Linear combinations, products and ratios of simplicial or spherical variates

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Abstract. The density level sets of the two types of measures under consideration are $l_{2,p}$-circles with $p = 1$ and $p = 2$, respectively. The intersection-percentage function (ipf) of such a measure reflects the percentages which the level set corresponding to the p-radius r shares for each $r > 0$ with a set to be measured. The geometric measure representation formulae in (Richter, 2009) is based upon these ipf's and will be used here for evaluating exact cdf's and pdf's for the linear combination, the product and the ratio of the components of two-dimensional simplicial or spherically distributed random vectors.

Keywords: linear combinations of random variables; products of random variables; ratios of random variables; spherical distributions; simplicial distributions; geometric measure representation; intersection-percentage function; $l_{2,p}$-generalized arc-length measure; generalized uniform distribution; exact statistical distributions; density generating function; Cauchy distribution.

MSC2010: 62E15, 62H10

1 Introduction

The distributions of the product $X_1 X_2$, the ratio X_1/X_2 and the linear combination $\alpha_1 X_1 + \alpha_2 X_2$ of the random variables X_1 and X_2 apply in both theoretical and practical fields of modern sciences. To illustrate this fact, we refer to some of the examples from (Nadarajah, 2005a) and (Nadarajah and Gupta, 2005).
Consider a share from a foreign stock exchange, which is quoted in a different currency. The profit of an investment in such a paper depends on both the price of a share in the local market and the exchange rate and can be modeled as the product of these two random variables. A rather theoretical example for the product comes up by using a regression model $Y = \alpha + \beta X_1$ for making a forecast. Here, the estimator of β is a random variable X_2 depending on Y and X_1.
The ratios of random variables are inter alia of interest in nuclear physics and in genetics. In the former case, the interest lies in the ratio of the mass difference X_1 between product and educt of a chemical reaction and the energy X_2, which is released in such a reaction. In genetics, Mendelian inheritance ratios can be modeled as the ratios of random variables.
Linear combinations of random variables are of great importance in economic multiple factor models. Consider for example the Arbitrage Pricing Theory, which models the price of a share as a linear combination of different microeconomic and macroeconomic factors.
The general interest in products, ratios and linear combinations of random variables is also reflected by the fact that various works deal with a detailed study of the distributions of these statistics, see for example (Nadarajah, 2005b).

A random vector $X = (X_1, X_2)$ with a density

$$\varphi_{g,p}(x) = C_{2,g,p}\; g\left(|x|_p^p\right), \; x \in \mathbb{R}^2,$$

is said to have a continuous $l_{2,p}$-symmetric distribution $\Phi_{g,p}$, where $C_{2,g,p}$ is a normalizing constant, g is a density generating function and $|(x_1,x_2)|_p^p := |x_1|^p + |x_2|^p$, $(x_1,x_2) \in \mathbb{R}^2$. The random vector X has a simplicial contoured distribution if $p=1$ and a spherically contoured distribution if $p=2$. Basic properties of continuous $l_{n,p}$-symmetric distributions have been studied, e.g., in (Fang et al., 1990) and in (Richter, 2009), where it was shown that a random vector $X \sim \Phi_{g,p}$ satisfies for $p>0$ the representation

$$X \stackrel{d}{=} \mathcal{R}_p \mathcal{U}_p\,. \tag{1}$$

Thereby, $\mathcal{R}_p = |X|_p$ is a generalized radius variable which is independent from a generalized uniform basis vector $\mathcal{U}_p$. Considering that $\mathbb{E}\mathcal{U}_p = 0_2$ and $\mathbb{E}\mathcal{U}_p\mathcal{U}_p^T = 1/(4-p)\, I_2$ for $p \in \{1,2\}$, see (Richter, 2009), the moments of X can be calculated using (1). To this end,

$$\mathbb{E}X = \mathbb{E}\mathcal{R}_p\, \mathbb{E}\mathcal{U}_p = 0_2$$

if $\mathbb{E}\mathcal{R}_p < \infty$, $p \in \{1,2\}$ and

$$\mathbb{E}XX^T = \mathbb{E}\mathcal{R}_p^2\, \mathbb{E}\mathcal{U}_p\mathcal{U}_p^T = \frac{1}{4-p}\, \mathbb{E}\mathcal{R}_p^2\, I_2$$

if $\mathbb{E}\mathcal{R}_p^2 < \infty$, $p \in \{1,2\}$ and where I_2 denotes the 2×2 unit matrix. In Tables 1 and 2, the density normalizing constants $C_{2,g,p}$ and the quantities $\mathbb{E}\mathcal{R}_p^2$ are given for selected subclasses of 2-dimensional spherical and simplicial distributions, respectively. These subclasses are chosen according to some often used density generating functions.

Type	$g(r^2)$	$C_{2,g,2}$	$\mathbb{E}(R_2^2)$
Kotz type	$r^{2M-2}\, e^{-\beta\, r^{2\gamma}}$; $M,\ \beta,\ \gamma \in \mathbb{R}_+$	$\frac{\gamma\, \beta^{M/\gamma}}{\pi\, \Gamma(M/\gamma)}$	$\frac{\Gamma((M+1)/\gamma)}{\Gamma(M/\gamma)\, \beta^{1/\gamma}}$
Multinormal	$e^{-\frac{1}{2}\, r^2}$	$\frac{1}{2\,\pi}$	2
Pearson type VII	$\left(1+\frac{r^2}{m}\right)^{-M}$; $M>1$, $m>0$	$\frac{M-1}{\pi\, m}$	$\frac{m}{M-2}$, $M>2$
Multivariate t	$\left(1+\frac{r^2}{m}\right)^{-m/2-1}$; $m>0$	$\frac{1}{2\,\pi}$	$\frac{2\,m}{m-2}$, $m>2$
Multivariate Cauchy	$\left(1+r^2\right)^{-3/2}$	$\frac{1}{2\,\pi}$	$-$
Pearson type II	$1_{[0,1]}(r^2)\,(1-r^2)^m$; $m>-1$	$\frac{m+1}{\pi}$	$\frac{1}{m+2}$

Table 1: Often used subclasses of 2-dimensional spherical distributions

Here, the condition $\mathbb{E}R_p < \infty$ is fulfilled in all considered cases unless for the Pearson type VII density generating function if $M \le 3/p$, $p \in \{1,2\}$. In case of a density generating function $g(r) = \exp(-r/p)$, $x \in \mathbb{R}$, X follows the Laplace distribution and the Gaussian distribution if $p=1$ and $p=2$, respectively, see Figure 1.

