Geometric approach to the skewed normal distribution

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Abstract

The representations of the skewed normal distribution given in Propositions 1-4 in M. Genton (ed., 2004) are considered here from a unified geometric point of view and are, based upon this, generalized in two respects. On the one hand, the four concrete representations motivate us for a unified and much more general algebraic-geometric representation of the skewed normal distribution (Theorems 1 and 2 as well as Remarks 2 and 3); on the other hand, the mentioned representations are generalized to the elliptically contoured case (Propositions and Corollaries 1c-4c).

MSC 2010: 60E99
Key words: skewed normal distribution, skewed elliptical distribution, geometric measure representation, stochastic representations, method of indivisibles, intersection percentage function

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[†] Supported by the Hermes-Junior-Preis of the University of Rostock

1 Introduction

The class of skewed normal distributions appeared in [5] for the first time as a subclass of special interest within a systematic approach to a more general class of distributions. Later on many authors published results on this and more general classes of skewed distributions. The book [7] reviews the state-of-the-art advances up to its appearance in the class of skew-elliptical distributions in one and more dimensions. Because of the wide spread development of this research area, there arose a need for finding as general and systematic approaches to it as possible. Several authors contributed to this direction of development. To mention three of them, we refer to [1], [2] and [3]. The recent paper [4] opens a new perspective for a further significant generalization of the class of skewed distributions.
In the spirit of [5], a random variable Z is called a skew-normal one with parameter $\alpha \in \mathbb{R}$, for brevity $Z \sim SN(\alpha)$, if it has the pdf

$$\widetilde{g}(z;\alpha) = 2\phi^{(1)}(z)\Phi^{(1)}(\alpha z), \quad z \in \mathbb{R}, \tag{1}$$

where $\phi^{(1)}$ and $\Phi^{(1)}$ are the standard normal pdf and the standard normal cdf, respectively, and $\mathbb{R}$ denotes the real line. Several results that explain to a certain extent the nature of the SN distribution and its relation to some other distributions are discussed in the very beginning of [7]. To be specific, we will refer to four of these results here as the following Propositions 1-4. These results establish a close connection between the one-dimensional skew-normal and an underlying two-dimensional normal distribution.
Thereby Φ denotes the standard Gaussian measure in the two-dimensional Euclidean space $\mathbb{R}^2$ and Φ_ρ the Gaussian measure with expectation vector zero and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $-1 < \rho < 1$. The sign $\sim$ is used if a vector on the left side of it is distributed according to the distribution indicated on the right side of it.

Proposition 1 *If* $(X,Y)^T \sim \Phi$, *then* $\mathfrak{L}(X \mid \alpha X > Y) = SN(\alpha)$.

Proposition 2 *If* $(X,Y)^T \sim \Phi_\rho$, *then* $\mathfrak{L}(Y \mid X > 0) = SN\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)$, $-1 < \rho < 1$.

Proposition 3 *If* $(X,Y)^T \sim \Phi$, *then* $\mathfrak{L}\left(\delta|X| + \sqrt{1-\delta^2}Y\right) = SN\left(\frac{\delta}{\sqrt{1-\delta^2}}\right)$, $-1 < \delta < 1$.

Proposition 4 *If* $(X,Y)^T \sim \Phi_\rho$, *then* $\mathfrak{L}(\max(X,Y)) = SN\left(\sqrt{\frac{1-\rho}{1+\rho}}\right)$, $-1 < \rho < 1$.

Making use of some vector-algebra, the conditional probability dealt with in the distributional statement of Proposition 1 may be reformulated as

$$P(X < z \mid \alpha X > Y) = 2\Phi(A_1(z)),\ z \in \mathbb{R},$$

and those of Propositions 2-4 as $P(Y < z \mid X > 0) = 2\Phi(A_2(z))$, $P(\delta|X| + \sqrt{1-\delta^2}Y < z) = 2\Phi(A_3(z))$, and $P(\max(X,Y) < z) = 2\Phi(A_4(z))$, $z \in \mathbb{R}$, respectively. Here, $A_i(z)$, $i \in \{1,2,3,4\}$, are suitably defined elements of the Borel σ-algebra $\mathfrak{B}^2$ in $\mathbb{R}^2$. The sets $A_i(z)$, $i \in \{1,2,3,4\}$, describe the events under consideration in Propositions 1-4 and will be described in detail in Section 2. It will turn out that there is some "similarity" between the sets $A_i(z)$, $i = 1,2,3,4$ and that it is possible to transform each of them by a true similarity, being actually an isometric transformation, into each other. Following this line, the aim of the present paper is to introduce a new possibility of comparing different representations for the skewed normal distribution with each other and to introduce a more general representation for the skew normal distribution which includes the four cited cases as special cases. First of all, we shall concentrate our consideration to the representations in Propositions 1 to 4. Our method of comparison is a geometric one. It is based upon a geometric measure representation for the two-dimensional normal distribution. This representation applies to the probabilities $\Phi(A_i(z)),\ i \in \{1,2,3,4\}$, and gives certain new information on these quantities. The geometric measure representation allows to look at the well known stochastic representation

$$\zeta \stackrel{d}{=} R \cdot U \tag{2}$$

of a standard Gaussian two-dimensional random vector ζ in a new, geometric, way. Here, R and U are independent and distributed according to the χ_2-distribution and the uniform distribution on the unit circle, respectively, and the sign $\stackrel{d}{=}$ indicates that the random elements on the left and right side of it are equally distributed. The geometric measure representation was proved with a more general multivariate setting in [9] first. It applied rather early to a problem from engineering in [12]. Several other of its applications to probability theory and mathematical statistics are reviewed in [11]. A slightly modification of the representation in [9], which is more convenient for the purposes of the present paper, was proved in [10] and will be discussed in Section 3. In Section 4, we apply this representation to the four sets $A_i(z)$, considered in Section 2. This allows us to reformulate and reprove the Propositions 1-4 from a unified geometric point of view. As a result, later it will be much easier to further compare the present four models of a $SN(\alpha)$-distributed random variable with other similar models and even with models from, until yet, not known distributions. Some of the lengthy calculations will be given in the Appendix. Moreover, a g-generalization, where g denotes a density generating function, of all four propositions will be given in Section 4. The necessary g-generalization of the geometric measure representation for spherically distributed random vectors was introduced in [10]. The geometric reformulations of Propositions 1-4 will be discussed in Section 5 and are the motivation for a more general geometrically formulated new theorem on the $SN(\alpha)$- distribution. To give some hints for possible further work on this topic, several concluding remarks and directions of future research are given in Section 6.

2 Vector-algebraic reformulations of Propositions 1 to 4

The aim of this section is to reformulate the (partly conditional) probabilities studied in the Propositions 1-4 in such a way that afterwards the geometric measure representation formula, which will be presented in Section 3, applies.

Because $P(\alpha X > Y) = \frac{1}{2}$, the conditional probability $P(X < z \mid \alpha X > Y)$ equals two times the probability $P(X < z, \alpha X > Y)$. But this may be written as $P(X < z \mid \alpha X > Y) = 2P((X,Y)^T \in \{(x,y)^T \in \mathbb{R}^2 : x < z, y < \alpha x\})$. Hence, the following proposition has been proved.

Proposition 1a *If* $(X,Y)^T \sim \Phi$*, then*

$$P(X < z \mid \alpha X > Y) = 2\Phi(A_1(z)),\ -\infty < z < \infty,$$

with

$$A_1(z) = \{(x,y)^T \in \mathbb{R}^2 : x < z, y < \alpha x\}.$$

For an illustration of the set $A_1(z)$, we refer to Figure 1.
Analogously, the conditional probability $P(Y < z \mid X > 0)$ may be written as $2P((X,Y)^T \in A_2^*(z))$ with $A_2^*(z) = \{(x,y)^T \in \mathbb{R}^2 : x > 0, y < z\}$. Assuming $(X,Y)^T \sim \Phi_\rho,\ -1 < \rho < 1$, the transformed random vector

$$(\xi,\eta)^T = D \cdot O \cdot (X,Y)^T$$

satisfies

$$(\xi,\eta)^T \in D \cdot O \cdot A_2^*(z)$$

iff $(X,Y)^T \in A_2^*(z)$. Here, $D = diag(\frac{1}{\sqrt{1+\rho}}, \frac{1}{\sqrt{1-\rho}})$ is a diagonal matrix, $O = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is an orthogonal one and

$$\begin{aligned} D \cdot O \cdot A_2^*(z) &= \{D \cdot O \cdot (s,t)^T : s > 0, t < z\} \\ &= \{(x,y)^T \in \mathbb{R}^2 : \sqrt{1+\rho}x + \sqrt{1-\rho}y > 0, \sqrt{1+\rho}x - \sqrt{1-\rho}y < \sqrt{2}z\}. \end{aligned}$$

Hence, the following proposition has been proved.

