A \in \mathfrak{A}_{(a,b)}$, let

$$CPC_{(a,b)}(A) = \{(x,y) \in \mathbb{R}^2 : \frac{(x,y)}{||(x,y)||_{(a,b)}} \in A\}$$

denote the central projection cone induced by the set A and

$$sector_{(a,b)}(A, \varrho) = CPC_{(a,b)}(A) \cap K_{(a,b)}(\varrho)$$

the ellipse-sector induced by the set A.

Definition 3.1. The measure $\mathfrak{U}_{(a,b)}|\mathfrak{A}_{(a,b)} \to \mathbb{R}^+$ defined by

$$\mathfrak{U}_{(a,b)}(A) = f'(1) \quad \text{with} \quad f(\varrho) = \lambda(sector_{(a,b)}(A, \varrho))$$

is called the $E_{(a,b)}$-generalized circumference- or arc-length measure.

For a further description of this measure we shall make use of suitably defined generalizations of trigonometric functions. Defining them, we adopt the basic ideas developed in [2] to the present situation.

Definition 3.2. The $E_{(a,b)}$-generalized sine- and cosine-values of the angle φ between the positive x-axis and the ray starting from the origin $(0, 0)$ and passing through the point (x, y) are defined as

$$\cos_{(a,b)}(\varphi) = \frac{d_{(a,b)}((0,0),(x,0))}{d_{(a,b)}((0,0),(x,y))} \quad \text{and} \quad \sin_{(a,b)}(\varphi) = \frac{d_{(a,b)}((0,0),(0,y))}{d_{(a,b)}((0,0),(x,y))},$$

respectively.

Notice that

$$\cos_{(a,b)}(\varphi) = \frac{x}{(x^2 + (ay/b)^2)^{1/2}} \quad \text{and} \quad \sin_{(a,b)}(\varphi) = \frac{y}{((bx/a)^2 + y^2)^{1/2}}.$$

Let us be given an angle $\varphi \in [0, \pi/2)$, then there exists a uniquely determined point $(x, y) \in E_{(a,b)}$ such that $y/x = \tan\varphi$. Because of

$$y = x \tan\varphi \quad \text{and} \quad ||(x, x\tan\varphi)||_{(a,b)} = 1$$

it follows

$$x^2 = \frac{1}{\frac{1}{a^2} + \frac{\tan^2\varphi}{b^2}} \quad \text{and} \quad y^2 = \frac{\tan^2\varphi}{\frac{1}{a^2} + \frac{\tan^2\varphi}{b^2}}.$$

Hence, the $E_{(a,b)}$-generalized sine and cosine functions may be represented as

$$\cos_{(a,b)}(\varphi) = \frac{b\cos\varphi}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/2}}$$

and

$$\sin_{(a,b)}(\varphi) = \frac{a \sin\varphi}{(a^2 \sin^2\varphi + b^2 \cos^2\varphi)^{1/2}}.$$

Alternatively,

$$\cos_{(a,b)}(\varphi) = \frac{\cos\varphi}{aN_{(a,b)}(\varphi)} \quad \text{and} \quad \sin_{(a,b)}(\varphi) = \frac{\sin\varphi}{bN_{(a,b)}(\varphi)}$$

where

$$N_{(a,b)}(\varphi) = ||(\cos\varphi, \sin\varphi)||_{(a,b)}.$$

Definition 3.3. The $E_{(a,b)}$-generalized elliptical polar-coordinate transformation

$$Pol_{(a,b)}|[0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$$

is defined by

$$x = ar\cos_{(a,b)}(\varphi) \quad \text{and} \quad y = br\sin_{(a,b)}(\varphi). \tag{PD2}$$

Let us denote the orthants in $\mathbb{R}^2$ in the usual anticlockwise ordering by Q_1 up to Q_4.

Theorem 3.4. *The map $Pol_{(a,b)}$ is almost one-to-one, its inverse $Pol_{(a,b)}^{-1}$ is given by*

$$r = ||(x,y)||_{(a,b)}, \ \text{arctan}\,|\frac{y}{x}| = \varphi \textit{ in } Q_1, \pi - \varphi \textit{ in } Q_2, \pi + \varphi \textit{ in } Q_3, 2\pi - \varphi \textit{ in } Q_4$$

and its Jacobian satisfies the representation

$$J(r,\varphi) = |\frac{d(x,y)}{d(r,\varphi)}| = \frac{r}{N^2_{(a,b)}(\varphi)}.$$

