Reverse triangle inequality. Antinorms and semi-antinorms

M. Moszyńska and W.-D. Richter

Abstract. The paper concerns a biunique correspondence between some positively homogeneous functions on R^n and some star-shaped sets with nonempty interior, symmetric with respect to the origin (Theorems 3.5 and 4.3).

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0. Introduction. The present paper is motivated by results obtained by the second author (see [12] and [13]). As can be seen in Remark 6.4, those results have roots in probability theory.

Our idea is to consider antinorms (that are "relatives" of norms) and semi-antinorms("relatives" of semi-norms) and characterize them geometrically in terms of their "generalized Minkowski balls". These characterizations (Theorems 3.5 and 4.3) are analogues of the well known relationship between norms and convex bodies symmetric w.r.t. the origin. The generalized Minkowski balls are some star bodies (in the case of antinorms) or some closed but possibly unbounded star-shaped sets with nonempty interior (in the case of semi-antinorms). In both cases they are symmetric at the origin. For basic relations between the family of convex and of star-shaped sets we refer to [8] and [7].

At the end of our paper we show that this approach cannot be replaced by that (see for instance [2]) using sectors of R^n instead of the symmetric star bodies (see Remarks 6.2 and 6.3). These two points of view are essentially different.

As is well known (see Proposition 1.1.6 in [16]), for any Minkowski space $(\mathrm{R}^n, \|\cdot\|)$, the set

$$B_{\|\cdot\|} := \{x \in \mathrm{R}^n \mid \|x\| \leq 1\} \tag{0.1}$$

is a convex body in R^n symmetric with respect to the origin, that is called the unit ball generated by the norm $\|\cdot\|$.

Conversely, for any convex body B in R^n, symmetric with respect to 0, the function $\|\cdot\|_B : \mathrm{R}^n \to \mathrm{R}_+$ defined by

$$\|x\|_B := \inf\{\alpha \geq 0 \mid x \in \alpha B\} \tag{0.2}$$

is a norm (see Proposition 1.1.8 in [16]). Moreover, these two maps, $\|\cdot\| \mapsto B_{\|\cdot\|}$ and $B \mapsto \|\cdot\|_B$, are mutually inverse.

The function defined by formula (0.2) is called a Minkowski functional or the gauge function of B (see [15]). Its restriction to $\mathrm{R}^n \setminus \{0\}$ coincides with the reciprocal of the radial function $\rho_B : \mathrm{R}^n \setminus \{0\} \to R_+$ of B that is defined by the formula

$$\rho_B(x) := \sup\{\lambda \geq 0 \mid \lambda x \in B\}. \tag{0.3}$$

The radial function of a convex body is positive and continuous (see [10]).

Let us now consider compact sets that are not necessarily convex, and observe that

- for any positively homogeneous function $g : \mathrm{R}^n \to \mathrm{R}_+$ the formula (0.1) (with $\|\cdot\|$ replaced by g) defines a star-shaped set B_g in R^n, symmetric with respect to 0;

- for any star-shaped set B in R^n symmetric with respect to 0 the formula (0.2) (with $\|\cdot\|_B$ replaced by g_B) defines a positively homogeneous function $g_B : \mathrm{R}^n \to \mathrm{R}_+$;

- the maps $g \mapsto B_g$ and $B \mapsto g_B$ are mutually inverse.

We shall study relationships between properties of B and properties of the function g_B, which is the *Minkowski functional of* B. We begin with *antinorms*, a counterpart of the notion of norm, with the triangle inequality replaced partially by the "reverse triangle inequality" (Section 3). Next, we pass to *semi-antinorms*, a counterpart of the notion of semi-norm, with the triangle inequality again replaced partially by the reverse triangle inequality (Section 4). It is a remarkable property of such functions that, in contrast to semi-norms, they may degenerate for various reasons (Section 5).

Let us notice that properties of the Minkowski ball are usually expressed in terms of linear space. When we use topological terminology (for instance, we say that B has nonempty interior instead of saying that it is absorbing), then we always refer to the Euclidean topology in R^n.

For any Minkowski unit ball B, the topology determined by the norm $\|\cdot\|_B$ coincides with the Euclidean topology. What are the topologies induced by semi-norms or other functionals is a separate problem, not considered in the present paper.

To avoid confusion, let us mention that in the literature the terms "reverse triangle inequality" and "antinorm" have been used in quite different meanings (see, for instance, [1], [3], [4], [11], and [9]).

1. Preliminaries. Let $(e_1, ..., e_n)$ be the canonical basis in R^n: $e_i = (\delta_i^1, ..., \delta_i^n)$, where

$$\delta_i^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

We use, in principle, standard terminology and notation following [15]. In particular, for any subset A of R^n, we write $\mathrm{conv}A$ and $\mathrm{aff}A$ for convex hull and affine hull of A, respectively; furthermore, $\mathrm{lin}x := \{\alpha x \mid \alpha \neq 0\}$, $\mathrm{pos}\ x := \{\alpha x \mid \alpha \geq 0\}$, $\mathrm{lin}A := \bigcup_{x \in A} \mathrm{lin}x$, and $\mathrm{pos}\ A := \mathrm{conv} \bigcup_{x \in A} \mathrm{pos}\ x = \mathrm{conv} \bigcup_{\alpha \geq 0} \alpha A$.

As usual, $\mathrm{cl}A$, $\mathrm{int}A$, and $\mathrm{bd}A$ are closure, interior, and boundary of A. If A is a subset of an affine subspace E of R^n with $\dim E < n$, then relative interior and relative boundary of A are $\mathrm{relint}A$ (interior of A with respect to E) and $\mathrm{relbd}A$ (boundary of A with respect to E).

For any affinely independent set $\{x_0, ..., x_k\}$ in R^n, with $k \leq n$, let $\Delta(x_0, ..., x_k)$ be the k-dimensional simplex with vertices $x_0, ..., x_k$, i.e. $\Delta(x_0, ..., x_k) =$

$\mathrm{conv}\{x_0, ..., x_k\}$; in particular, if $x \neq y$, then $\Delta(x, y)$ is the segment with end-points x, y.