Proposition 2a *If* $(X,Y)^T \sim \Phi_\rho,\ -1 < \rho < 1$, *then*

$$P(Y < z \mid X > 0) = 2\Phi(A_2(z)),\ -\infty < z < \infty,$$

with

$$A_2(z) = \{(x,y)^T \in \mathbb{R}^2 : \sqrt{1+\rho}x + \sqrt{1-\rho}y > 0, \sqrt{1+\rho}x - \sqrt{1-\rho}y < \sqrt{2}z\}.$$

For an illustration of the set $A_2(z)$, we refer to Figure 2.

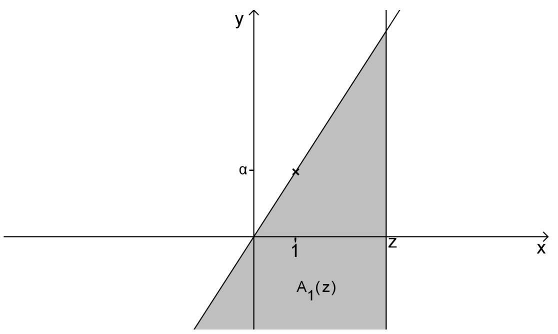

Figure 1: The set $A_1(z)$ for $z > 0, \alpha > 1$.

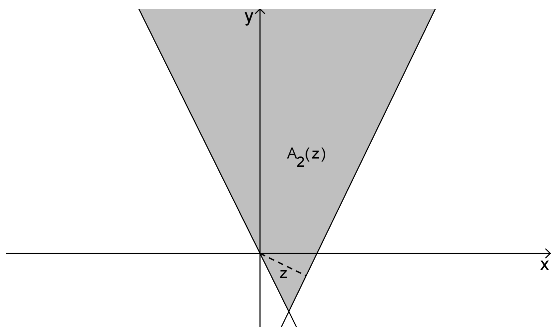

Figure 2: The set $A_2(z)$ for $z > 0, \rho > 0$.

The reformulation of Proposition 3 follows immediately from the equation $P(\delta|X| + \sqrt{1-\delta^2}Y < z) = 2P(\delta X + \sqrt{1-\delta^2}Y < z, X > 0)$.

Proposition 3a *If* $(X,Y)^T \sim \Phi$, *then*

$$P(\delta|X| + \sqrt{1-\delta^2}Y < z) = 2\Phi(A_3(z)),\ -\infty < z < \infty,\ -1 < \delta < 1,$$

with

$$A_3(z) = \{(x,y)^T \in \mathbb{R}^2 : x > 0, \delta x + \sqrt{1-\delta^2}y < z\}.$$

For an illustration of the set $A_3(z)$, we refer to Figure 3.
The probability $P(max(X,Y) < z)$ can be written as $P((X,Y)^T \in A_4^*(z))$ with $A_4^*(z) = \{(x,y)^T \in \mathbb{R}^2 : x < z, y < z\}$. Using the same transformation method with the same matrices D and O as for Proposition 2a, we get

$$D \cdot O \cdot A_4^*(z) = \{(x,y)^T \in \mathbb{R}^2 : \sqrt{\frac{1+\rho}{2}}x + \sqrt{\frac{1-\rho}{2}}y < z, \sqrt{\frac{1+\rho}{2}}x - \sqrt{\frac{1-\rho}{2}}y < z\}.$$

Considering the subset of the set above which is bounded by the lines $y = 0$ and $\sqrt{\frac{1+\rho}{2}}x + \sqrt{\frac{1-\rho}{2}}y = z$, and a symmetry consideration, the following Proposition has been proved.

Proposition 4a *If* $(X,Y)^T \sim \Phi_\rho,\ -1 < \rho < 1$*, then*

$$P(max(X,Y) < z) = 2\Phi(A_4(z)),\ -\infty < z < \infty,$$

with

$$A_4(z) = \{(x,y)^T \in \mathbb{R}^2 : y > 0, \sqrt{\frac{1+\rho}{2}}x + \sqrt{\frac{1-\rho}{2}}y < z\}.$$

For an illustration of the set $A_4(z)$, we refer to Figure 4.

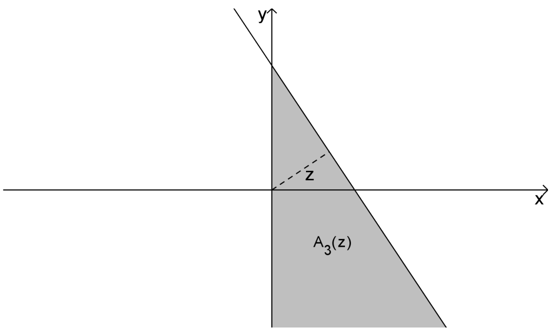

Figure 3: The set $A_3(z)$ for $z > 0, \delta > 0$.

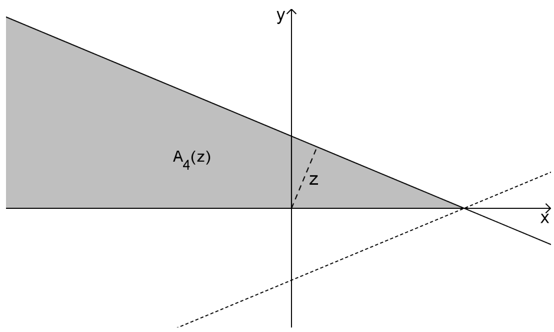

Figure 4: The set $A_4(z)$ for $z > 0, \rho > 0$.

3 The geometric-measure theoretic approach to the two-dimensional Gaussian law and its generalization

To simplify matters for the reader who is possibly not yet familiar with the geometric measure representation of the multivariate standard Gaussian law, we give here a short introduction to this representation in the case of dimension two. First of all, let us recall the famous principle of Cavalieri (1635) for comparing the area content of two regions R_1, R_2 of dimension two. Let R_1 and R_2 be located between two parallel lines in the Euclidean plane $\mathbb{R}^2$ as in Figure 5. If every line l parallel to and between these two lines intersects both R_1 and R_2 in line segments of equal lengths, then the two regions have equal area contents.
The line segments $l \cap R_1$ and $l \cap R_2$ are called the indivisibles of the sets R_1 and R_2, respectively, and the principle of Cavalieri is often called the method of indivisibles, too. A modification of this method which uses arc segments of circles $S(r) = \{(x,y)^T \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ as indivisibles is due to Torricelli, see Figure 6.

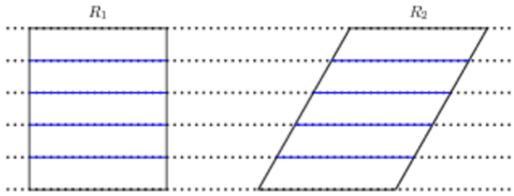

Figure 5: The principle of Cavalieri

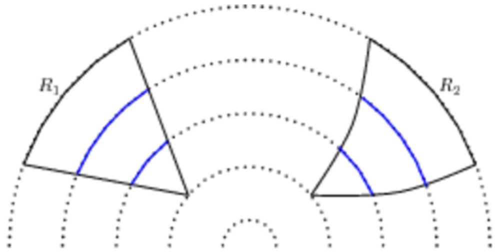

Figure 6: The modification of Torricelli

The method of weighted indivisibles was introduced in [9] for the (n-dimensional) Gaussian law and extended in [10] to the case of spherical distributions. The correctness of this method is proved, using modern measure and integration theory including the theorem of Fubini. The weights of the indivisibles are the values which the density function

$$\phi(x,y) = \frac{1}{2\pi} exp\{-\frac{x^2}{2} - \frac{y^2}{2}\},\ (x,y) \in \mathbb{R}^2,$$

attains on the indivisibles, i.e. $\frac{1}{2\pi} exp\{-\frac{r^2}{2}\}$ on $S(r)$, times the Jacobian r of the well known polar coordinate transformation. It turns out that the standard Gaussian measure Φ satisfies the representation formula