Proof. We start with checking the generalized Pythagoras theorem

$$||(x,y)||^2_{(a,b)} = (\frac{x}{a})^2 + (\frac{y}{b})^2 = r^2[\frac{\cos^2\varphi}{a^2N^2(\varphi)} + \frac{\sin^2\varphi}{b^2N^2(\varphi)}] \equiv r^2,$$
$$N(\varphi) := N_{(a,b)}(\varphi),$$

the relations for $\arctan|\frac{y}{x}|$ being as in the case of usual polar coordinates. For calculating now the Jacobian, define

$$x' := \frac{\partial}{\partial\varphi}x, \quad \text{and} \quad x_r := \frac{\partial}{\partial r}x.$$

From

$$x' = r(\frac{\cos\varphi}{N(\varphi)})' = r\frac{-\sin\varphi N(\varphi) - \cos\varphi N'(\varphi)}{N^2(\varphi)} = -y - xl, \quad l = (\ln N(\varphi))'$$

and

$$y' = r(\frac{\sin\varphi}{N(\varphi)})' = r\frac{\cos\varphi N(\varphi) - \sin\varphi N'(\varphi)}{N^2(\varphi)} = x - yl; \quad x_r = \frac{x}{r}, \quad y_r = \frac{y}{r}$$

it follows that

$$J(r,\varphi) = |\begin{vmatrix} -y - xl & \frac{x}{r} \\ x - yl & \frac{y}{r} \end{vmatrix}| = |-\frac{x^2+y^2}{r}|$$

$$= \frac{r^2}{r}(\cos^2\varphi + \sin^2\varphi)\frac{1}{N^2(\varphi)} = \frac{r}{N^2(\varphi)} \qquad \square$$

Let us denote the restriction of the map $Pol_{(a,b)}$ to the case $r = 1$ by $Pol^*_{(a,b)}(\varphi) = Pol_{(a,b)}(1,\varphi)$ and its inverse by $Pol^{*-1}_{(a,b)}|\mathfrak{A}_{(a,b)} \to \mathfrak{B}([0,2\pi))$.

Theorem 3.5. *The $E_{(a,b)}$-generalized arc-length measure $\mathfrak{U}_{(a,b)}$ satisfies for all $A \in \mathfrak{A}_{(a,b)}$ the equations*

$$\mathfrak{U}_{(a,b)}(A) = \int\limits_{Pol^{*-1}_{(a,b)}(A)} \frac{d\varphi}{N^2_{(a,b)}(\varphi)} \tag{a}$$

and

$$\mathfrak{U}_{(a,b)}(A) = 2\lambda(sector_{(a,b)}(A,1)). \tag{b}$$

Notice that equation (b) is much more general than the equation $A(1) = \frac{1}{2}U_{(a,b)}(1)$ from Section 2.

Proof. Changing first Cartesian with $E_{(a,b)}$-generalized elliptical polar coordinates and changing then the order of integration, we get

$$\lambda(sector_{(a,b)}(A,\varrho)) = \int\limits_{sector_{(a,b)}(A,\varrho)} d(x,y) = \int\limits_{r=0}^{\varrho} \int\limits_{\varphi\in Pol^{*-1}_{(a,b)}(A)} \frac{r}{N^2_{(a,b)}(\varphi)} d(\varphi,\, r)$$

$$= \int\limits_0^{\varrho} r dr \int\limits_{Pol^{*-1}_{(a,b)}(A)} \frac{d\varphi}{N^2_{(a,b)}(\varphi)}.$$

Taking the derivative w.r.t. ϱ yields (a), for $\varrho = 1$, and (b) follows then immediately. $\square$

The next theorem states the equivalence of the global and local approaches to the generalized arc-length measure considered in this paper. To this end, let us assume w.l.g. that A belongs to the upper half-ellipse, $A \in \mathfrak{A}^+_{(a,b)}$, say.

Theorem 3.6. *The $||.||_{(\frac{1}{b},\frac{1}{a})}$-arc-length of a set $A \in \mathfrak{A}^+_{(a,b)}$ coincides with the $E_{(a,b)}$-generalized arc-length of A, i.e.*

$$AL_{(\frac{1}{b},\frac{1}{a})}(A) = \int\limits_{G(A)} ||(1, \varphi'(x))||_{(\frac{1}{b},\frac{1}{a})} dx = \mathfrak{U}_{(a,b)}(A), \quad A \in \mathfrak{A}^+_{(a,b)}$$

with

$$G(A) = \{x \in [-1,1] : (x, \varphi(x)) \in A\} \quad \textit{and} \quad \varphi(x) = b\sqrt{1 - (\frac{x}{a})^2}\,.$$