The Minkowski sum $A_1 + A_2 := \{a_1 + a_2 \mid a_i \in A_i,\ i = 1, 2\}$ of subsets A_1, A_2 of R^n will be called their *direct sum* and denoted by $A_1 \oplus A_2$ if $\mathrm{lin}A_1 \oplus \mathrm{lin}A_2 = \mathrm{R}^n$ (compare [15], p. 142).

For any convex polytope P in R^n, an $(n-1)$-dimensional face of P is called *facet*; the collection of facets of P will be denoted by $\mathcal{F}(P)$.

We would like to warn the reader that different authors use sometimes essentially different definitions for some basic notions of convex geometry, such as body, convex body, and star body. Of course, it is necessary to decide which definitions we are going to follow. Thus, when defining these notions, we give appropriate references to the bibliography.

- A nonempty subset A of R^n is a *body* whenever A is compact and is equal to the closure of its interior: $\mathrm{cl}\ \mathrm{int}A = A$ (see [6] or [10]).

- Clearly, the set A is convex whenever $\Delta(x, y) \subset A$ for any pair of distinct points $x, y \in A$.

It is well known that a compact, convex subset A of R^n is a body if and only if $\mathrm{int}A \neq \emptyset$.

Let us note that Schneider in [15] refers to a larger family of convex sets as convex bodies. However, we follow [15] (and [10]) when speaking about star-shaped sets and star bodies.

- A subset A of R^n is *star-shaped with respect to a point* $a \in A$ whenever for every $x \in A \setminus \{a\}$ the segment $\Delta(a, x)$ is contained in A. The kernel, $\ker A$, of a set A consists of all points $a \in A$ such that A is star-shaped with respect to a.

  It can be shown that $\ker A$ is convex; hence, if A is symmetric with respect to 0 and $\ker A \neq \emptyset$, then $0 \in \ker A$ and formula (0.3) applies.

- The set A is a *star body* if A is a body with $\ker A \neq \emptyset$. (Gardner in [6] deals with a much larger family of star bodies, which is useful in geometric tomography.)

It is evident that every convex set is star-shaped with respect to any of its points, and thus every convex body is a star body.

- A subset C of R^n is a *cone with vertex* 0 whenever $tx \in C$ for every $x \in C$ and $t \geq 0$ ([6]).
- Any closed convex cone C in R^n containing no half-space, with vertex 0 and non-empty interior, will be called a *sector of* R^n.
- A finite collection $\mathcal{C}$ of sectors in R^n will be called a *fan* if its members have pairwise disjoint interiors and their union is R^n (compare "complete fan" in [5], Def. 1.7).
- The generalized ball B_g (determined by a Minkowski functional g) is *radially concave with respect to a sector* C whenever $C \setminus B_g$ is convex.
- B_g is *radially concave with respect to a fan* if B_g is radially concave with respect to every sector in this fan.

Consider the following example of fan. Let $H_i = \{x = (x_1, ..., x_n) \in \mathrm{R}^n \mid x_i = 0\}$, let

$$H_i^+ = \{x = (x_1, ..., x_n) \mid x_i \geq 0\}$$

and $H_i^- = -H_i^+$.

We can define the fan $\mathcal{C}(n)$ by induction on the dimension n as follows:

- For $n = 2$:

$$C_1(2) := H_1^+ \cap H_2^+, \quad C_2(2) := H_1^- \cap H_2^+,$$
$$C_3(2) := H_1^- \cap H_2^-, \quad C_4(2) := H_1^+ \cap H_2^-.$$

  Then $\mathcal{C}(2) := \{C_i(2) \mid i = 1, ..., 4\}$ is a fan.

- For $n \geq 3$:

  Assume that we already have the fan $\mathcal{C}(n-1) = \{C_i(n-1) \mid i \in \{1, ..., 2^{n-1}\}\}$. Let

$$C_i(n) := \begin{cases} C_i(n-1) \oplus \mathrm{pos} e_n & \text{if } i \in \{1, ..., 2^{n-1}\} \\ C_i(n-1) \oplus \mathrm{pos}(-e_n) & \text{if } i \in \{2^{n-1}+1, ..., 2^n\} \end{cases}.$$

  Then the family $\mathcal{C}(n)$ defined by

$$\mathcal{C}(n) := \{C_i(n) \mid i \in \{1, ..., 2^n\}\}$$

  is a fan.

This fan will be referred to as *canonical fan*.

Let us now consider a function $g : \mathrm{R}^n \to \mathrm{R}_+$. Recall that

- g is positively homogeneous if and only if for every $t \in \mathrm{R}$ and $x \in \mathrm{R}^n$

$$g(tx) = |t|g(x);$$

- g is non-degenerate if and only if for every $x \in \mathrm{R}^n$

$$g(x) = 0 \Longleftrightarrow x = 0;$$

- g is subadditive if and only if for every $x, y \in \mathrm{R}^n$

$$g(x + y) \leq g(x) + g(y). \tag{1.1}$$

  The inequality (1.1) is often referred to as the triangle inequality.

- g is a semi-norm provided it is positively homogeneous and subadditive;

- a semi-norm is a norm provided that it is non-degenerate.

A well known characterization of norms was mentioned in the Introduction. For the completeness of our presentation, let us give a characterization of semi-norms, although it is probably well known, too. We use definitions (0.1) and (0.2) with $\| \cdot \|$ replaced by g.

PROPOSITION 1.1. *Let $g : \mathrm{R}^n \to \mathrm{R}_+$ be a positively homogeneous function and let B_g be the corresponding generalized unit ball. Then the following are equivalent:*

(i) *g is a semi-norm;*

(ii) *$V := g^{-1}(0)$ is a k-dimensional linear subspace of R^n with $k \in \{0, ..., n\}$, the function $g|V^\perp$ is a norm in $V^\perp$, and B_g is invariant under the translations along V, i.e.,*

$$B_g + V = B_g. \tag{1.2}$$

(See Fig. 1.)

Proof. (i) $\Longrightarrow$ (ii):

Since g is subadditive, it follows that V is a linear subspace. The function $g|V^{\perp}$ is non-degenerate and thus is a norm. Finally, if $x \in B_g$ and $y \in V$, then $g(x) \leq 1$ and $g(y) = 0$, whence $g(x+y) \leq g(x) + g(y) \leq 1$, i.e. $x + y \in B_g$; thus $B_g + V \subset B_g$. The converse inclusion is evident.