$$\Phi(A) = \frac{1}{2\pi} \int_0^\infty l(A \cap S(r)) e^{-\frac{r^2}{2}}\, dr,\ A \in \mathfrak{B}^2,$$

where $l(\cdot)$ denotes the Euclidean arc length. It is common to rewrite this representation as

$$\Phi(A) = \int_0^\infty \mathcal{F}(A,r) r e^{-\frac{r^2}{2}}\, dr,\ A \in \mathfrak{B}^2, \tag{3}$$

with the so-called intersection percentage function (ipf) of the set A:

$$\mathcal{F}(A,r) = \omega([\frac{1}{r}A] \cap S),\ A \in \mathfrak{B}^2, r > 0,$$

where

$$\omega(M) = l(M)/2\pi,\ M \in \mathfrak{B}(1) = \mathfrak{B}^1 \cap S,$$

denotes the uniform probability distribution on $S = S(1)$ and

$$\frac{1}{r}A = \{(\frac{x}{r}, \frac{y}{r})^T : (x,y)^T \in A\},\ r > 0.$$

Formula (3) will be called the geometric measure representation of Φ or its indivisibles representation. Note that equations (2) and (3) are closely connected because $U \sim \omega$ and they "reflect the two sides of one and the same medal". Formula (3) was extended to the class of spherical distributions in [10]. A two-dimensional random vector is called spherically distributed with continuous density generating function (dgf) $g : \mathbb{R}^+ \to \mathbb{R}^+$ if its density is

$$\phi(x,y;g) = C(g) g(x^2+y^2),\ (x,y) \in \mathbb{R}^2,$$

where the normalizing constant is

$$C(g) = 1/\left(2\pi \int_0^\infty r g(r^2)\, dr\right)$$

and the integral $I(g) = \int_0^\infty rg(r^2)\,dr$ is assumed to satisfy the inequalities $0 < I(g) < \infty$. The uniquely defined one-dimensional marginal distribution may be considered as a generalization of the normal distribution and its pdf and cdf will be denoted by $\phi^{(1)}(\cdot; g)$ and $\Phi^{(1)}(\cdot; g)$, respectively. Note that the marginal variables are uncorrelated but not independent in general. The probability measure $\Phi(\cdot; g)$ corresponding to the density $\phi(\cdot; g)$ allows the indivisibles representation

$$\Phi(A; g) = 2\pi C(g) \int_0^\infty \mathcal{F}(A, r) rg(r^2)\,dr,\ A \in \mathfrak{B}^2,$$

i.e.

$$\Phi(A; g) = \frac{1}{I(g)} \int_0^\infty \mathcal{F}(A, r) rg(r^2)\,dr,\ A \in \mathfrak{B}^2. \tag{4}$$

If a random vector $(X, Y)^T$ has the density $\phi(\cdot; g)$, then the transformed vector $(\xi, \eta)^T = M \cdot (X, Y)^T$ has the density

$$(x, y) \mapsto C(g)|det(M)|^{-1} g((x, y)(M^{-1})^T M^{-1}(x, y)^T).$$

This pdf and the corresponding cdf will be denoted by $\phi_\rho(\cdot; g)$ and $\Phi_\rho(\cdot; g)$, respectively, if the symmetric matrix MM^T equals $\left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$ with $-1 < \rho < 1$.
A considerable generalization of the method of indivisibles was proved in [11] and applied to the skewed distribution theory in [4] through exploiting the corresponding stochastic representation, which is a generalization of (2). Formula (3) applies to Propositions 1a-4a and all the results may be extended under much more general model assumptions, using formula (4). The latter will be done in the second part of the following section.

4 Geometric-measure theoretic reformulations of Propositions 1 to 4 and their generalization

4.1 The Gaussian case

In this section, at first we combine the results from Section 2 with the representation formula (3) of the standard Gaussian measure given in Section 3. For $i \in \{1, 2, 3, 4\}$, we have to determine the ipf of the set $A_i(z),\ z \in \mathbb{R}$. The corresponding elementary geometric considerations can be found in the Appendix. As announced above, it is possible to reprove the Propositions 1-4 in a new, geometric way by taking derivatives in the resulting geometric integral representations. The corresponding partly tedious calculations are also shifted to the Appendix.

Proposition 1b *If $(X, Y)^T \sim \Phi$, then*

$$P(X < z \mid \alpha X > Y) = 2 \int_0^\infty \mathcal{F}(A_1(z), r) re^{-\frac{r^2}{2}}\,dr$$

with

$$\mathcal{F}(A_1(z),r)=\begin{cases}\frac{1}{2} & \text{if } z>0,\ \alpha\geq 0,\ r\leq z\\ \frac{1}{2}-\frac{1}{\pi}f(z,r) & \text{if } z>0,\ \alpha\geq 0\ \ z<r\leq z\sqrt{\alpha^2+1}\\ \frac{1}{2}-\frac{1}{2\pi}[f(z,r)+B_\alpha] & \text{if } z>0,\ \alpha\geq 0\ \ r>z\sqrt{\alpha^2+1}\\ \frac{1}{2} & \text{if } z>0,\ \alpha<0,\ r\leq z\sqrt{\alpha^2+1}\\ \frac{1}{2}-\frac{1}{2\pi}[f(z,r)+B_\alpha] & \text{if } z>0,\ \alpha<0,\ r>z\sqrt{\alpha^2+1}\\ 0 & \text{if } z\leq 0,\ \alpha\geq 0\ \ r\leq -z\sqrt{\alpha^2+1}\\ \frac{1}{2\pi}[f(-z,r)-B_\alpha] & \text{if } z\leq 0,\ \alpha\geq 0\ \ r>-z\sqrt{\alpha^2+1}\\ 0 & \text{if } z\leq 0,\ \alpha<0,\ r\leq -z\\ \frac{1}{\pi}f(-z,r) & \text{if } z\leq 0,\ \alpha<0,\ -z<r\leq -z\sqrt{\alpha^2+1}\\ \frac{1}{2\pi}[f(-z,r)-B_\alpha] & \text{if } z\leq 0,\ \alpha<0,\ r>-z\sqrt{\alpha^2+1},\end{cases}$$

where $f(z,r)=\arccos\left(\frac{z}{r}\right)$ *and* $B_\alpha=\arctan(\alpha)$.

Reproving Proposition 1 based upon Proposition 1b makes it necessary to take the derivatives w.r.t. z of parameter integrals wherein both the integrand and the integral limits may depend on z. The Leibniz integral rule applies in all cases where it is needed in this paper. Below we use the pdf $\widetilde{g}$ from equation (1).

Corollary 1b *If* $(X,Y)^T\sim\Phi$, *then*

$$\frac{d}{dz}P(X<z\mid \alpha X>Y)=\widetilde{g}(z;\alpha),\ -\infty<z<\infty.$$

Now we consider the situation which we dealt with in Propositions 2 and 2a.

Proposition 2b *If* $(X,Y)^T\sim\Phi_\rho,\ -1<\rho<1$, *then*

$$P(Y<z\mid X>0)=2\int_0^\infty \mathcal{F}(A_2(z),r)re^{-\frac{r^2}{2}}\,dr$$

with

$$\mathcal{F}(A_2(z),r)=\begin{cases}\frac{1}{2} & \text{if } z>0,\ \ \rho\geq 0,\ \ r\leq z\\ \frac{1}{2}-\frac{1}{\pi}f(z,r) & \text{if } z>0,\ \ \rho\geq 0,\ \ z<r\leq \frac{z}{\sqrt{1-\rho^2}}\\ \frac{1}{2}-\frac{1}{2\pi}\left[f(z,r)+C_\rho\right] & \text{if } z>0,\ \ \rho\geq 0,\ \ r>\frac{z}{\sqrt{1-\rho^2}}\\ \frac{1}{2} & \text{if } z>0,\ \ \rho<0,\ \ r\leq\frac{z}{\sqrt{1-\rho^2}}\\ \frac{1}{2}-\frac{1}{2\pi}\left[f(z,r)-C_\rho\right] & \text{if } z>0,\ \ \rho<0,\ \ r>\frac{z}{\sqrt{1-\rho^2}}\\ 0 & \text{if } z\leq 0,\ \ \rho\geq 0,\ \ r\leq -\frac{z}{\sqrt{1-\rho^2}}\\ \frac{1}{2\pi}\left[f(-z,r)-C_\rho\right] & \text{if } z\leq 0,\ \ \rho\geq 0,\ \ r>-\frac{z}{\sqrt{1-\rho^2}}\\ 0 & \text{if } z\leq 0,\ \ \rho<0,\ \ r\leq -z\\ \frac{1}{\pi}f(-z,r) & \text{if } z\leq 0,\ \ \rho<0,\ \ -z<r\leq -\frac{z}{\sqrt{1-\rho^2}}\\ \frac{1}{2\pi}\left[f(-z,r)+C_\rho\right] & \text{if } z\leq 0,\ \ \rho<0,\ \ r>-\frac{z}{\sqrt{1-\rho^2}},\end{cases}$$

where $C_\rho=\arccos\sqrt{1-\rho^2}$.