Proof. It follows from the definition of φ that

$$\varphi'(x) = -\frac{bx/a^2}{\sqrt{1-(x/a)^2}}$$

and

$$||(1,\varphi'(x))||_{(\frac{1}{b},\frac{1}{a})} = (b^2 + a^2\frac{b^2x^2}{a^4(1-(x/a)^2)})^{1/2} = b(1+\frac{x^2}{a^2-x^2})^{1/2}$$
$$= \frac{ab}{\sqrt{a^2-x^2}}.$$

Hence,

$$AL_{(\frac{1}{b},\frac{1}{a})}(A) = ab\int\limits_{G(A)} \frac{dx}{\sqrt{a^2-x^2}}.$$

Changing variables $x = \frac{\cos\varphi}{N(\varphi)} = a\cos_{(a,b)}(\varphi)$ with $N(\varphi) = N_{(a,b)}(\varphi)$ gives

$$1/\sqrt{a^2-x^2} = 1/\sqrt{a^2 - \cos^2\varphi/N^2(\varphi)} = \sqrt{N^2(\varphi)/(a^2N^2(\varphi) - \cos^2\varphi)}$$
$$= \sqrt{\frac{\cos^2\varphi/a^2 + \sin^2\varphi/b^2}{\cos^2\varphi + \frac{a^2}{b^2}\sin^2\varphi - \cos^2\varphi}} = \frac{b}{a}\frac{N(\varphi)}{\sin\varphi}.$$

With the notation $\frac{dx}{d\varphi} = x'(\varphi)$, it follows $dx = x'(\varphi)d\varphi$ where

$$x'(\varphi) = \frac{-\sin\varphi N(\varphi) - \cos\varphi N'(\varphi)}{N^2(\varphi)}$$

and

$$N'(\varphi) = \frac{1}{2}(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2})^{-1/2}[\frac{-2\cos\varphi\sin\varphi}{a^2} + \frac{2\sin\varphi\cos\varphi}{b^2}]$$
$$= \frac{1}{N(\varphi)}(-\cos\varphi\sin\varphi/a^2 + \cos\varphi\sin\varphi/b^2).$$

Hence,

$$x'(\varphi) = \frac{-\sin\varphi(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}) - \cos\varphi(\frac{-\cos\varphi\sin\varphi}{a^2} + \frac{\cos\varphi\sin\varphi}{b^2})}{N^3(\varphi)},$$

such that

$$x'(\varphi)N^3(\varphi) = -\frac{\sin\varphi}{b^2}[\sin^2\varphi + \cos^2\varphi] \quad \text{and} \quad x'(\varphi) = -\frac{\sin\varphi}{b^2N^3(\varphi)}.$$

Recognize that in the range of integration x increases when φ decreases. It follows that

$$AL_{(\frac{1}{b},\frac{1}{a})}(A) = ab \int\limits_{Pol^{*-1}_{(a,b)}(A)} \frac{bN(\varphi)}{a\sin\varphi}\frac{\sin\varphi}{b^2N^3(\varphi)}d\varphi$$
$$= \int\limits_{Pol^{*-1}_{(a,b)}(A)} \frac{d\varphi}{N^2(\varphi)} = \mathfrak{U}_{(a,b)}(A).$$

□

Remark 3.7. The derivatives of the generalized trigonometric functions $\sin_{(a,b)}$ and $\cos_{(a,b)}$ which have been used in the proof of Theorem 3.6 satisfy the equations

$$\cos'_{(a,b)}(\varphi) = -\frac{\sin_{(a,b)}(\varphi)}{\frac{b}{a}\cos^2\varphi + \frac{a}{b}\sin^2\varphi} \quad \text{and} \quad \sin'_{(a,b)}(\varphi) = \frac{\cos_{(a,b)}(\varphi)}{\frac{b}{a}\cos^2\varphi + \frac{a}{b}\sin^2\varphi}.$$

Proof. It follows from the proof of Theorem 3.6 that

$$(a\cos_{(a,b)}(\varphi))' = -\frac{\sin\varphi}{bN(\varphi)} \cdot \frac{1}{bN^2(\varphi)}.$$

The first equation in the remark follows now from the definition of $\sin_{(a,b)}$ and

$$abN^2(\varphi) = \frac{b}{a}\cos^2\varphi + \frac{a}{b}\sin^2\varphi.$$

The second equation in the remark follows analogously □

4 Geometric measure representations

Definition 4.1. The normalized generalized arc-length measure defined by

$$\omega_{(a,b)}(A) = \mathfrak{U}_{(a,b)}(A)/\mathfrak{U}_{(a,b)}(E_{(a,b)})$$

is called the $E_{(a,b)}$-generalized uniform probability distribution on $\mathfrak{A}_{(a,b)}$.