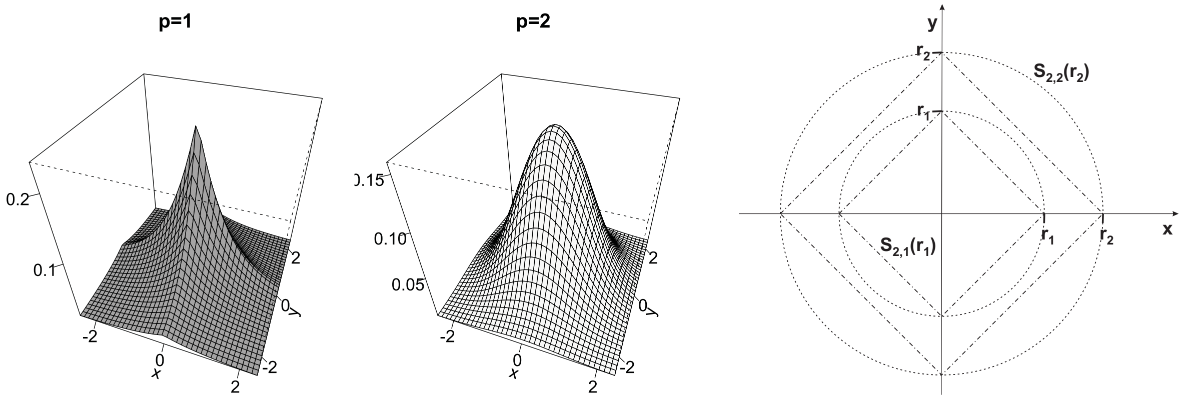

Figure 1: The density of the p-generalized Gaussian distribution. The level sets of $\varphi_{g,p}$ are circles w.r.t. the l_1-norm and the Euclidean norm if $p = 1$ and $p = 2$, respectively.

Notice, that the p-generalized Student distribution was introduced in (Richter, 2007) for arbitrary $p > 0$. Analogously to the spherical case, a multivariate version of the 1-generalized Student distribution can be obtained as a special case of the Pearson type VII distribution by setting $M = 2+m$. Table 2 contains additionally to the multivariate 1-generalized Student distribution the multivariate 1-generalized Cauchy distribution, which is motivated by Definition 8 and Corollary 9 and which is analogously to the spherical case the multivariate 1-generalized Student distribution with $m = 1$.

Type	$g(r)$	$C_{2,g,1}$	$\mathbb{E}(R_1^2)$
Kotz type	$r^{M-1}\, e^{-\beta\, r^{\gamma}}$; $M,\, \beta,\, \gamma \in \mathbb{R}_+$	$\frac{\gamma\, \beta^{(M+1)/\gamma}}{4\,\Gamma((M+1)/\gamma)}$	$\frac{\Gamma((M+3)/\gamma)}{\Gamma\left(\frac{M+1}{\gamma}\right)\beta^{2/\gamma}}$
Multivariate Laplace	$e^{-\frac{1}{2}r}$	$\frac{1}{16}$	24
Pearson type VII	$\left(1+\frac{r}{m}\right)^{-M}$; $M > 2,\, m > 0$	$\frac{(M-1)(M-2)}{4\,m^2}$	$\frac{6m^2}{(M-3)(M-4)}$, $M > 4$
Multivariate 1-generalized t	$\left(1+\frac{r}{m}\right)^{-m-2}$; $m > 0$	$\frac{m+1}{4\,m}$	$\frac{6\,m^2}{(m-1)(m-2)}$, $m > 2$
Multivariate 1-generalized Cauchy	$(1+r)^{-3}$	$\frac{1}{2}$	–
Pearson type II	$1_{[0,1]}(r)\,(1-r)^m$; $m > -1$	$\frac{(m+1)(m+2)}{4}$	$\frac{6}{(m+3)(m+4)}$

Table 2: Some subclasses of 2-dimensional simplicial distributions

In this paper, we present exact representations for the cdf and the pdf of the product, the ratio and the linear combination of the components of $X \sim \Phi_{g,p}$, $p \in \{1, 2\}$. These representations are essentially based upon the geometric measure representation of $\Phi_{g,p}$ introduced in (Richter, 2009). The mentioned functions of random variables have been studied by several authors, but only a few of them assumed a similar sample distribution. In (Nadarajah and Gupta, 2005), the product and the ratio for the elliptically Pearson type VII distribution are considered. This distribution is a special

case of $\Phi_{g,2}$ for a Pearson type VII density generating function g. Furthermore, the works (Harter, 1951), (Press, 1969) and (Nadarajah and Kotz, 2005) deal with the bivariate Student distribution, which is again a special case of the elliptically Pearson type VII distribution. It should be mentioned at this point that the case of a random vector $X = (X_1, X_2)$ with independent t-variates X_1 and X_2 as was dealt with, e.g., in (Chapman, 1950), (Ruben, 1960), (Ghosh, 1975) and (Walker and Saw, 1978) is not a special case of a continuous $l_{2,p}$-symmetric distribution.
The paper is structured as follows. In the preliminary Section 2, we refer to the necessary basics from measure theory. Here, a geometric representation for $\Phi_{g,p}$ on the basis of a non-Euclidean geometry will be introduced. In the Sections 3-5, we prove representations for the cdf and the pdf of the product, the linear combination and the ratio of random variables from a simplicial and a spherical sample distribution, respectively. In this connection, we prove in Section 5 for fixed $p > 0$ the robustness of the statistic $R(X_1, X_2) = X_1/X_2$ w.r.t. the density generating function of the sample distribution $(X_1, X_2) \sim \Phi_{g,p}$ and generalize in this way the Standard Cauchy distribution and the robustness result from (Arnold and Brockett, 1992). In the final Section 6, the cdf representations from the Sections 3 and 4 will be used to compute numerical quantile approximations for the product and the linear combination of random variables from a simplicial or a spherical sample distribution.