(ii)$\Longrightarrow$ (i):

By the assumption, g is positively homogeneous. It remains to show that it is subadditive. If either $x, y \in V$ or $x, y \in V^{\perp}$, then by (ii),

$$g(x+y) \leq g(x) + g(y).$$

Let $x \in V^{\perp}$ and $y \in V$; then by (0.2) combined with (ii),

$$g(x+y) = \inf\{\alpha \geq 0 \mid x + y \in \alpha B_g + V\}$$

$$= \inf\{\alpha \geq 0 \mid x \in \alpha B_g\} = g(x) = g(x) + g(y). \Box$$

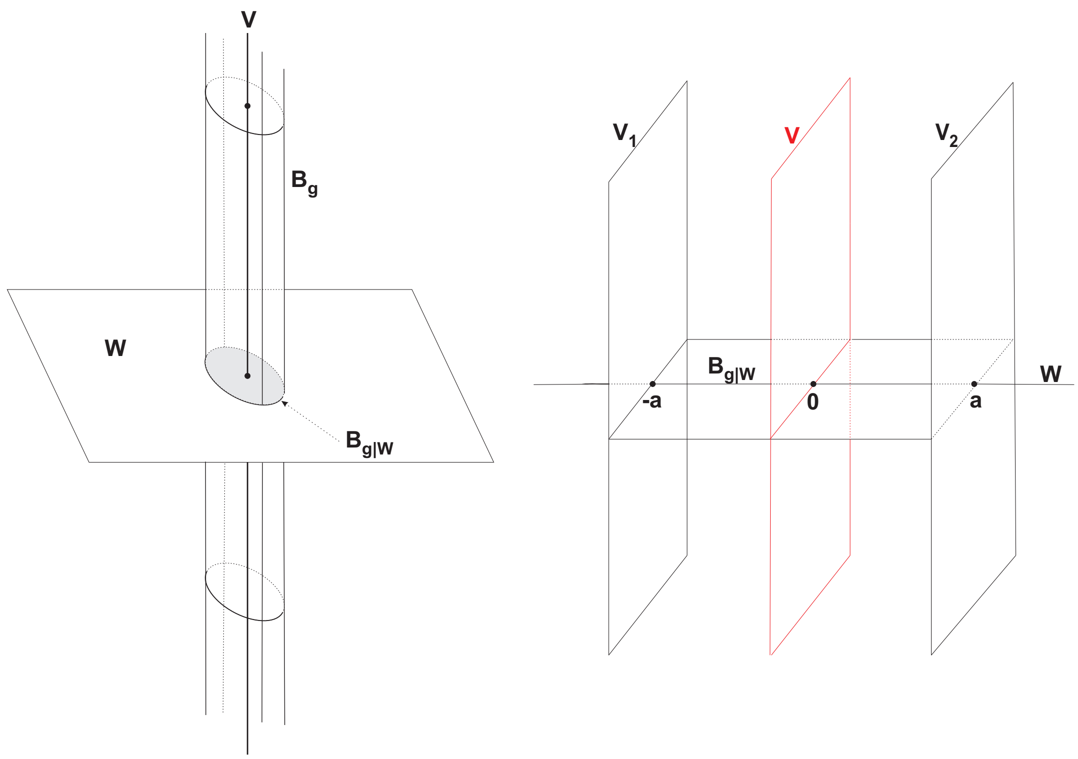

Figure 1: n=3, $W = V^{\perp}$; Left: k=1, $B_{g|W}$-plane convex set, B_g-cylinder over $B_{g|W}$; Right: k=2, $B_{g|W} = \Delta(-a, a)$, B_g-strip bounded by the planes $V_1, V_2 || V$

2. Convex polytopes; the triangle equality case. We begin this section with two well known examples.

EXAMPLE 2.1. Let $\| \cdot \|$ be a generalized l_1-norm in R^n:

$$\|(x_1, ..., x_n)\| := \Sigma_{i=1}^n t_i |x_i|$$

for some $t_1, ..., t_n > 0$.

Then the corresponding unit ball is the convex polytope $P_{(t_i)}$ with vertices $\frac{e_i}{t_i}$ and $\frac{-e_i}{t_i}$ for $i = 1, ..., n$.

If $x, y \in C_i(n)$ for some i and $\|\cdot\| = \|\cdot\|_{P_{(t_i)}}$, then

$$\|x + y\|_{P(t_i)} = \|x\|_{P(t_i)} + \|y\|_{P(t_i)}. \tag{2.1}$$

EXAMPLE 2.2. Let $\|\cdot\|$ be the max-norm in R^n:

$$\|(x_1, ..., x_n)\| := \max\{|x_i| \mid i = 1, ..., n\}.$$

Then the corresponding unit ball is the cube $Q := [-1, 1]^n$, with centre 0 and edges of length 2, each parallel to e_i for some $i \in \{1, ..., n\}$. For any facet F of this cube, if $x, y \in \mathrm{pos}F$ and $\|\cdot\| = \|\cdot\|_Q$, then

$$\|x + y\|_Q = \|x\|_Q + \|y\|_Q. \tag{2.2}$$

Proposition 2.4 below is a generalization of Examples 2.1 and 2.2. In its proof we make use of the following statement.