In the same way as Corollary 1b was derived from Proposition 1b, the following Corollary can be proved.

Corollary 2b *If* $(X,Y)^T \sim \Phi_\rho,\ -1<\rho<1,$ *then*

$$\frac{d}{dz}P(Y<z \mid X>0) = \widetilde{g}(z;\frac{\rho}{\sqrt{1-\rho^2}}),\ -\infty<z<\infty.$$

The upcoming two statements will continue our consideration from Propositions 3 and 3a.

Proposition 3b *If* $(X,Y)^T \sim \Phi$, *then*

$$P(\delta|X|+\sqrt{1-\delta^2}Y<z) = 2\int_0^\infty \mathcal{F}(A_3(z),r)re^{-\frac{r^2}{2}}\,dr,$$

where we can get $\mathcal{F}(A_3(z),r)$ *from* $\mathcal{F}(A_2(z),r)$ *just by substituting the parameter* ρ *by the parameter* δ*.*

Corollary 3b *If* $(X,Y)^T \sim \Phi$, *then*

$$\frac{d}{dz}P(\delta|X|+\sqrt{1-\delta^2}Y<z) = \widetilde{g}(z;\frac{\delta}{\sqrt{1-\delta^2}}),\ -1<\delta<1,\ -\infty<z<\infty.$$

We now turn over to the situation of Propositions 4 and 4a.

Proposition 4b *If* $(X,Y)^T \sim \Phi_\rho,\ -1<\rho<1,$ *then*

$$P(\max(X,Y)<z) = 2\int_0^\infty \mathcal{F}(A_4(z),r)re^{-\frac{r^2}{2}}\,dr$$

with

$$\mathcal{F}(A_4(z),r) = \begin{cases} \frac{1}{2} & \text{if } z>0,\ \ 0<r\le z \\ \frac{1}{2}-\frac{1}{\pi}f(z,r) & \text{if } z>0,\ \ z<r\le \frac{\sqrt{2}z}{\sqrt{1+\rho}} \\ \frac{1}{2}-\frac{1}{2\pi}\left[f(z,r)+D_\rho\right] & \text{if } z>0,\ \ r>\frac{\sqrt{2}z}{\sqrt{1+\rho}} \\ 0 & \text{if } z\le 0,\ \ 0<r\le -\frac{\sqrt{2}z}{\sqrt{1+\rho}} \\ \frac{1}{2\pi}\left[f(-z,r)-D_\rho\right] & \text{if } z\le 0,\ \ r>-\frac{\sqrt{2}z}{\sqrt{1+\rho}}, \end{cases}$$

where $D_\rho = \arccos\sqrt{\frac{1+\rho}{2}}$*.*

Corollary 4b *If* $(X,Y)^T \sim \Phi_\rho,\ -1<\rho<1,$ *then*

$$\frac{d}{dz}P(max(X,Y)<z) = \widetilde{g}(z;\sqrt{\frac{1-\rho}{1+\rho}}),\ -\infty<z<\infty.$$

4.2 The spherical case

In the second part of this section, we present significant generalizations of Propositions 1b-4b. These generalizations extend all the results known so far for the normal distributions Φ and Φ_ρ to the much more general case of arbitrary spherical distributions $\Phi(\cdot;g)$ and $\Phi_\rho(\cdot;g)$, respectively. Here, the dgf g satisfies the assumption $0<I(g)<\infty$. For a discussion of several classes of dgf's, we refer to [8]. It follows immediately from formula (3) and Propositions 1a and 1b that Proposition 1b may be generalized as follows. Note that the ipf is taken over from Proposition 1b to Proposition 1c.

Proposition 1c *If* $(X,Y)^T \sim \Phi(\cdot; g)$, *then*

$$P(X < z \mid \alpha X > Y) = \frac{2}{I(g)} \int_0^\infty \mathcal{F}(A_1(z), r) r g(r^2)\, dr,\ z \in \mathbb{R}.$$

Analogously, the following generalizations of Propositions 2b-4b hold with the ipf in each c-Proposition being always the same as in the corresponding b-Proposition.

Proposition 2c *If* $(X,Y)^T \sim \Phi_\rho(\cdot; g),\ -1 < \rho < 1$, *then*

$$P(Y < z \mid X > 0) = \frac{2}{I(g)} \int_0^\infty \mathcal{F}(A_2(z), r) r g(r^2)\, dr,\ z \in \mathbb{R}.$$

Proposition 3c *If* $(X,Y)^T \sim \Phi(\cdot; g)$, *then*

$$P(\delta|X| + \sqrt{1-\delta^2} < z) = \frac{2}{I(g)} \int_0^\infty \mathcal{F}(A_3(z), r) r g(r^2)\, dr,\ z \in \mathbb{R}.$$

Proposition 4c *If* $(X,Y)^T \sim \Phi_\rho(\cdot; g),\ -1 < \rho < 1$, *then*

$$P(\max(X,Y) < z) = \frac{2}{I(g)} \int_0^\infty \mathcal{F}(A_4(z), r) r g(r^2)\, dr,\ z \in \mathbb{R}.$$

Again looking through the proofs of Corollaries 1-4, we find out by very slight modifications that the following corollaries of Propositions 1c-4c are true.

Corollary 1c *If* $(X,Y)^T \sim \Phi(\cdot; g)$, *then*

$$\frac{d}{dz} P(X < z \mid \alpha X > Y) = \frac{1}{\pi I(g)} \cdot \int_{-\infty}^{\alpha z} g(t^2 + z^2)\, dt,\ z \in \mathbb{R}.$$

Corollary 2c *If* $(X,Y)^T \sim \Phi_\rho(\cdot; g),\ -1 < \rho < 1$, *then*

$$\frac{d}{dz} P(Y < z \mid X > 0) = \frac{1}{\pi I(g)} \cdot \int_{-\infty}^{\frac{\rho}{\sqrt{1-\rho^2}} z} g(t^2 + z^2)\, dt,\ z \in \mathbb{R}.$$

Corollary 3c *If* $(X,Y)^T \sim \Phi(\cdot; g)$, *then*

$$\frac{d}{dz} P(\delta|X| + \sqrt{1-\delta^2} < z) = \frac{1}{\pi I(g)} \cdot \int_{-\infty}^{\frac{\delta}{\sqrt{1-\delta^2}} z} g(t^2 + z^2)\, dt,\ z \in \mathbb{R}.$$

Corollary 4c *If* $(X,Y)^T \sim \Phi_\rho(\cdot; g),\ -1 < \rho < 1$, *then*

$$\frac{d}{dz} P(\max(X,Y) < z) = \frac{1}{\pi I(g)} \cdot \int_{-\infty}^{\sqrt{\frac{1-\rho}{1+\rho}} z} g(t^2 + z^2)\, dt,\ z \in \mathbb{R}.$$

It was shown in [6] (see also formulas (3.3) and (3.5) in [7]) that

$$\frac{1}{\pi \cdot I(g)} \int_{-\infty}^{\nu} g(t^2 + z^2) dt = 2 f(z) F(\nu z),\ \ z \in \mathbb{R},$$

where f is the pdf of a suitably chosen one-dimensional elliptically contoured distribution and F the cdf of a suitably chosen (possibly different) one-dimensional elliptically contoured distribution as well. The skewness parameter ν is chosen in Corollaries 1c-4c as α, $\rho/\sqrt{1-\rho^2}$, $\delta/\sqrt{1-\delta^2}$ and $\sqrt{(1-\rho)/(1+\rho)}$, respectively. Hence, each of the stochastic representations of the skewed normal distribution, stated in Propositions 1-4, has been extended in Corollaries 1c-4c to a stochastic representation of a much more general skewed elliptically contoured distribution.