2 Geometric measure representation

In this section, we shall make use of the geometric measure representation formula for continuous $l_{2,p}$-symmetric distributions following from (Richter, 2009). The exact representations for the cdf and the pdf of the product, the ratio and the linear combination of the components of $X \sim \Phi_{g,p}$ will be essentially based upon this representation of $\Phi_{g,p}$. In the general case $p > 0$, the mentioned formula in (Richter, 2009) makes use of a non-Euclidean geometry if $p \neq 2$. In this context, a sector of the $l_{2,p}$-unit circle $S_{2,p} := \{x \in \mathbb{R}^2 : |x|_p = 1\}$ has to be measured w.r.t. the $l_{2,p}$ generalized arc-length measure $\mathfrak{U}_p$, which is according to (Richter, 2009) generated by the dual $l_{2,q}$-norm, if $p \geq 1$. Here, the function $|.|_p$ is defined as $|(x_1, x_2)|_p = (|x_1|^p + |x_2|^p)^{1/p}$, $(x_1, x_2) \in \mathbb{R}^2$, which is a norm iff $p \geq 1$. In this case

$$\mathfrak{U}_p(D) = \int\limits_{G(D)} |N(x)|_q \, dx$$

holds, where $1/p + 1/q = 1$, $N(x)$ is the normal vector to the $l_{2,p}$-circle $S_{2,p}$ at the point x and $G(D) = \{x_1 \in \mathbb{R} : |x_1|^p \leq 1 \wedge (x_1, x_2) \in D\}$. If $0 < p < 1$, $\mathfrak{U}_p$ is generated analogously but with $|.|_q$ being then a semi-anti-norm defined in (Moszyńska and Richter, 2011). A certain characteristic property of $\mathfrak{U}_p$ is demonstrated in Figure 2.

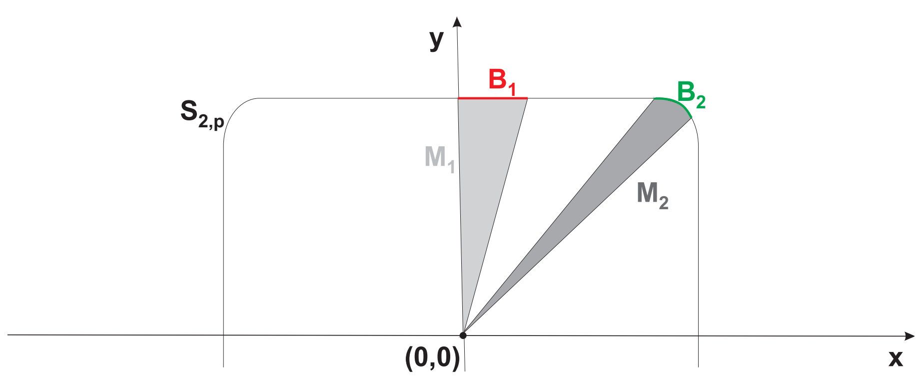

Figure 2: Sets M_1, M_2 having the same area content correspond to arcs B_1, B_2 having the same $\mathfrak{U}_p$-lengths but different Euclidean lengths, in general.

The following geometric measure representation formula of $\Phi_{g,p}$ uses for all $p > 0$ the function

$$r \to \mathfrak{F}_p(A, r) := \frac{\mathfrak{U}_p\left(r^{-1}\left[A \cap S_{2,p}(r)\right]\right)}{\mathfrak{U}_p(S_{2,p})},\ r > 0,$$

which is called the $l_{2,p}$-circle intersection-percentage function (ipf) of the set $A \in \mathfrak{B}(\mathbb{R}^2)$. It reflects the percentages which the level sets $S_{2,p}(r) := \{x \in \mathbb{R}^2 : |x|_p = r\}$ share for each $r > 0$ with the set A w.r.t. the arc-length measure $\mathfrak{U}_p$. For this reason $\mathfrak{F}_p$ can be understood as a basic function in context of a generalized method of indivisibles for the measure $\Phi_{g,p}$.

Theorem 1. The $l_{2,p}$-symmetric distribution with density-generating function g satisfies the representation

$$\Phi_{g,p}(A) = \frac{1}{I_{2,g,p}} \int_0^\infty \mathfrak{F}_p(A, r)\, r\, g(r^p)\, dr, \quad A \in \mathfrak{B}(\mathbb{R}^2), \tag{2}$$

where $I_{2,g,p} := \int\limits_0^\infty r\, g(r^p)\, dr$.