LEMMA 2.3. *Let E be a k-dimensional linear subspace of R^n and $B \subset E$ be a star-shaped subset of E, symmetric at 0. If $\phi : E \to \mathrm{R}^k$ is a linear isomorphism, then for every $x \in E$*

$$g_{\phi(B)}(\phi(x)) = g_B(x).$$

Proof. Since g_B (and so also $g_{\phi(B)}$) is positively homogeneous, by (0.2) it follows that

$$\begin{aligned} g_{\phi(B)}(\phi(x)) &= \inf\{\alpha \geq 0 \mid \phi(x) \in \alpha \cdot \phi(B)\} \\ &= \inf\{\alpha \geq 0 \mid x \in \alpha \cdot B\} = g_B(x). \end{aligned}$$

□

PROPOSITION 2.4. *Let P be a centrally symmetric, n-dimensional convex polytope in R^n and let $F \in \mathcal{F}(P)$. If $x, y \in \mathrm{pos}F$, then*

$$\|x + y\|_P = \|x\|_P + \|y\|_P. \tag{2.3}$$

Proof. Let $x, y \in \mathrm{pos}F$. We may assume that $0, x, y$ are affinely independent, because otherwise equality (2.3) follows from the positive homogenity of norm. Consider the plane $E := \mathrm{aff}\{0, x, y\}$. Let $\phi : E \to \mathrm{R}^2$ be a linear isomorphism such that $\phi(E \cap \mathrm{pos}F) = C_1(2)$ and put $x_0 := \phi(x), y_0 := \phi(y), F_0 := \phi(E \cap F)$, (see Fig.2). Then F_0 is a side of a convex polygon P_0 in R^2, symmetric w.r.t. 0 and, according to Example 2.1,

$$\|x_0 + y_0\|_{P_0} = \|x_0\|_{P_0} + \|y_0\|_{P_0}.$$

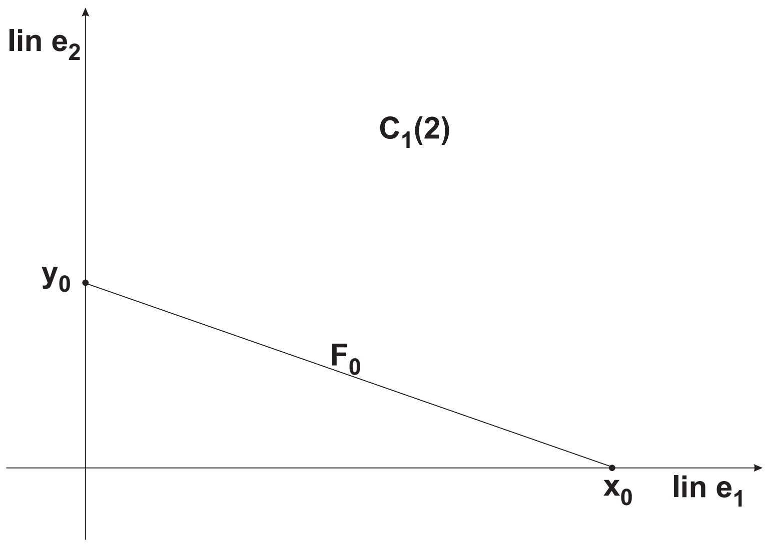

Figure 2:

In view of Lemma 2.3, this is equivalent to

$$\|x + y\|_{P\cap E} = \|x\|_{P\cap E} + \|y\|_{P\cap E}$$

and thus to (2.3) as well. □

3. Antinorms. We are now going to consider a counterpart of the notion of norm, with subadditivity replaced by a (partial) "superadditivity".

DEFINITION 3.1. (i) A function $g : \mathrm{R}^n \to \mathrm{R}_+$ is *superadditive in a sector* C if

$$g(x + y) \geq g(x) + g(y) \text{ for every } x, y \in C. \tag{3.1}$$

(ii) g is *superadditive in a fan* $\mathcal{C}$ if it is superadditive in every sector in the fan $\mathcal{C}$.

DEFINITION 3.2. (i) A function $g : \mathrm{R}^n \to \mathrm{R}_+$ is an *antinorm* if g is continuous, positively homogeneous, non-degenerate, and superadditive in some fan.

(ii) The inequality (3.1) is referred to as the *reverse triangle inequality.*

THEOREM 3.3. *Let B be a star body in R^n symmetric with respect to* 0, *with positive and continuous radial function, and let g_B be the corresponding Minkowski functional.*

Then for any sector C the following conditions are equivalent:

(i) *the set B is radially concave with respect to the sector C;*

(ii) *for any points $x, y \in C$*

$$g_B(x+y) \geq g_B(x) + g_B(y); \tag{3.2}$$

equality holds if and only if either $x = y$ or $\Delta(x', y') \subset \mathrm{bd}B$ for $x' \in \mathrm{pos}x \cap \mathrm{bd}B$ and $y' \in \mathrm{pos}y \cap \mathrm{bd}B$.

Proof.

(i) $\Longrightarrow$(ii):

Assume (i). Since the radial function ρ_B is positive and continuous (see [10]), for every $x \in C \setminus \{0\}$ there is a unique point $x' \in (\mathrm{pos}x) \cap \mathrm{bd}B$.

Take $x, y \in C$. If $x = y$, then obviously the equality in (3.2) holds. Let $x \neq y$.

By (0.2), for any $t > 0$ and $z \in \mathrm{R}^n$

$$g_{tB}(z) = \frac{1}{t} g_B(z).$$

Hence, if $t > 0$, then each of the two conditions (i) and (ii) is equivalent to the corresponding condition (i') or (ii') with B replaced by tB. Consequently, without loss of generality we may assume that

$$x \in B \text{ and } y \in \mathrm{bd}B,$$

that is,

$$g_B(x) \leq g_B(y) = 1.$$

Let $x' \in (\mathrm{pos}x) \cap (\mathrm{bd}B)$ and $y' = y \in (\mathrm{pos}y) \cap (\mathrm{bd}B)$. Then, $\Delta(x', y')$ is contained in a facet of a convex polytope B' symmetric with respect to 0.

Therefore, by Proposition 2.4,

$$g_{B'}(x+y) = g_{B'}(x) + g_{B'}(y). \tag{3.3}$$

Since $B \cap \mathrm{pos}\{x,y\} \subset B' \cap \mathrm{pos}\{x,y\}$, by (0.2) it follows that

$$g_B(x+y) \geq g_B(x) + g_B(y).$$

Equality holds if and only if $g_B(x+y) = g_{B'}(x+y)$, i.e., $(\mathrm{pos}\ \frac{x+y}{2}) \cap \mathrm{bd}B = (\mathrm{pos}\ \frac{x+y}{2}) \cap \mathrm{bd}B'$, and consequently $\Delta(x', y') \subset \mathrm{bd}B$. Thus (ii) is satisfied.