5 Geometric representation of the skewed normal distribution

In this Section, we discuss a generalization of the considered representations in the previous sections. Let $(X,Y)^T \sim \Phi$. We have studied so far four cases, which can be written in the following way:

1. $2P(X < z, \alpha X > Y) = P(Z < z)$,
2. $2P(\sqrt{\frac{1+\rho}{2}}X - \sqrt{\frac{1-\rho}{2}}Y < z, \sqrt{1+\rho}X + \sqrt{1-\rho}Y > 0) = P(Z < z)$,
3. $2P(\delta X + \sqrt{1-\delta^2}Y < z, X > 0) = P(Z < z)$,
4. $2P(\sqrt{\frac{1+\rho}{2}}X + \sqrt{\frac{1-\rho}{2}}Y < z, Y > 0) = P(Z < z)$,

where $Z \sim SN(\nu)$ with the appropriate skewness parameter ν in each case.
These four representations of the skewed normal distribution are special cases of the general stochastic representation

$$2P(aX + bY < 0, cX + dY < e) = P(Z < z),$$

which holds true for a skewed normally distributed random variable Z with $Z \sim SN(\nu)$ if the quintuple (a,b,c,d,e) and z satisfy the conditions

$$z = \frac{e}{\sqrt{c^2 + d^2}} \tag{5}$$

and

$$\nu = \begin{cases} \frac{ac+bd}{ad-bc}, \text{ if } \ ad - bc < 0 \\ -\frac{ac+bd}{ad-bc}, \text{ if } \ ad - bc > 0. \end{cases} \tag{6}$$

Under the same assumptions, it holds

$$P(cX + dY < e | aX + bY < 0) = P(Z < z).$$

Thus, the following theorem has already been motivated by these four examples.

Theorem 1 *If $(X,Y)^T \sim \Phi$, then*

$$\mathfrak{L}\left(\frac{cX + dY}{\sqrt{c^2 + d^2}} | aX + bY < 0\right) = SN(\nu)$$

for all quadruples (a,b,c,d) satisfying (6).

Remark 1 *Theorem 1 follows from Theorem 2.*

Table 1 summarizes our study of the four cases considered in the previous sections and presents the quadruples (a,b,c,d) and the skewness parameter ν corresponding to the Propositions 1-4. The statement of Theorem 1 may be reformulated as follows.

Remark 2 *If $(X,Y)^T \sim \Phi$, then*

$$2P\left(\frac{cX + dY}{\sqrt{c^2 + d^2}} < z, aX + bY < 0\right) = P(Z < z)$$

with $Z \sim SN(\nu)$ if the quintuple (a,b,c,d,e) and z satisfy conditions (5) and (6).

	a	b	c	d	ν
Proposition 1	$-\alpha$	1	1	0	α
Proposition 2	$-\sqrt{1+\rho}$	$-\sqrt{1-\rho}$	$\frac{\sqrt{1+\rho}}{\sqrt{2}}$	$-\frac{\sqrt{1-\rho}}{\sqrt{2}}$	$\frac{\rho}{\sqrt{1-\rho^2}}$
Proposition 3	-1	0	δ	$\sqrt{1-\delta^2}$	$\frac{\delta}{\sqrt{1-\delta^2}}$
Proposition 4	0	-1	$\frac{\sqrt{1+\rho}}{\sqrt{2}}$	$\frac{\sqrt{1-\rho}}{\sqrt{2}}$	$\sqrt{\frac{1+\rho}{1-\rho}}$

Table 1: Parameters a, b, c, d and ν, corresponding to Propositions 1-4.

Another way to formulate this result makes use of more geometric quantities. Let

$$H_1(a,b) = \{(x,y)^T \in \mathbb{R}^2 : ax + by < 0\}$$

and

$$H_2(c,d,e) = \{(x,y)^T \in \mathbb{R}^2 : cx + dy < e\}$$

denote two half spaces of $\mathbb{R}^2$ and let the cone

$$C(a,b,c,d,e) = H_1(a,b) \cap H_2(c,d,e)$$

be their intersection. Let us recall that a set C is called a cone with vertex in $v \in \mathbb{R}^2$ iff for all $x \in C - v$ and $\lambda \geq 0$ follows that $v + \lambda x \in C$.
Note that $(-\frac{be}{ad-bc}, \frac{ae}{ad-bc})^T$ is the vertex of the cone $C(a,b,c,d,e)$, the origin belongs to the boundary $\partial H_1(a,b)$ and that $\partial H_2(c,d,e)$ has distance $\frac{|e|}{\sqrt{c^2+d^2}}$ from $(0,0)^T$. If $(a,b)^T$ and $(c,d)^T$ are linear independent vectors from $\mathbb{R}^2$, then the lines $\partial H_1(a,b)$ and $\partial H_2(c,d,e)$ are not parallel. This assumption is equivalent to the condition

$$ad - bc \neq 0, \tag{7}$$

which has already been assumed to be satisfied within the condition (6). Therefore, the following Theorem 2 may be considered just as a reformulation of Theorem 1.

Theorem 2 *If (5), (6) and (7) are satisfied, then*

$$2\frac{d}{dz}\Phi(C(a,b,c,d,e)) = \widetilde{g}(z,\nu), z \in \mathbb{R}.$$

Proof We take into account that Φ is a spherical distribution. Hence, if O is an orthogonal 2×2-matrix and $A \in \mathfrak{B}^2$, then

$$\Phi(OA) = \Phi(A). \tag{8}$$

We note that the cone $C(a,b,c,d,e)$ can be rewritten as follows:

$$C(a,b,c,d,e) = C^*(\theta,\phi,z) := \left\{(x,y)^T \in \mathbb{R}^2 \ : \ \cos(\theta)x + \sin(\theta)y < 0, \ \cos(\phi)x + \sin(\phi)y < z\right\},$$

where

$$\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}, \ \sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}, \ \cos(\phi) = \frac{c}{\sqrt{c^2+d^2}}, \ \sin(\phi) = \frac{d}{\sqrt{c^2+d^2}} \tag{9}$$

and z is given by (5). The angles θ and ϕ are unique. From the equations (9), it follows by trigonometric addition theorems that $\sin(\theta-\phi)>0$ is equivalent to $ad-bc<0$ and apart from that

$$-\frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}=\frac{ac+bd}{ad-bc}. \tag{10}$$

Case A: Let $ad-bc<0$, then $\sin(\theta-\phi)>0$. Defining $O_1:=\begin{pmatrix}\cos(\phi) & \sin(\phi)\\ -\sin(\phi) & \cos(\phi)\end{pmatrix}$, we check with the help of trigonometric addition theorems that

$$O_1C^*(\theta,\phi,z)=\left\{(x,y)^T\in\mathbb{R}^2\ :\ x\ <z\ ,\ y\ <-\frac{\cos(\theta-\phi)}{\sin(\theta-\phi)}x\right\}. \tag{11}$$

We recall that if $Z\sim SN(\nu)$, then it holds

$$\int_{-\infty}^{z}2\phi^{(1)}(x)\Phi^{(1)}(\nu x)\ dx=P(Z<z). \tag{12}$$

Further, we have

$$2\Phi(\underbrace{\left\{(x,y)^T\in\mathbb{R}^2\ :\ x<z,\ y<\nu x\right\}}_{=:\widetilde{C}(\nu,z)})=\int_{-\infty}^{z}2\phi^{(1)}(x)\Phi^{(1)}(\nu x)\ dx. \tag{13}$$

It follows that

$$2\Phi(C(a,b,c,d,e))=2\Phi(C^*(\theta,\phi,z))\overset{(8)}{=}2\Phi(O_1C^*(\theta,\phi,z))\underset{(6)}{\overset{(10)}{=}}2\Phi(\widetilde{C}(\nu,z))\underset{(12)}{\overset{(13)}{=}}P(Z<z).$$

Case B: Let $ad-bc>0$, then $\sin(\theta-\phi)<0$. We define $O_2:=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and consider $O_2O_1C^*(\theta,\phi,z)$ instead of $O_1C^*(\theta,\phi,z)$ in case A. Note that the condition $\sin(\theta-\phi)<0$ changes the representation (11) of the set $O_1C^*(\theta,\phi,z)$ for case B. Analogously to case A , we make use of (6), (8), (10), (12) and (13) to complete the proof in case B. ■

It is worth noting that there is a close connection between the parameter ν given by (6) and the angle ψ between the vectors $(-a,-b)^T$ and $(c,d)^T$, which can be considered as the opening angle of the cone $C(a,b,c,d,e)$. For clarification, we note that $(-a,-b)^T$ is a normal vector of $\partial H_1(a,b)$ which is directed into the half space $H_1(a,b)$, and $(c,d)^T$ is a normal vector of $\partial H_2(c,d,e)$ which is directed away from $H_2(c,d,e)$. Then

$$\nu=\cot(\psi). \tag{14}$$

Remark 3 *In other words, if $Z\sim SN(\cot\psi)$, then its cdf allows the representation*

$$P(Z<z)=2\Phi(C(a,b,c,d,e))$$

for all (a,b,c,d,e) satisfying (5), (6) and (14).