Proof. See (Richter, 2009). □

Theorem 1 can be used to evaluate the cdf of a statistic $T(X)$, where $X = (X_1, X_2) \sim \Phi_{g,p}$ and $T : \mathbb{R}^2 \to \mathbb{R}$, because

$$P(T(X) < t) = P(X \in T^{-1}(-\infty, t))$$

$$= P^X(A(t)) = \Phi_{g,p}(A(t))\,, \tag{3}$$

where

$$A(t) = \{(x_1, x_2) \in \mathbb{R}^2 : T(x_1, x_2) < t\}\,.$$

Furthermore, the $l_{2,p}$-circle ipf $\mathfrak{F}_p$ can be reduced to a ratio of Euclidean arc-lengths, iff $p \in \{1, 2\}$. In this case holds according to (Richter, 2009)

$$\mathfrak{U}_1(A) = \frac{1}{\sqrt{2}}\,\mathfrak{U}(A),\ A \in \mathfrak{B}(S_{2,1})\,.$$

Thus,

$$\mathfrak{F}_2(A, r) = \frac{\mathfrak{U}_2\left(r^{-1}\left[A \cap S_{2,2}(r)\right]\right)}{\mathfrak{U}_2(S_{2,2})} = \frac{\mathfrak{U}\left(r^{-1}A \cap S_{2,2}\right)}{2\pi},\ A \in \mathfrak{B}(\mathbb{R}^2),$$

and

$$\mathfrak{F}_1(A, r) = \frac{\mathfrak{U}_1\left(r^{-1}\left[A \cap S_{2,1}(r)\right]\right)}{\mathfrak{U}_1(S_{2,1})} = \frac{\mathfrak{U}\left(r^{-1}A \cap S_{2,1}\right)}{4\sqrt{2}},\ A \in \mathfrak{B}(\mathbb{R}^2)\,.$$

This means that although the geometric representation of $\Phi_{g,1}$ is based upon a non-Euclidean geometry, the ipf $\mathfrak{F}_1$ can be evaluated using the Euclidean arc-length measure $\mathfrak{U}$. Taking into account that $\mathfrak{U}_p$ does not equal a multiple of $\mathfrak{U}$ if $p \notin \{1, 2\}$, the continuous spherical and the continuous simplicial distributions can be considered as the Euclidean case and the pseudo-Euclidean case, respectively.

3 Product

In this section, we derive representations for the cdf and the pdf of $P(X) = X_1 \cdot X_2$, where $X = (X_1, X_2) \sim \Phi_{g,p}$ and $p \in \{1,2\}$. The resulting integral representations involve an arbitrary density generating function $g : \mathbb{R}_+ \to \mathbb{R}_+$, which can be used to model distributions with light and even with heavy tails.

Notice, that the distribution of the product $P(X)$ in the cases that X follows the elliptically symmetric Pearson type VII distribution or X follows the bivariate t-distribution were already considered in (Nadarajah and Gupta, 2005) and in (Harter, 1951), respectively. In this context, the bivariate t-distribution is a special case of the elliptically symmetric Pearson type VII distribution, which is itself a special case of $\Phi_{g,2}$ for g being a Pearson type VII density generating function, i.e.

$$g(r) = \left(1 + \frac{r}{m}\right)^{-M}, \; m > 0, \; M > \frac{2}{p}. \tag{4}$$

For this reason, we will compare the pdf representation of $P(X)$ from Corollary 3 numerically to the corresponding pdf representations from (Nadarajah and Gupta, 2005) and (Harter, 1951).

Theorem 2 is an immediate consequence of the equations (2) and (3). In this context, the central projected intersections of the set

$$A(t) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \cdot x_2 < t\}$$

(see Figure 3) with the density level sets $S_{2,p}(r)$ will be measured w.r.t. $\mathfrak{U}$ and will be divided by $\mathfrak{U}(S_{2,p})$, where $p \in \{1,2\}$.

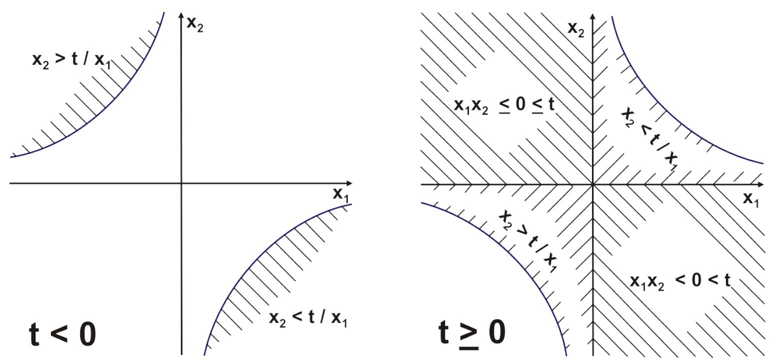

Figure 3: The set $A(t)$ in the case of the product.

Theorem 2. (a) If $X \sim \Phi_{g,1}$, then

$$F_P(t) := P(P(X) < t) = \begin{cases} 1 - \frac{1}{I_{2,g,1}} \int\limits_{2\sqrt{t}}^{\infty} \left(\sqrt{\frac{r^2}{4} - t}\right) g(r)dr & , t \geq 0 \\ \frac{1}{I_{2,g,1}} \int\limits_{2\sqrt{|t|}}^{\infty} \left(\sqrt{\frac{r^2}{4} + t}\right) g(r)dr & , t < 0 \end{cases}.$$

(b) If $X \sim \Phi_{g,2}$, then

$$F_P(t) := P(P(X) < t) = \begin{cases} 1 - \frac{1}{I_{2,g,2}} \int\limits_{\sqrt{2t}}^{\infty} \left(\frac{1}{2} - \frac{2}{\pi}\alpha\right) r \; g(r^2)dr & , t \geq 0 \\ \frac{1}{I_{2,g,2}} \int\limits_{\sqrt{2|t|}}^{\infty} \left(\frac{1}{2} - \frac{2}{\pi}\alpha\right) r \; g(r^2)dr & , t < 0 \end{cases},$$

where $\alpha = \frac{1}{4} \arccos\left(1 - \frac{8t^2}{r^4}\right)$.