To prove the converse implication, suppose (i) is not satisfied. Then there exist $x, y \in C \cap \mathrm{bd}B$ such that $\frac{x+y}{2} \in \mathrm{int}B$, whence $g_B(\frac{x+y}{2}) < 1$.

Thus, $g_B(x+y) < 2 = g_B(x) + g_B(y)$, because $g_B(x) = 1 = g_B(y)$. Hence (ii) is not satisfied.

This completes the proof. □

We are now ready to give a geometric characterization of antinorms in terms of their generalized balls (Theorem 3.5). Let us begin with the following

LEMMA 3.4. *For every star body B in R^n symmetric with respect to 0, the radial function ρ_B is positive if and only if for every $x \in \mathrm{R}^n \setminus \{0\}$*

$$g_B(x) = \frac{1}{\rho_B(x)}. \tag{3.4}$$

If $\rho_B(x) = 0$ for some $x \neq 0$, then $g_B(x)$ is not defined.

Proof. The proof is based on (0.2) (with $\|\cdot\|_B$ replaced by g_B) and (0.3). Its details are left to the reader. □

THEOREM 3.5. *For any function $g : \mathrm{R}^n \to \mathrm{R}_+$ the following conditions are equivalent:*

(i) *g is an antinorm;*

(ii) *B_g is a star body symmetric with respect to 0, with positive and continuous radial function, and radially concave with respect to every sector in some fan.*

Proof. (i) $\Longrightarrow$ (ii): Since g is positively homogeneous, it follows that B_g is star-shaped at 0 and symmetric with respect to 0. Since g is non-degenerate, by Lemma 3.4 the radial function ρ_{B_g} is positive, and it is continuous because g is continuous. Furthermore, for every sector C in some fan, g satisfies condition (ii), and so also (i), of Theorem 3.3, whence B_g is concave with respect to C.

Let us show that B_g is a star body. Since the radial function of B_g is continuous, it follows that $B_g = \mathrm{cl}\ \mathrm{int} B_g$; hence it remains to prove that B_g is compact.

By definition (see (0.1)),

$$B_g = g^{-1}([0;1]).$$

Let S^{n-1} be the Euclidean unit sphere. Take a sequence $(x^{(k)})_{k\in \mathrm{N}}$ in B_g. Since g is non-degenerate, we may assume that $x^{(k)} \neq 0$ for every k, whence $x^{(k)} = t_k \cdot u_k$ for some $t_k > 0$ and $u_k \in S^{n-1}$. Since g is positively homogeneous, it follows that $g(x^{(k)}) = t_k g(u_k) \leq 1$, whence $t_k \leq \frac{1}{g(u_k)} < \infty$ and thus $(t_k)_{k\in\mathrm{N}}$ has a convergent subsequence. By the compactness of S^{n-1} we may assume that $(u_k)_{k\in\mathrm{N}}$ is convergent in S^{n-1}. Hence, $(x^{(k)})_{k\in\mathrm{N}}$ has a convergent subsequence, which proves the compactness of B_g.

The converse implication (ii) $\Longrightarrow$ (i) is a direct consequence of Theorem 3.3. □

REMARK 3.6. In view of Proposition 2.4, the Minkowski functional of an arbitrary convex polytope in R^n is simultanuously a norm and an antinorm. Thus the family of norms and the family of antinorms have nonempty intersection.

4. Semi-antinorms. Following partially the convention that is commonly used for convex functions (see [15] or [14]), we extend the range R_+ of positively homogeneous functions to $\bar{\mathrm{R}}_+ := R_+ \cup \{\infty\}$. We admit the following rules:

$$\infty+\infty = \infty, \quad \infty+t = t+\infty = \infty \quad \text{and} \quad \infty\cdot t = t\cdot\infty = \infty \quad \text{for any} \quad t > 0.$$

Then, a function $f : \mathrm{R}^n \to \bar{\mathrm{R}}_+$ is said to be *continuous* if for every $x \in \mathrm{R}^n$ and every sequence $(x^{(k)})_{k\in\mathrm{N}}$ in R^n

$$\lim_k x^{(k)} = x \Longrightarrow \lim_k f(x^{(k)}) = f(x);$$

thus, in particular, if $f(x) = \infty$, then $\lim_k f(x^{(k)}) = \infty$.
We are now going to extend the notion of antinorm to that of semi-antinorm.

DEFINITION 4.1. A function $g : \mathrm{R}^n \to \mathrm{R}_+$ is a *semi-antinorm* provided g is continuous, positively homogeneous, and there exists a fan $\mathcal{C}$ for R^n such that g is superadditive in C for every sector $C \in \mathcal{C}$.

The following theorem is an analogue of Theorem 3.3.

THEOREM 4.2. *Let B be a star-shaped set in R^n (not necessarily compact), symmetric with respect to 0, with $\mathrm{cl\ int}B = B$ and with positive and continuous radial function $\rho_B : \mathrm{R}^n \to \bar{\mathrm{R}}_+$. Let g_B be the corresponding Minkowski functional. Then for every sector C the following conditions are equivalent:*
(i) *the set B is radially concave with respect to C;*
(ii) *for every $x, y \in C$*

$$g_B(x + y) \geq g_B(x) + g_B(y) \tag{4.1}$$

with equality if and only if either $x = y$ or $\Delta(x', y') \subset \mathrm{bd}B$ for $x' \in \mathrm{pos}x \cap \mathrm{bd}B$ and $y' \in \mathrm{pos}y \cap \mathrm{bd}B$.

Proof. The proof is analoguous to that of Theorem 3.3. □

As an analogue of Theorem 3.5, we obtain a geometric characterization of semi-antinorms, Theorem 4.3. Its proof is analogous to that of Theorem 3.5.

THEOREM 4.3. *For every function $g : \mathrm{R}^n \to \mathrm{R}_+$ the following conditions are equivalent:*
(i) *g is a semi-antinorm;*
(ii) *B_g is a set star-shaped at 0, symmetric with respect to 0, with positive and continuous radial function $\rho_{B_g} : \mathrm{R}^n \to \bar{\mathrm{R}}_+$, and it is radially concave with respect to C for every sector C in some fan.*

The following statement is related to Proposition 1.1 characterizing seminorms.