Against the backdrop of Theorem 2 and formula (14), we again want to focus on the sets $A_i(z),\ i\in\{1,2,3,4\}$, which were defined in Section 2. Looking at all the sets $A_i(z),\ i\in\{1,2,3,4\}$, at the same time, we can figure out some similarity between them. Each of these sets can be considered as a cone $C(a,b,c,d,e)$, where the appropriate parameters a,b,c,d are

i	a	b	c	d
1	$-\nu$	1	1	0
2	$-\sqrt{1+\frac{\nu}{\sqrt{1+\nu^2}}}$	$-\sqrt{1-\frac{\nu}{\sqrt{1+\nu^2}}}$	$\frac{1}{\sqrt{2}}\sqrt{1+\frac{\nu}{\sqrt{1+\nu^2}}}$	$-\frac{1}{\sqrt{2}}\sqrt{1-\frac{\nu}{\sqrt{1+\nu^2}}}$
3	-1	0	$\frac{\nu}{\sqrt{1+\nu^2}}$	$\frac{1}{\sqrt{1+\nu^2}}$
4	0	-1	$\frac{1}{\sqrt{1+\nu^2}}$	$\frac{\nu}{\sqrt{1+\nu^2}}$

Table 2: Chosen parameters a, b, c, d, so that ν is one and the same for each set $A_i(z)$, $i \in \{1, 2, 3, 4\}$

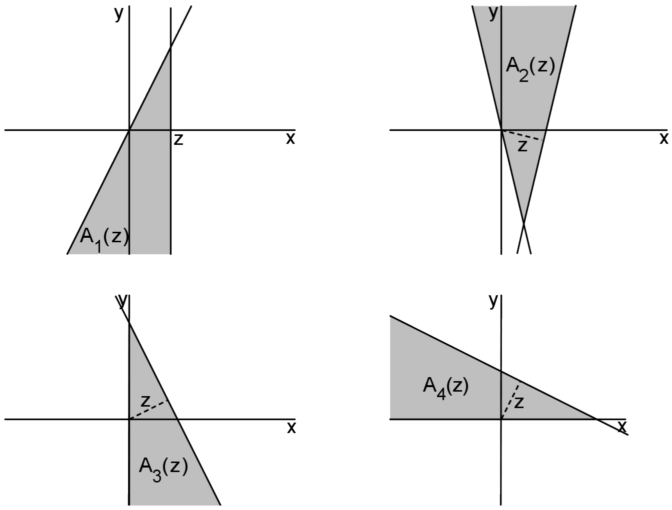

Figure 7: The sets $A_i(z)$, $i \in \{1, 2, 3, 4\}$, with appropriate parameters.

given by Table 1 and the parameter e is set equal to z. However, we can choose the parameters alternatively in such a way that the corresponding parameter ν, given by (6), is one and the same for each set $A_i(z)$, $i \in \{1, 2, 3, 4\}$, see Table 2.
Due to formula (14), the opening angle ψ of each of the cones $A_i(z)$, $i \in \{1, 2, 3, 4\}$, is one and the same if the parameters are chosen according to Table 2. For an illustration, we refer to Figure 7, where the parameter ν equals 2 and the value of z is 1 for each set. As suggested in Figure 7, it is possible to map each set $A_i(z)$, $i \in \{1, 2, 3, 4\}$, onto each other via an orthogonal transformation, that is a rotation, a mirroring or a composition of them. Hence, there is even an isometry between the sets. To show some examples for this, the transformations of $A_i(z)$, $i \in \{2, 3, 4\}$, onto the set $A_1(z)$ will be given in the Example in the Appendix. The importance of the latter statement arises from formula (8), which was the starting point of the proof of Theorem 2. In fact, the last paragraph may be considered as a discussion of Theorem 2 for the special cones $A_i(z)$, $i \in \{1, 2, 3, 4\}$, but also gives an overview about the relations between the Propositions 1-4 from a geometric point of view.

6 Concluding remarks and directions of future research

As announced in the Introduction, this paper presents a more general geometric-measure theoretic representation of the skewed normal law than how it would just follow from Propositions 1-4, see Theorem 2. An equivalent stochastic representation of the skewed normal law is given in Theorem 1. These results may be considered as just some first representations of this geometric-stochastic type. Among other similar possible results, which may be proved in the future, are representations based upon one of the assumptions that $(X,Y)^T$ is distributed according to the Φ_ρ-distribution, an arbitrary elliptically contoured distribution or an $l_{2,p}$-symmetric distribution. In the higher dimensional case, one may ask whether intersections of half spaces are the natural generalization of the cone $C(a,b,c,d,e)$.

Acknowledgements
The authors are grateful to the reviewer for the valuable remarks.
The second author would like to express his gratitude to M.D. Branco and R.B. Arellano-Valle for the stimulating discussion at the beginning of the work on the present paper during his stay at the University of Sao Paulo, in August 2010.
All authors are grateful to R.B. Arellano-Valle and A. Azzalini for their valuable hints to this work, given during the stay of A. Azzalini in Rostock in September 2011 and the stay of Th. Günzel and K. Schicker in Santiago de Chile and St. Scheutzow and J. Venz in Padua in October 2011.

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Appendix: Proofs

Remark 4 *The following proofs of Propositions 1b, 2b and 4b together with the corresponding Corollaries are shortened versions. To understand the method in proving these Propositions and Corollaries in more detail, see the proofs of Proposition and Corollary 3b first.*

Proof of Proposition 1b We have to calculate the ipf $\mathcal{F}$ of the set $A_1(z)$. By using the theorem of Pythagoras, we distinguish for $z > 0,\ \alpha \geq 0$ between three cases. For $r \leq z$, the ipf is obviously $\frac{1}{2}$. For $z < r \leq z\sqrt{\alpha^2+1}$, we get $\mathcal{F}$ by considering the angle ψ between the $x-$axis and the line segment between the origin and the intersection of the line $x = z$ with the circle $S(r)$. We state that $\cos(\psi) = \frac{z}{r}$. If $r > z\sqrt{\alpha^2+1}$, we use vertically opposed angles and trigonometric functions to get $\mathcal{F}$. In case of $z \leq 0,\ \alpha \geq 0$ and in case of $\alpha < 0$, one can get the ipf by similar calculations. ∎

Proof of Corollary 1b Using Proposition 1b for $z > 0,\ \alpha \geq 0$, one will get

$$\frac{d}{dz}P(X < z \mid \alpha X > Y) = \frac{d}{dz}\left(-\frac{1}{\pi}\int\limits_{z}^{\infty} \arccos\left(\frac{z}{r}\right) re^{-r^2/2}\,dr\right)$$
$$+\frac{d}{dz}\left(-\frac{1}{\pi}\int\limits_{z}^{z\sqrt{\alpha^2+1}} \arccos\left(\frac{z}{r}\right) re^{-r^2/2}\,dr\right) + \frac{d}{dz}\left(-\frac{1}{\pi}\int\limits_{z\sqrt{\alpha^2+1}}^{\infty} \arctan\left(\alpha\right) re^{-r^2/2}\,dr\right).$$

Using the Leibniz integral rule, it follows that

$$\frac{d}{dz}P(X < z \mid \alpha X > Y) = \frac{1}{\pi}\int\limits_{z}^{\infty} \frac{1}{\sqrt{r^2 - z^2}} r e^{-r^2/2}\, dr + \frac{1}{\pi}\int\limits_{z}^{z\sqrt{\alpha^2+1}} \frac{1}{\sqrt{r^2 - z^2}} r e^{-r^2/2}\, dr.$$