Proof. One has

$$\frac{1}{r}A(t) = \left\{\left(\frac{x_1}{r}, \frac{x_2}{r}\right) : x_1 \cdot x_2 < t\right\} = \left\{(x_1, x_2) : x_1 \cdot x_2 < \frac{t}{r^2}\right\},\ t \in \mathbb{R},$$

and the ipf is

$$\mathfrak{F}_p(A(t), r) = \frac{1}{\mathfrak{U}_2(S_{2,p})}\mathfrak{U}_2\left(\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \cdot x_2 < \frac{t}{r^2}, |x_1|^p + |x_2|^p = 1\right\}\right),\ p \in \{1, 2\}.$$

Consequently,

$$\mathfrak{F}_1(A(t), r) = \begin{cases} 1 - \left(\sqrt{\frac{1}{4} - \frac{t}{r^2}}\right) \mathrm{I}_{[2\sqrt{t}, \infty)}(r) & , t \geq 0 \\ \left(\sqrt{\frac{1}{4} + \frac{t}{r^2}}\right) \mathrm{I}_{\left(2\sqrt{|t|}, \infty\right)}(r) & , t < 0 \end{cases}$$

(see Figure 4) and

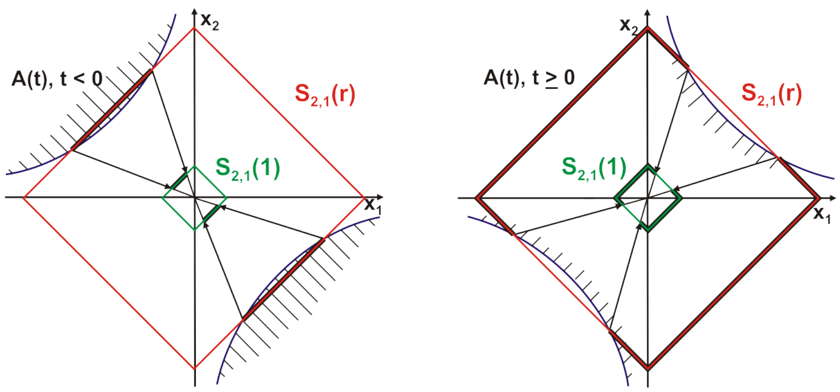

Figure 4: The set $\frac{1}{r}A(t) \cap S_{2,1}$ in the case of the product.

$$\mathfrak{F}_2(A(t), r) = \begin{cases} 1 - \frac{\pi - 4\alpha}{2\pi}\ \mathrm{I}_{\left[\sqrt{2t}, \infty\right)}(r) & , t \geq 0 \\ \frac{2(\pi/2 - 2\alpha)}{2\pi}\ \mathrm{I}_{\left[\sqrt{2|t|}, \infty\right)}(r) & , t < 0 \end{cases},$$

where $\alpha = \beta = \arccos\left(\frac{z}{r}\right)$ are angles from $(0, 2\pi)$ (refer to Figure 5).

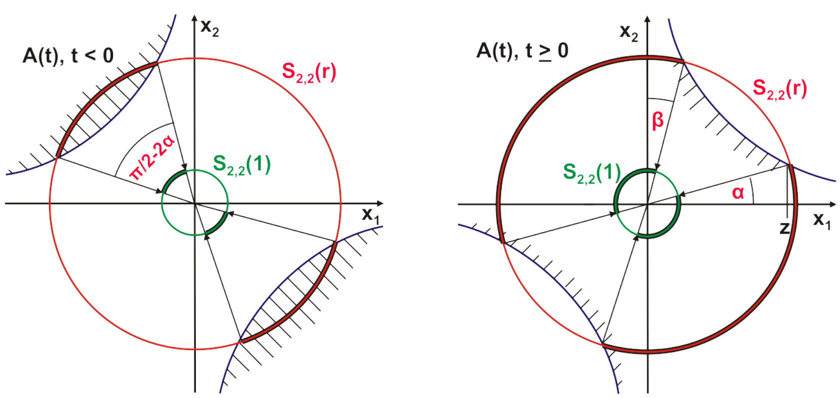

Figure 5: The set $\frac{1}{r}A(t) \cap S_{2,2}$ in the case of the product.

□

Making use of Leibniz' integral rule, one obtains the following corollary presenting the pdf of $P(X)$ as an immediate consequence of Theorem 2.

Corollary 3. (a) If $X \sim \Phi_{g,1}$, then the pdf of $P(X)$ is

$$f_P(t) = \frac{1}{I_{2,g,1}} \int_{2\sqrt{|t|}}^{\infty} \frac{g(r)}{\sqrt{r^2 - 4|t|}} dr, \ t \in \mathbb{R}.$$

(b) If $X \sim \Phi_{g,2}$, then the pdf of $P(X)$ is

$$f_P(t) = \frac{2}{\pi \ I_{2,g,2}} \int_{\sqrt{2|t|}}^{\infty} \frac{r g(r^2)}{\sqrt{r^4 - 4t^2}} \ dr, \ t \in \mathbb{R}. \tag{5}$$

Proof. If $p = 2$ and $t > 0$, it holds

$$\begin{aligned} \frac{d}{dt}\left(1 - \int_{\sqrt{2t}}^{\infty} \left(\frac{1}{2} - \frac{2}{\pi}\alpha\right) r g(r^2) \ dr\right) &= \int_{\sqrt{2t}}^{\infty} \frac{2}{\pi} \ r g(r^2) \ \left(\frac{d}{dt}\alpha(t)\right) \ dr \\ &= \frac{2}{\pi} \int_{\sqrt{2t}}^{\infty} \frac{r g(r^2)}{\sqrt{r^4 - 4t^2}} \ dr. \end{aligned}$$

In the case $t < 0$ we obtain

$$\frac{d}{dt} \int_{\sqrt{-2t}}^{\infty} \left(\frac{1}{2} - \frac{2}{\pi}\alpha\right) r g(r^2) \ dr = \frac{2}{\pi} \int_{\sqrt{-2t}}^{\infty} \frac{r g(r^2)}{\sqrt{r^4 - 4t^2}} \ dr.$$