PROPOSITION 4.4. *Let $g : R^n \to R_+$ be a positively homogeneous* *function and let B_g be the corresponding generalized unit ball. If $V := g^{-1}(0)$ is a linear subspace of R^n, the function $g|V^{\perp}$ is an antinorm in $V^{\perp}$ and $B_g + V = B_g$, then g is a semi-antinorm.*

Proof. Since $g|V^{\perp}$ is an antinorm, by Theorem 3.5 it follows that $B_{g|V^{\perp}}$ is a star body in $V^{\perp}$, symmetric with respect to 0, with positive and continuous radial function, and it is radially concave with respect to every sector in some fan $\mathcal{C}_0$ in $V^{\perp}$. By the assumption, $B_g + V = B_g$, whence $B_g = B_{g|V^{\perp}} \oplus V$, and thus B_g is a star body in R^n symmetric with respect to 0, with positive and continuous radial function. Moreover, B_g is radially concave with respect to every sector of the fan $\mathcal{C} := \{C \oplus V | C \in \mathcal{C}_0\}$. Hence, in view of Theorem 4.3, the function g is a semi-antinorm. □

As we shall show in Section 5, the converse implication does not hold (see Remark 5.3); thus Proposition 4.4 is not a strict analogue of Proposition 1.1 concerning semi-norms. However, Proposition 4.4 combined with Propositions 2.4 and 1.1 yields the following analogue of Remark 3.6.

REMARK 4.5. Any cylinder over a convex polytope corresponds to a semi-norm and to a semi-antinorm. Thus, the family of semi-norms and the family of semi-antinorms have non-empty intersection.

5. Examples of antinorms and semi-antinorms.

Let $p \neq 0$ and $n \geq 2$. For every $x = (x_1, ..., x_n) \in \mathrm{R}^n$, let

$$\|x\|_{n,p} := (\Sigma_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}. \tag{5.1}$$

Formula (5.1) defines the so called "$l_{n,p}$-norm", which is positively homogeneous and for $p > 0$ is non-degenerate; for $p \geq 1$ it is subadditive (see [16], Proposition 1.1.16) and thus is a norm.

EXAMPLE 5.1.

(a) For $p \in (0, 1)$ the function $g := \| \cdot \|_{n,p} : \mathrm{R}^n \to \mathrm{R}^+$ is an antinorm. Indeed, it remains to prove that it is superadditive (continuity is evident).

For $n = 2$ it can be derived (by means of standard calculation) from Theorem 3.3 for the canonical fan $\mathcal{C}(2)$. (For $p = \frac{1}{2}$, see Fig.3.)

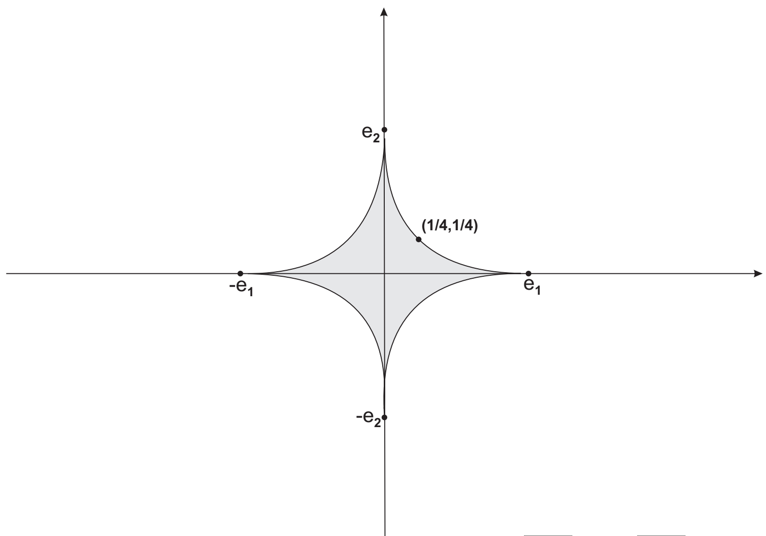

Figure 3: n=2, p=$\frac{1}{2}$, $B_g = \{(x_1, x_2) \mid \sqrt{|x_1|} + \sqrt{|x_2|} \leq 1\}\}$

Let $n \geq 3$ and let $x, y \in \mathrm{R}^n$. We may assume that x, y are linear independent and (by symmetry)that $x, y \in C_1(n)$. Let $\phi : \mathrm{R}^2 \to \mathrm{lin}(x, y)$ be a linear isomorphism mapping $C_1(2)$ onto a subset of $C_1(n)$. In view of Lemma 2.3, the reverse triangle inequality for x, y is equivalent to the reverse triangle inequality for $\phi^{-1}(x), \phi^{-1}(y)$.

(b) For $p < 0$, let

$$\|x\|_{n,p} := \begin{cases} 0 & \text{if } x = (x_1, ..., x_n) \in \bigcup_{i=1}^{n} H_i \\ (\Sigma_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} & \text{otherwise} \end{cases} . \tag{5.2}$$

The function $g := \| \cdot \|_{n,p}$ is continuous. Moreover, it is a semi-antinorm. The proof of superadditivity is the same as in (a).

In particular, let $n = 2$; then $g(x) = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$ and

$$B_g = \{(x_1, x_2) \in \mathrm{R}^2 \mid |x_1|^p + |x_2|^p \geq 1\}.$$

For $p = -1$, see Fig.4.