Expending this expression with $\frac{\sqrt{2\pi}e^{-z^2/2}}{\sqrt{2\pi}e^{-z^2/2}}$ and using the substitution $t = \sqrt{r^2 - z^2}$ afterwards, we get

$$\frac{d}{dz}P(X < z \mid \alpha X > Y) = 2\phi^{(1)}(z) \cdot \Phi^{(1)}\left(\alpha z\right) = \tilde{g}(z;\alpha).$$

In an analogously way, the result is proved for $z \leq 0,\ \alpha \geq 0$ and for $\alpha < 0$ by using the same rules and substitutions given above. ∎

Proof of Proposition 2b We have to calculate the ipf $\mathcal{F}$ of the set $A_2(z)$. For $z \geq 0$ and $\rho < 0$, we distinguish between two cases. If $r \leq \frac{z}{\sqrt{1-\rho^2}}$, the ipf reduced obviously to $\frac{1}{2}$. If $r > \frac{z}{\sqrt{1-\rho^2}}$, it follows with the help of trigonometric functions that $\psi = \arccos\left(\frac{z}{r}\right) - \arccos(\sqrt{1-\rho^2})$ describes the part of the circle which is additional outside the set $A_2(z)$. The ipf for $z \geq 0$, $\rho \geq 0$ and for $z < 0$ can be obtained by similar calculations. ∎

Proof of Corollary 2b Using Proposition 2b for $z \geq 0$ and $z < 0$ with $\rho \geq 0$ and $\rho < 0$, respectively, one will get the claim of the Corollary using the Leibniz integral rule and the substitution $t = \sqrt{r^2 - z^2}$ again. ∎

Proof of Proposition 3b

We perform the proof for the case $\delta \geq 0$. In case of $\delta < 0$, we can get the result by similar calculations.

1. **Case ($z > 0$, $\delta \geq 0$, $0 < r \leq z$)** For this case it is obvious, that 50 percent of the sphere is in $A_3(z)$, so $\mathcal{F}(A_3(z), r) = \frac{1}{2}$.

2. **Case ($z > 0$, $\delta \geq 0$, $z < r \leq \frac{z}{\sqrt{1-\delta^2}}$)** It is

$$\cos\beta = \frac{z}{r}$$

and because of the symmetry of the circle, it follows

$$\mathcal{F}(A_3(z), r) = \frac{1}{2} - \frac{2\arccos\left(\frac{z}{r}\right)r}{2\pi r} = \frac{1}{2} - \frac{\arccos\left(\frac{z}{r}\right)}{\pi}$$

3. **Case ($z > 0$, $\delta \geq 0$, $\frac{z}{\sqrt{1-\delta^2}} < r$)** The following geometrical aspects are to consider:

$$\beta'' = \arccos\left(\frac{z}{\frac{z}{\sqrt{1-\delta^2}}}\right) = \arccos\left(\sqrt{1-\delta^2}\right),\ \beta' = \arccos\left(\frac{z}{r}\right),\ \beta = \pi - \beta' - \beta''.$$

Now it follows

$$\begin{aligned}\mathcal{F}(A_3(z), r) &= \frac{1}{2} - \frac{\arccos\left(\sqrt{1-\delta^2}\right)r}{2\pi r} - \frac{\arccos\left(\frac{z}{r}\right)r}{2\pi r} \\ &= \frac{1}{2} - \frac{\arccos\left(\sqrt{1-\delta^2}\right)}{2\pi} - \frac{\arccos\left(\frac{z}{r}\right)}{2\pi}.\end{aligned}$$

For an illustration of cases 1-3, see Figure 8.

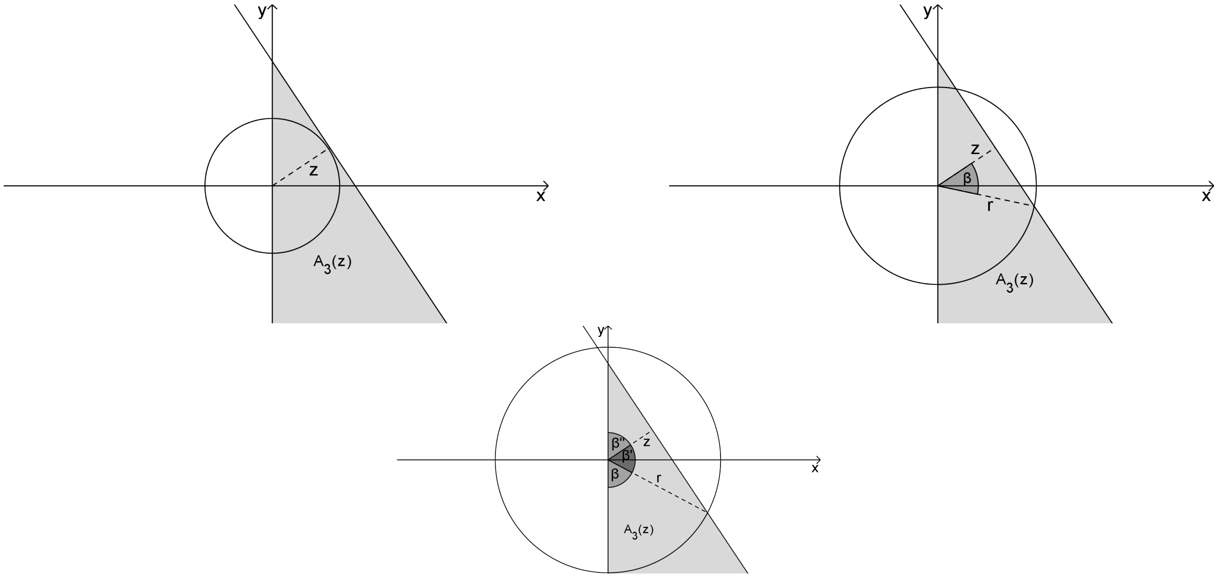

Figure 8: Illustration of cases 1-3, Proposition 3b

4. **Case** $(z \leq 0,\ \delta \geq 0,\ 0 < r \leq -\frac{z}{\sqrt{1-\delta^2}})$ For this case, there is nothing in $A_3(z)$, so $\mathcal{F}(A_3(z), r) = 0$.

5. **Case**$(z \leq 0,\ \delta \geq 0,\ -\frac{z}{\sqrt{1-\delta^2}} < r)$ With the geometrical aspects

$$\beta' = \arccos \frac{-z}{-z}\sqrt{1-\delta^2} = \arccos \sqrt{1-\delta^2},\ \beta'' = \arccos \frac{-z}{r},\ \beta = \beta'' - \beta',$$

it follows

$$\mathcal{F}(A_3(z), r) = \frac{\arccos \frac{-z}{r} - \arccos \sqrt{1-\delta^2}}{2\pi}.$$

For an illustration of cases 4-5, see Figure 9. ■

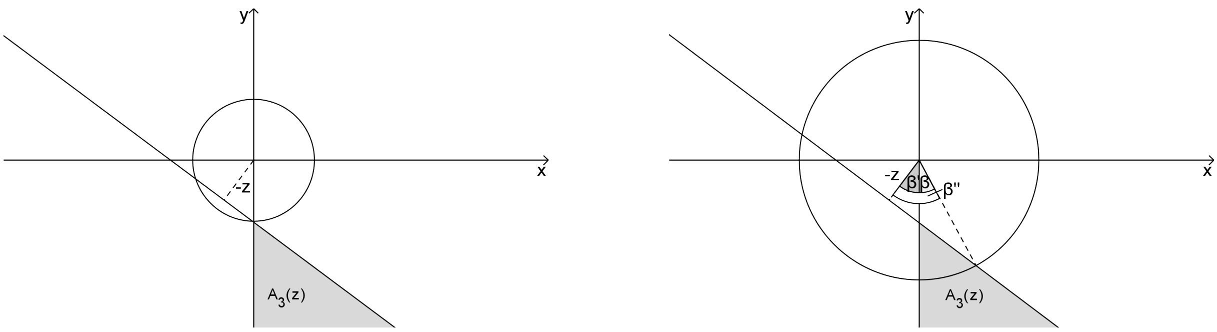

Figure 9: Illustration of cases 4-5, Proposition 3b

Proof of Corollary 3b Using Proposition 3b for $z > 0,\ \delta \geq 0$, one will get

$$\frac{d}{dz}P(\delta|X|+\sqrt{1-\delta^2}Y<z)=2\frac{d}{dz}\int\limits_0^{\infty}\mathcal{F}(A_3(z),r)re^{-\frac{r^2}{2}}dr$$

$$=\frac{d}{dz}\Big[\underbrace{\int\limits_0^{\infty}re^{-\frac{r^2}{2}}dr}_{=1}-\frac{2}{\pi}\int\limits_z^{\frac{z}{\sqrt{1-\delta^2}}}\arccos{(\frac{z}{r})}re^{-\frac{r^2}{2}}dr$$

$$-\frac{1}{\pi}\int\limits_{\frac{z}{\sqrt{1-\delta^2}}}^{\infty}\left(\arccos{(\sqrt{1-\delta^2})}+\arccos{(\frac{z}{r})}\right)re^{-\frac{r^2}{2}}dr\Big](*).$$