If $p = 1$ and $t \neq 0$ it follows analogously

$$\frac{d}{dt} \int_{2\sqrt{|t|}}^{\infty} \sqrt{\frac{r^2}{4} - |t|} \ g(r) \ dr = \begin{cases} -\int_{2\sqrt{|t|}}^{\infty} \frac{1}{\sqrt{r^2-4|t|}} \ g(r) \ dr & , t > 0 \\ \int_{2\sqrt{|t|}}^{\infty} \frac{1}{\sqrt{r^2-4|t|}} \ g(r) \ dr & , t < 0 \end{cases}.$$

□

According to (Harter, 1951), the pdf of $|P(X)|$ satisfies the infinite series representation

$$f_{|P(X)|}(x) = \frac{1}{\pi \, \Gamma(m/2)} \sum_{j=0}^{\infty} (-1)^j m^{m/2+j} |x|^{-(1+j+m/2)} \, \Gamma^2 \left(\frac{2+2j+m}{4}\right) / \Gamma(1+j), \ x > \frac{m}{2}, \tag{6}$$

if X follows a bivariate t-distribution with m degrees of freedom and $\mathbb{E}X = (0,0)^T$. In Table 3, the pdf of $P(X)$ was evaluated numerically by using the representation from (6), where $m \in \{1,2,3\}$. Here, the sum in (6) was approximated by its first $n+1$ summands $s_0, \ldots, s_n$, i.e. by $S_n := \sum_{i=0}^{n} s_i$. In this connection, the choice $n = 110$ turned out to be adequate since as shown by (Harter, 1951), $s_j \to 0$ as $j \to \infty$ and in the cases being considered it holds $|s_{110}| < 10^{-12}$, i.e. the first 12 decimals of S_{n+1} and S_n are equal. This comes up with the approximation accuracy $\varepsilon = 10^{-12}$ of the matlab integration routine `quadgk` used for the integral approximation in (5), see (Schweizer, 2009). An alternative evaluation of f_P according to (5) delivered function values $f_P(x)$ which are in all cases equal to those from Table 3 w.r.t. the first 12 decimals, where $x \in \{2, 2.5, 3, 5, 10\}$ and $m \in \{1,2,3\}$.

x	$f_P(x)$ according to (6)		
	m=1	m=2	m=3
2	0.037305297971	0.041953265484	0.042634381445
2.5	0.027922690158	0.029954412695	0.029394799534
3	0.021910871109	0.022472285023	0.021301533047
5	0.010863820256	0.009555457949	0.007986272535
10	0.004042085415	0.002740025112	0.001818192382
20	0.001467330744	0.000737031782	0.000369318257

Table 3: Numerical approximations of $f_P(x)$ according to equation (6).

Following (Nadarajah and Gupta, 2005), the pdf of $P(X)$ satisfies the representation

$$f_{P(X)}(z) = \frac{N-1}{\pi} m^{M-1} B(M,M) z^{-M} {}_2F_1\left(\frac{M}{2}, \frac{M}{2}, M+\frac{1}{2}, 1-\frac{m^2}{4z^2}\right),\ |z| > m/2, \tag{7}$$

if X is elliptically symmetric Pearson type VII distributed and has uncorrelated components X_1 and X_2. Here, $B(x,y)$ and ${}_2F_1(a,b,c,d)$ denote the Beta function and the Gauss hypergeometric function, respectively. In Table 4, the pdf of $P(X)$ was evaluated numerically by using the representation from (7), where ${}_2F_1(a,b,c,d)$ was approximated on the basis of the infinite series expansion

$${}_2F_1(a,b,c,z) = \sum_{k=0}^{\infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{\Gamma(k+1)},\ |z| < 1. \tag{8}$$

It turned out, that

$$S_n := \sum_{k=0}^{n} \frac{\Gamma\left(M+\frac{1}{2}\right)}{\Gamma\left(\frac{M}{2}\right)\Gamma\left(\frac{M}{2}\right)} \frac{\Gamma\left(\frac{M}{2}+k\right)\Gamma\left(\frac{M}{2}+k\right)}{\Gamma\left(M+\frac{1}{2}+k\right)} \frac{\left(1-\frac{m^2}{4z^2}\right)^k}{\Gamma(k+1)}$$

is a suitable approximation of ${}_2F_1\left(\frac{M}{2}, \frac{M}{2}, M+\frac{1}{2}, 1-\frac{m^2}{4z^2}\right)$, if $n = 10^5$, because larger values of n as $n = 10^6$ or $n = 10^7$ do not change the first 12 decimals of S_n. An alternative evaluation of f_p according to (5) delivered again function values $f_P(x)$ which are equal to those from Table 4 w.r.t. the first 12 decimals, where $x \in \{2,3,5,10\}$, $M \in \{2,3,5\}$ and $m \in \{1,2,3\}$.

x	M	$f_P(x)$ according to (7)		
		m=1	m=2	m=3
2	2	0.027997005191	0.041953265484	0.049246102471
3	2	0.013874506807	0.022472285023	0.027968843656
5	2	0.005480050225	0.009555457949	0.012630861213
10	2	0.001474063564	0.002740025112	0.003832066486
2	3	0.008844911928	0.022196764425	0.033641690440
3	3	0.003125386104	0.008889343865	0.014797842950
5	3	0.000784390397	0.002513021730	0.004608985646
10	3	0.000110391527	0.000392195198	0.000787865794
2	5	0.000533125098	0.003731628903	0.009383913172
3	5	0.000095972645	0.000838374512	0.002487752602
5	5	0.000009742272	0.000105097115	0.000370174984
10	5	0.000000375841	0.000004871136	0.000020162708

Table 4: Numerical approximations of $f_P(x)$ according to equation (7).

Both the numerical results from Table 3 and from Table 4 indicate on the one hand, that the representations of f_P from the equations (6) and (7) are almost equivalent to the representation from equation (5) in the mentioned special cases of Pearson type VII density generating functions. On the other hand, Corollary 3 enables one to evaluate the pdf of the product statistic for a much greater class of sample distributions than the ones considered in (Harter, 1951) and in (Nadarajah and Gupta, 2005).