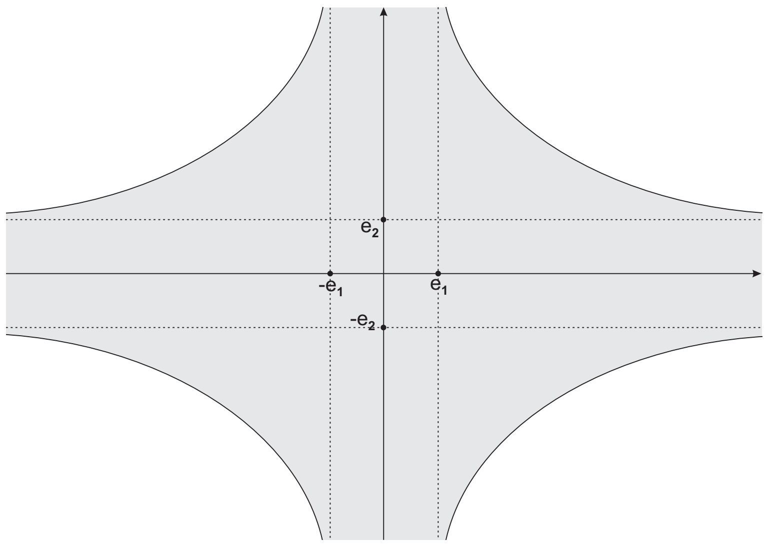

Figure 4: n=2, p=-1, $B_g = \{(x_1, x_2) \mid |x_1 x_2| \leq |x_1| + |x_2|\}\}$

EXAMPLE 5.2.(a) For $n = 2$, let

$$g^{(2)}(x) := \sqrt{|x_1 x_2|}.$$

Then the boundary of the corresponding unit ball is the union of two hyperbolae whose common asymptotes are the coordinate axes. (See Fig. 5.)

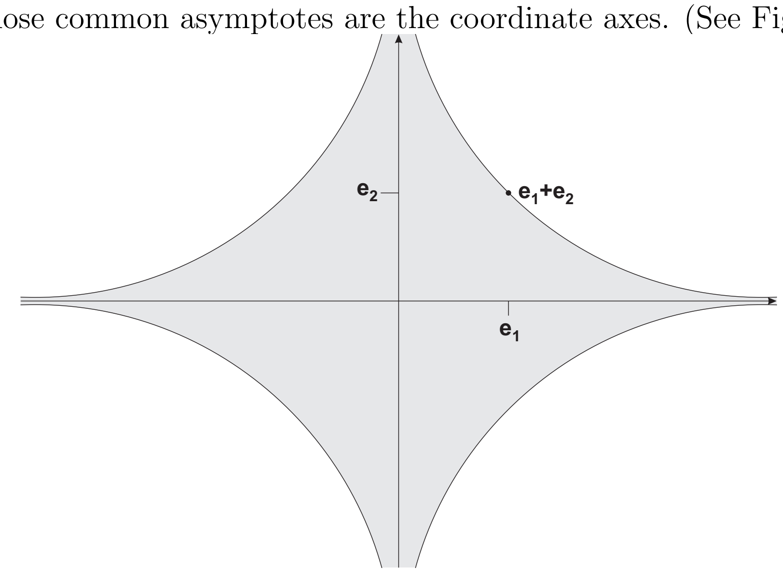

Figure 5: n=2, $B_g = \{(x_1, x_2) \mid |x_1 \cdot x_2| \leq 1\}\}$

The function $g = g^{(2)}$ is a semi-antinorm, since it is positively homogeneous, continuous, and, by Theorem 4.3, superadditive with respect to the canonical fan. Moreover, the function g is degenerate because the inverse image of 0 is $\mathrm{line}_{e_1} \cup \mathrm{line}_{e_2}$.

This example can be generalized to arbitrary $n \geq 2$: let

$$g^{(n)}(x) := \sqrt{\Sigma_{i<j}|x_i x_j|}.$$

By the same reasoning as in Example 5.1 (b), we infer that $g = g^{(n)}$ is superadditive with respect to the canonical fan. Evidently it is positively homogeneous and continuous, and thus it is a semi-antinorm. Moreover, this function is degenerate because the inverse image of 0 is a union of 1-dimensional linear subspaces.

(b) This example can be modified as follows. Let

$$\tilde{g}(x) := (\prod_{i=1}^{n} |x_i|)^{1/n}.$$

To prove that the function $g = \tilde{g}$ is superadditive with respect to the canonical fan, we may restrict our consideration to $C_1(n)$ and apply Theorem 17, p. 35 in [2].

Furthermore, it is continuous and positively homogeneous, whence it is a semi-antinorm. Moreover, the inverse image of 0 is the union of the hyperplanes $H_1, ..., H_n$.

Let us now return to Proposition 4.4.

REMARK 5.3. Examples 5.1(b) and 5.2 show that the implication converse to that in Proposition 4.4 does not hold, because in Examples 5.1(b) and 5.2(b)

$$g^{-1}(0) = \bigcup_{i=1}^{n} H_i$$

while in Example 5.2(a)

$$g^{-1}(0) = \bigcup_{i=1}^{n} \mathrm{line}_{e_i}.$$

Hence, $g^{-1}(0)$ is not a subspace of R^n.

6. Final remarks. The following theorem concerning the triangle inequality is a counterpart of Theorem 3.3, which concerns the reverse triangle inequality.

THEOREM 6.1. *Under the assumptions of Theorem 3.3, for any sector C in* R^n *the following conditions are equivalent:*

(i) *the set $B \cap C$ is convex;*

(ii) *for every $x, y \in C$*

$$g_B(x+y) \leq g_B(x) + g_B(y).$$

Proof. We follow the proof of Theorem 3.3 until the formula (3.3). Furthermore, since $B \cap C$ is convex and x', y' belong to the intersection of the boundaries of B and B', it follows that $(\mathrm{pos}\{x, y\}) \cap B \supset (\mathrm{pos}\{x, y\}) \cap B'$, $g_B(x) = g_{B'}(x)$, and $g_B(y) = g_{B'}(y)$. Consequently, in view of (0.2) and (3.3),

$$g_B(x+y) \leq g_{B'}(x+y) = g_B(x) + g_B(y).$$

The equality case is as in proof of Theorem 3.3.

To prove the converse implication, suppose (i) is not satisfied. Then there exist $x, y \in C \cap \mathrm{bd}B$ such that $\frac{x+y}{2} \in C \setminus B$, whence $g_B(\frac{x+y}{2}) > 1$. Thus $g_B(x+y) > 2 = g_B(x) + g_B(y)$, because $g_B(x) = 1 = g_B(y)$. Hence (ii) is not satisfied. This completes the proof. $\square$

REMARK 6.2. Let us notice that a continuous, positively homogeneous, and non-degenerate function $g : \mathrm{R}^n \to \mathrm{R}_+$ may satisfy the triangle inequality in each sector of some fan although g is not a norm. Equivalently, a star body B in R^n symmetric with respect to 0, with positive and continuous radial function, need not be convex while its intersection with every sector in some fan is convex. For such an example derived from two ellipses, see Fig. 6.