The following representation formula is a consequence of Theorem 3.5 and (2).

Corollary 4.2. $\omega_{(a,b)}(A) = \frac{1}{2\pi_{(a,b)}} \int\limits_{Pol^{*-1}_{(a,b)}(A)} \frac{d\varphi}{N^2_{(a,b)}(\varphi)}, A \in \mathfrak{A}_{(a,b)}$.

Remark 4.3. The $E_{(a,b)}$-generalized uniform probability distribution on $\mathfrak{A}_{(a,b)}$ satisfies also the representation

$$\omega_{(a,b)}(A) = \frac{\lambda(sector_{(a,b)}(A,1))}{\lambda(K_{(a,b)})}$$

with $\lambda(K_{(a,b)}) = \lambda(K_{(a,b)}(1)) = A(1) = \pi_{(a,b)}$.

Example 4.4. (a) Let a random vector (ξ, η) follow the elliptically contoured Gaussian density function

$$\varphi_{(a,b)}(x,y) = \frac{1}{2\pi_{(a,b)}} \exp\{-\frac{1}{2}||(x,y)||^2_{(a,b)}\}, \quad (x,y) \in \mathbb{R}^2, \quad a > 0, \quad b > 0.$$

The normalized vector $(X,Y) := (\xi,\eta)/||(\xi,\eta)||_{(a,b)}$ takes its values on $E_{(a,b)}$. Its distribution can be represented as

$$\begin{aligned} P((X,Y) \in A) &= P((\xi,\eta) \in CPC_{(a,b)}(A)) \\ &= \int\limits_{CPC_{(a,b)}(A)} \varphi_{(a,b)}(x,y)d(x,y), \quad A \in \mathfrak{A}_{(a,b)}. \end{aligned}$$

Because $Pol^{-1}_{(a,b)}(CPC_{(a,b)}(A))$ may be written as $[0,\infty)\times\tilde{A}$ with $\tilde{A} = Pol^{*-1}_{(a,b)}(A)$, it follows

$$\begin{aligned} P((X,Y) \in A) &= \frac{1}{2\pi_{(a,b)}} \int\limits_{r=0}^{\infty} \int\limits_{\tilde{A}} r \exp\{-\frac{1}{2}r^2\} \frac{d(r,\varphi)}{N^2_{(a,b)}(\varphi)} \\ &= \frac{1}{2\pi_{(a,b)}} \int\limits_{r=0}^{\infty} re^{-\frac{1}{2}r^2} dr \int\limits_{Pol^{*-1}_{(a,b)}(A)} \frac{d\varphi}{N^2_{(a,b)}(\varphi)}. \end{aligned}$$

Hence, (X,Y) follows the $E_{(a,b)}$-generalized uniform distribution, i.e., $(X,Y) \sim \omega_{(a,b)}$.
(b) Assume that a random vector (ξ,η) follows the uniform distribution on $K_{(a,b)}$, i.e., (ξ,η) has the probability density function

$$f(x,y) = \frac{1}{\pi_{(a,b)}} I_{K_{(a,b)}}(x,y), \quad (x,y) \in R^2$$

where

$$I_M(x) = \begin{cases} 1 & \text{if} \quad x \in M, \\ 0 & \text{otherwise} \end{cases}$$

denotes the indicator function of the set M. The normalized vector $(X,Y) := (\xi,\eta)/||(\xi,\eta)||_{(a,b)}$ satisfies the equation $||(X,Y)||_{(a,b)} = 1$ and, for all $A \in \mathfrak{A}_{(a,b)}$, its distribution allows the representation

$$P((X,Y) \in A) = P((\xi,\eta) \in sector_{(a,b)}(A,1)) = \frac{1}{\pi_{(a,b)}} \int\limits_{sector_{(a,b)}(A,1)} d(x,y)$$

$$= \frac{1}{\pi_{(a,b)}} \int\limits_{r=0}^{1} \int\limits_{\varphi \in Pol^{*-1}_{(a,b)}(A)} \frac{r d(r,\varphi)}{N^2_{(a,b)}(\varphi)}$$

$$= \frac{1}{2\pi_{(a,b)}} \int\limits_{Pol^{*-1}_{(a,b)}(A)} \frac{d\varphi}{N^2_{(a,b)}(\varphi)}.$$

Hence, $(X,Y) \sim \omega_{(a,b)}$.