4 Linear Combination

In this section, we obtain representations for the cdf and the pdf of $L(X) = \alpha_1 X_1 + \alpha_2 X_2$, where $X = (X_1, X_2) \sim \Phi_{g,p}$, $p \in \{1, 2\}$ and $\alpha_1\alpha_2 \neq 0$. Let us temporarily assume that the density generating function g coincides with the density generator, i.e. $\varphi_{g,p}(x) = g(|x|_p^p)$, $x \in \mathbb{R}^2$. Notice, that in the case that $p = 2$

$$(\alpha_1, \alpha_2)\, X^T \stackrel{d}{=} |(\alpha_1, \alpha_2)|_2\, X_1 \tag{9}$$

and that X_1 is spherically symmetric distributed with the characteristic generator of the random vector (X_1, X_2), see (Fang et al., 1990). For this reason, the linear combination $L(X)$ is again spherically symmetric distributed, but with a density generator g_2 which is in general different from g. Following (Fang et al., 1990), the density generator g_1 of X_1 satisfies

$$g_1(r) = \int\limits_0^\infty y^{-\frac{1}{2}}\, g\,(y + r)\; dy \quad .$$

Hence, X_1 has a density $\varphi_{X_1}(x) = g_1(x^2)$ and $|(\alpha_1, \alpha_2)|_2\, X_1$ has a density

$$\varphi_{L(X)}(x) = |(\alpha_1, \alpha_2)|_2^{-1}\; g_1(x^2/|(\alpha_1, \alpha_2)|_2^2) \quad .$$

Therefrom, the density generator g_2 of $L(X)$ satisfies $g_2(r) = |(\alpha_1, \alpha_2)|_2^{-1}\, g_1(r/|(\alpha_1, \alpha_2)|_2^2)$ and

$$g_2(r) = \frac{1}{|(\alpha_1, \alpha_2)|_2} \int\limits_0^\infty y^{-\frac{1}{2}}\, g\left(y + \frac{r}{|(\alpha_1, \alpha_2)|_2^2}\right)\, dy \quad . \tag{10}$$

For the subclasses of 2-dimensional spherical distributions considered in this paper, see Table 1, the integral in (10) can be solved exactly only in exceptional cases like the well known Gaussian case, where the density generating function is $g(r) = \exp(-c\,r)$. Furthermore, using formula (10) for deriving the cdf of $L(X)$ would lead to a rather complicate two dimensional integral. In Theorem 4, however, we give a one dimensional integral representation formula for this case. In the case that $X = (X_1, X_2) \sim \Phi_{g,1}$, the situation is even more complicated. Here, the random variables X_1 and X_2 are again l_1-norm symmetric distributed but in contrast to the spherical case, to the best of our knowledge, there is no l_1-analogue of formula (9) known.
However, the cdf and the pdf of $L(X)$ can be obtained geometrically with equation (2) if $X \sim \Phi_{g,p}$ and $p \in \{1, 2\}$. Analogously to Section 3, the resulting integral representations depend on a density generating function $g : \mathbb{R}_+ \to \mathbb{R}_+$ modeling both heavy and light tail distributions. The mentioned representations can be simplified in many cases by specifying g, e.g. by choosing $g(r)$ as a density generating function from the Pearson-VII type. According to the equations (2) and (3), the following Theorem 4 is essentially based upon the intersection percentage function of the set

$$A(t) = \{(x_1, x_2) \in \mathbb{R}^2 : \alpha_1 x_1 + \alpha_2 x_2 < t\}, t \in \mathbb{R},$$

which is for every $t \in \mathbb{R}$ a half-space of $\mathbb{R}^2$ (see Figure 6).

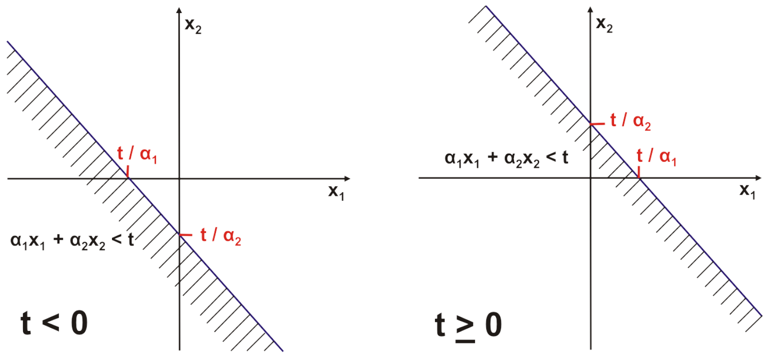

Figure 6: The set $A(t)$ in the case of the linear combination $(\alpha_1, \alpha_2 > 0)$.

Theorem 4. (a) If $X \sim \Phi_{g,1}$, then

$$F_L(t) := P(L(X) < t) = \begin{cases} \frac{1}{2} + \frac{1}{2\,I_{2,g,1}} \left[\int\limits_0^{t_1} r g(r) dr + \int\limits_{t_1}^{t_2} \beta_1(r) g(r) dr + \int\limits_{t_2}^{\infty} \frac{t}{|\alpha_1|+|\alpha_2|} g(r) dr \right] & , t \geq 0 \\ \frac{1}{2\,I_{2,g,1}} \left[\int\limits_{t_1}^{t_2} \beta_2(r) g(r) dr + \int\limits_{t_2}^{\infty} \left(r + \frac{t}{|\alpha_1|+|\alpha_2|} \right) g(r) dr \right] & , t < 0 \end{cases} ,$$

where

$$\beta_1(r) = \begin{cases} \frac{t \max(|\alpha_1|,|\alpha_2|) - r \min(\alpha_1,\alpha_2)^2}{|\alpha_1^2 - \alpha_2^2|} & , \alpha_1 \neq \alpha_2 \\ 0 & , \alpha_1 = \alpha_2 \end{cases} , \quad \beta_2(r) = \begin{cases} \frac{t \max(|\alpha_1|,|\alpha_2|) + r \max(\alpha_1,\alpha_2)^2}{|\alpha_1^2 - \alpha_2^2|} & , \alpha_1 \neq \alpha_2 \\ 0 & , \alpha_1 = \alpha_2 \end{cases}$$

and

$$t_1 := \frac{|t|}{\max(|\alpha_1|, |\alpha_2|)} , \quad t_2 := \frac{|t|}{\min(|\alpha_1|, |\alpha_2|)} .$$