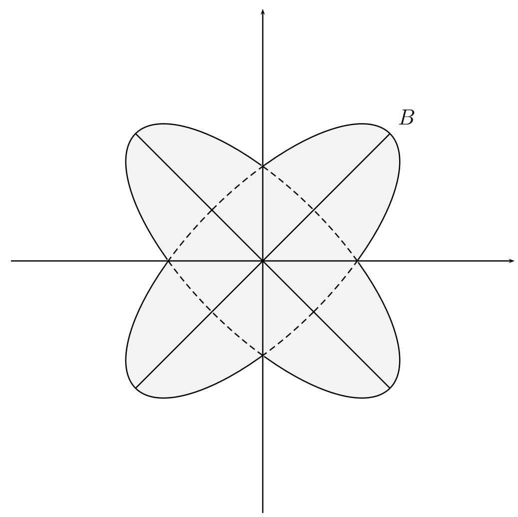

Figure 6:

Then the gauge function g_B is not subadditive, and thus it is not a norm. In view of Theorem 3.5, it is not an antinorm, either.

REMARK 6.3. A star body B satisfying the assumptions of Theorem 6.1 (and so also of Theorem 3.3) may be radially concave with respect to every sector in some fan but convex with respect to every sector in another fan. (See Fig. 7.)

Indeed, consider the canonical fan $\mathcal{C}(2)$ and let $\mathcal{C}'$ be its image under the rotation of R^2 about 0 by $\frac{\pi}{4}$. Let B be the star body whose boundary is the union of the eight segments

$$\Delta(4e_1, e_1+e_2),\ \Delta(e_1+e_2, 4e_2),\ \Delta(4e_2, -e_1+e_2),\ \Delta(-e_1+e_2, -4e_1),$$

$$\Delta(-4e_1, -e_1-e_2),\ \Delta(-e_1-e_2, -4e_2),\ \Delta(-4e_2, e_1-e_2),\ \Delta(e_1-e_2, 4e_1).$$

Then the corresponding function g_B is an antinorm (it is superadditive in $\mathcal{C}(2)$), but in view of Theorem 6.1 it is subadditive in each sector of the fan $\mathcal{C}'$, because the set $B \cap C$ is convex for every sector C of this fan.

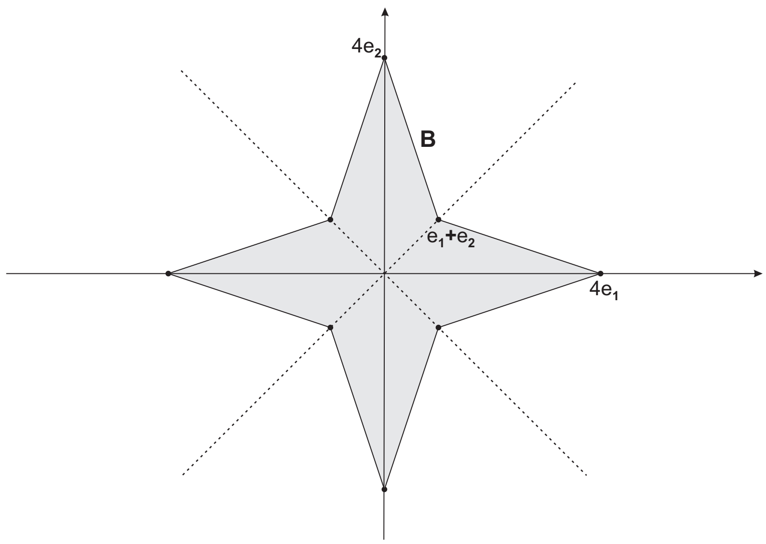

Figure 7: Sectors of the fan $\mathcal{C}'$ are marked with dotted lines

REMARK 6.4. For $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the $l_{2,q}$-unit ball, i.e. the set $\{(x_1, x_2) \in \mathrm{R}^2 \mid (|x_1|^q + |x_2|^q)^{1/q} \leq 1\}$, was used in [12] to define the arc-length of the $l_{2,p}$-circle which was assumed to be the supremum of the suitably defined "integral sums". This approach is based on the triangle inequality, which cannot be used if $0 < p < 1$. Let the set $S(q)$ for $q > 0$ be defined by

$$S(q) := \{(x_1, x_2) \in \mathrm{R}^2 \mid \frac{1}{|x_1|^q} + \frac{1}{|x_2|^q} \geq 1\}.$$

For $0 < p < 1$ and $\frac{1}{p} - \frac{1}{q} = 1$, the set $S(q)$ was used in [13] as the unit ball, to define the arc-length of the $l_{2,p}$-circle to be a certain integral. The results of the present paper show that for $p \in (0, 1)$ the reverse triangle inequality can be used to prove that this integral is just the infimum of all suitable "integral sums". To this end, let us notice that if $r := -q$, then $S(q) = B_g$ for $g := \| \cdot \|_r$. In view of Example 5.1(b), the function g is a semi-antinorm, whence $S(q)$ is the generalized ball generated by a semi-antinorm.

Furthermore, let us recall that according to Example 5.1(a) the $l_{2,p}$-circle, $p \in (0,\, 1)$ corresponds to an antinorm.

Hence, for $p \in (0, 1)$ and q satisfying $\frac{1}{p} - \frac{1}{q} = 1$, in [13] the arc-length of the $l_{2,p}$-circle corresponding to an antinorm was measured with respect to the semi-antinorm g. These considerations have the following probabilistic background. For $n = 2$ and $p > 0$, the density level-sets of the p-generalized normal distribution are $l_{2,p}$-circles and the measuring them in the way described above allows to prove powerful geometric measure representation formulae for these distributions.

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M. Moszyńska, Institute of Mathematics, University of Warsaw, Banacha 2,02-097 Warszawa, Poland,

W .-D. Richter, Institute of Mathematics, University of Rostock, Ulmenstrasse 69, Haus 3, D-18057 Rostock, Germany,