Using Leibniz' rule under the integral sign, it follows that

$$(*)=-\frac{2}{\pi}\Big[\int\limits_z^{\frac{z}{\sqrt{1-\delta^2}}}\left(\frac{d}{dz}\arccos{(\frac{z}{r})}\right)re^{-\frac{r^2}{2}}dr$$

$$+\arccos{(\sqrt{1-\delta^2})}\frac{z}{1-\delta^2}e^{-\frac{z^2}{2(1-\delta^2)}}-\underbrace{\arccos{(1)}}_{=0}ze^{-\frac{z^2}{2}}\Big]$$

$$-\frac{1}{\pi}\Big[\int\limits_{\frac{z}{\sqrt{1-\delta^2}}}^{\infty}\left(\frac{d}{dz}\arccos{(\frac{z}{r})}\right)re^{-\frac{r^2}{2}}dr-2\arccos{(\sqrt{1-\delta^2})}\frac{z}{1-\delta^2}e^{-\frac{z^2}{2(1-\delta^2)}}\Big]$$

$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\cdot\Big[\frac{\sqrt{2}}{\sqrt{\pi}}\int\limits_z^{\frac{z}{\sqrt{1-\delta^2}}}\frac{1}{\sqrt{r^2-z^2}}re^{-\frac{r^2-z^2}{2}}dr\;+\;\frac{\sqrt{2}}{\sqrt{\pi}}\int\limits_z^{\infty}\frac{1}{\sqrt{r^2-z^2}}re^{-\frac{r^2-z^2}{2}}dr\Big](**).$$

With the substitution

$$t=\sqrt{r^2-z^2},\;\frac{dt}{dr}=\frac{r}{\sqrt{r^2-z^2}}$$

prove that

$$(**)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\cdot 2\Big[\frac{1}{\sqrt{2\pi}}\int\limits_0^{\frac{\delta}{\sqrt{1-\delta^2}}z}e^{-\frac{t^2}{2}}dt\;+\;\underbrace{\frac{1}{\sqrt{2\pi}}\int\limits_0^{\infty}e^{-\frac{t^2}{2}}dt}_{=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{0}e^{-\frac{t^2}{2}}dt}\Big]$$

$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\cdot 2\Big[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\frac{\delta}{\sqrt{1-\delta^2}}z}e^{-\frac{t^2}{2}}dt\Big]=2\phi^{(1)}(z)\Phi^{(1)}\left(\frac{\delta}{\sqrt{1-\delta^2}}z\right),\text{ for }0<z<\infty.$$

In an analogous way, the result is proved for $-\infty<z\leq 0,\ \delta\geq 0$ and in case of $\delta<0$ by using the same rules and substitutions given above. ∎

Proof of Proposition 4b
To calculate $\mathcal{F}(A_4(z),r)$, we have to distinguish between the cases $(z>0,\ 0\le r\le z)$, $(z>0,\ z<r\le \frac{\sqrt{2}z}{\sqrt{1+\rho}})$, $(z>0,\ r>\frac{\sqrt{2}z}{\sqrt{1+\rho}})$, $(z\le 0,\ 0<r\le -\frac{\sqrt{2}z}{\sqrt{1+\rho}})$ and $(z\le 0,\ r>-\frac{\sqrt{2}z}{\sqrt{1+\rho}})$. In the first case, the ipf is obviously $\frac{1}{2}$. In all other cases, we have to consider suitable trigonometric relations being similar to those in the proof of Proposition 3b for deriving the ipf, stated in the proposition. ∎

Proof of Corollary 4b Using Proposition 4b for $z>0$, one will get

$$\begin{aligned}\frac{d}{dz}P(\max(X_0,X_1)\le z) =& \frac{d}{dz}\left[\underbrace{\int_0^\infty re^{-\frac{r^2}{2}}\,dr}_{=1} - \frac{2}{\pi}\int_z^{\frac{\sqrt{2}z}{\sqrt{1+\rho}}} \arccos\left(\frac{z}{r}\right) re^{-\frac{r^2}{2}}\,dr\right]\\ &-\frac{1}{\pi}\frac{d}{dz}\int_{\frac{\sqrt{2}z}{\sqrt{1+\rho}}}^\infty \left(\arccos\left(\frac{z}{r}\right)+\arccos\sqrt{\frac{1+\rho}{2}}\right) re^{-\frac{r^2}{2}}\,dr.\end{aligned}$$

Using Leibniz' integral rule, it follows that

$$\frac{d}{dz}P(\max(X_0,X_1)\le z) = \frac{2}{\pi}\int_z^{\frac{\sqrt{2}z}{\sqrt{1+\rho}}} \frac{re^{-\frac{r^2}{2}}}{\sqrt{r^2-z^2}}\,dr + \frac{1}{\pi}\int_{\frac{\sqrt{2}z}{\sqrt{1+\rho}}}^\infty \frac{re^{-\frac{r^2}{2}}}{\sqrt{r^2-z^2}}\,dr.$$

With the substitution $t=\sqrt{r^2-z^2}$, it follows that

$$\frac{d}{dz}P(\max(X_0,X_1)\le z) = 2\phi^{(1)}(z)\Phi^{(1)}\left(\sqrt{\frac{1-\rho}{1+\rho}}z\right),\ 0<z<\infty.$$

In an analogous way, the result is proved for $-\infty<z\le 0$ by using the same rules and substitutions given above. ∎

Remark 5 *Looking through the proof of Corollary 3b once more, it can be seen that only small changes are necessary to prove Corollary 3c. Namely, one has just to substitute in several integrals the function* $r\to e^{-\frac{r^2}{2}}$ *by the function* $r\to g(r^2)$*. The same holds true for Corollaries 1c, 2c and 4c.*

Example: We show the transformation of $A_i(z),\ i\in\{2,3,4\}$, onto the set $A_1(z)$. We recall the definition of the sets $A_i(z),\ i\in\{1,2,3,4\}$, in Section 2. We now write $A_1(z;\alpha):=A_1(z),\ A_2(z;\rho):=A_2(z),\ A_3(z;\delta):=A_3(z),\ A_4(z;\rho)=A_4(z)$, where α,ρ,δ are the corresponding parameters to the respective sets with reference to the definition of the sets in Section 2.
If $M_2:=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\rho} & -\sqrt{1-\rho}\\ -\sqrt{1-\rho} & -\sqrt{1+\rho}\end{pmatrix}$, then $A_1(z;\alpha)=M_2\cdot A_2(z;\rho)$, where $\alpha=\frac{\rho}{\sqrt{1-\rho^2}}$.
If $M_3:=\begin{pmatrix}\delta & \sqrt{1-\delta^2}\\ -\sqrt{1-\delta^2} & \delta\end{pmatrix}$, then $A_1(z;\alpha)=M_3\cdot A_3(z;\delta)$, where $\alpha=\frac{\delta}{\sqrt{1-\delta^2}}$.
If $M_4:=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\rho} & \sqrt{1-\rho}\\ \sqrt{1-\rho} & -\sqrt{1+\rho}\end{pmatrix}$, then $A_1(z;\alpha)=M_4\cdot A_4(z;\rho)$, where $\alpha=\sqrt{\frac{1-\rho}{1+\rho}}$.
The matrices $M_i,\ i\in\{2,3,4\}$, are orthogonal matrices.