Theorem 4.5. *For arbitrary fixed $a > 0, b > 0$, the Lebesgue measure λ in $\mathbb{R}^2$ satisfies the following (second) disintegration formula:*

$$\lambda(M) = \int\limits_0^\infty \mathfrak{U}_{(a,b)}(M \cap E_{(a,b)}(r))\, dr, \quad M \in \mathfrak{B}^2.$$

Notice that this formula is much more general than that in Corollary 2.3. Its proof makes use of the standard technique coming from the measure extension theorem and follows the proofs of similar statements in [2] and [3]. It will therefore be omitted, here.
Let us recall that the $l_{2,p}$-sphere intersection-percentage-function (i.p.f.) plays an important role in dealing with $l_{2,p}$-symmetric distributions. Something similar will be considered now in the present situation.

Definition 4.6. The $E_{(a,b)}$-based i.p.f. $\mathfrak{F}_{(a,b)}(M,.)$ of an arbitrary set $M \in \mathfrak{B}^2$ is defined as

$$r \to \frac{\mathfrak{U}_{(a,b)}(M \cap E_{(a,b)}(r))}{\mathfrak{U}_{(a,b)}(E_{(a,b)}(r))} =: \mathfrak{F}_{(a,b)}(M,r).$$

Remark 4.7. Based upon the notion of the i.p.f., the equation in Theorem 4.5 may be reformulated as

$$\lambda(M) = 2\pi_{(a,b)} \int_0^\infty \mathfrak{F}_{(a,b)}(M,r)\, r\, dr, \quad M \in \mathfrak{B}^2.$$

Remark 4.8. The thin-layers property mentioned in the Introduction is

$$\lambda(\{(x,y) \in \mathbb{R}^2 : r < ||(x,y)||_{(a,b)} < r+\varepsilon\}) \sim 2\pi_{(a,b)} r\varepsilon, \quad \varepsilon \to 0.$$

Proof. The $E_{(a,b)}$-based i.p.f. $\varrho \to \mathfrak{F}_{(a,b)}(M,\varrho)$ of the set

$$M = \{(x,y) \in \mathbb{R}^2 : r < ||(x,y)||_{(a,b)} < r+\varepsilon\}$$

satisfies the representation

$$\mathfrak{F}_{(a,b)}(M,\varrho) = I_{(r,r+\varepsilon)}(\varrho), \quad \varrho > 0.$$

By Remark 4.7,

$$\lambda(M) = 2\pi_{(a,b)} \int_r^{r+\varepsilon} \varrho d\varrho = 2\pi_{(a,b)}(r\varepsilon + \frac{\varepsilon^2}{2}). \qquad \square$$

The proof of the following geometric measure representation for the elliptically contoured Gaussian law follows the proofs of analogous results in [2] and [3] and will therefore also be omitted, here.

Theorem 4.9. *If* $(\xi,\eta) \sim \varphi_{(a,b)}$ *then*

$$P((\xi,\eta) \in A) = \int_0^\infty \mathfrak{F}_{(a,b)}(A,r) r e^{-\frac{r^2}{2}} dr.$$

Remark 4.10. If the random vector (ξ,η) follows the elliptically contoured Gaussian density, $(\xi,\eta) \sim \varphi_{(a,b)}$, then it allows the stochastic representation

$$(\xi,\eta) \stackrel{d}{=} R \cdot (X,Y)$$

where (X,Y) follows the $E_{(a,b)}$-generalized uniform distribution, $(X,Y) \sim \omega_{(a,b)}$, and R is a nonnegative random variable.

Corollary 4.11. *(a) The random variable R follows the Chi-distribution with two d.f., $R \sim \chi_2$.*
(b) The random elements R and (X, Y) are independent.

Proof. (a) Let $x > 0$, then

$$P(R < x) = P(R < x, (X, Y) \in E_{(a,b)}) = P((\xi, \eta) \in K_{(a,b)}(x)).$$

Theorem 4.9 applies with

$$\mathfrak{F}_{(a,b)}(K_{(a,b)}(x),\, r) = I_{(0,x)}(r), r > 0$$

such that

$$P(R < x) = \int_0^x re^{-\frac{r^2}{2}}\, dr.$$

(b) Let $x > 0$ and $A \in \mathfrak{A}_{(a,b)}$. As in Example 4.4 (a),

$$\begin{aligned} P(R < \varrho, (X, Y) \in A) &= \int_{sector_{(a,b)}} \varphi_{(a,b)}(x, y) d(x, y) \\ &= \int_{r=0}^{\varrho} re^{-\frac{r^2}{2}}\, dr \frac{1}{2\pi_{(a,b)}} \int_{Pol^{*-1}_{(a,b)}(A)} \frac{d\varphi}{N^2_{(a,b)}(\varphi)} \\ &= P(R < \varrho) P((X, Y) \in A). \end{aligned}$$

□

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