(b) If $X \sim \Phi_{g,2}$, then

$$F_L(t) := P(L(X) < t) = \begin{cases} 1 - \frac{1}{I_{2,g,2}} \left(\frac{1}{\pi} \int\limits_{t^*}^{\infty} arccos\left(\frac{t^*}{r}\right) r \; g(r^2) dr \right) & , t \geq 0 \\ \frac{1}{I_{2,g,2}} \left(\frac{1}{\pi} \int\limits_{t^*}^{\infty} arccos\left(\frac{t^*}{r}\right) r \; g(r^2) dr \right) & , t < 0 \end{cases} ,$$

where

$$t^* := \frac{|t|}{\sqrt{\alpha_1^2 + \alpha_2^2}} .$$

Proof. Without loss of generality let $\alpha_1 \geq \alpha_2 > 0$. If $p = 1$, the ipf is

$$\mathfrak{F}_1(A(t), r) = \begin{cases} 1 & , t \geq 0 \wedge r < \frac{t}{|\alpha_1|} \\ \frac{1}{2} + \frac{t|\alpha_1| - r\alpha_2^2}{2r\left(\alpha_1^2 - \alpha_2^2\right)} & , t \geq 0 \wedge \frac{t}{|\alpha_1|} \leq r < \frac{t}{|\alpha_2|} \\ \frac{1}{2} + \frac{t}{2r(|\alpha_1|+|\alpha_2|)} & , t \geq 0 \wedge \frac{t}{|\alpha_2|} \leq r \\ 0 & , t < 0 \wedge r \leq \left|\frac{t}{\alpha_1}\right| \\ \frac{t|\alpha_1| + r\alpha_1^2}{2r\left(\alpha_1^2 - \alpha_2^2\right)} & , t < 0 \wedge \left|\frac{t}{\alpha_1}\right| < r \leq \left|\frac{t}{\alpha_2}\right| \\ \frac{1}{2} + \frac{t}{2r(|\alpha_1|+|\alpha_2|)} & , t < 0 \wedge \left|\frac{t}{\alpha_2}\right| < r \end{cases} ,$$

see Figure 7.

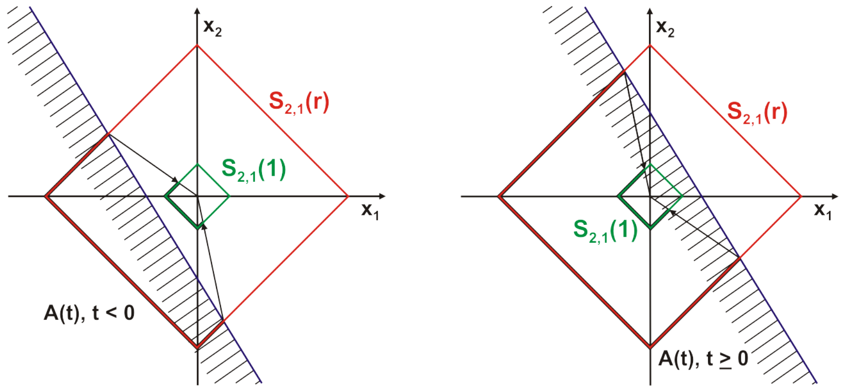

Figure 7: The set $\frac{1}{r}A(t) \cap S_{2,1}$ in the case of the linear combination.

In the case $p = 2$, the ipf is

$$\mathfrak{F}_2(A(t), r) = \begin{cases} 0 & , t < 0 \wedge 0 < r < t^* \\ \frac{1}{\pi} \cdot arccos\left(\frac{|t|}{r\sqrt{\alpha_1^2+\alpha_2^2}}\right) & , t < 0 \wedge r \geq t^* \\ 1 & , t \geq 0 \wedge 0 < r < t^* \\ 1 - \frac{1}{\pi} \cdot arccos\left(\frac{|t|}{r\sqrt{\alpha_1^2+\alpha_2^2}}\right) & , t \geq 0 \wedge r \geq t^* \end{cases},$$

see Figure 8.

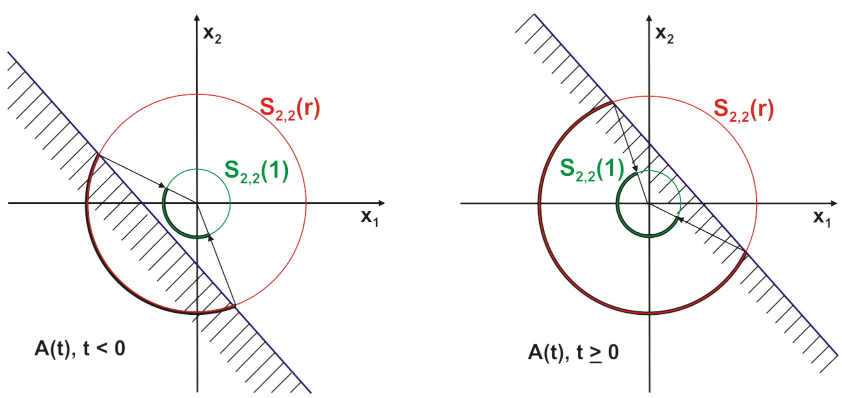

Figure 8: The set $\frac{1}{r}A(t) \cap S_{2,2}$ in the case of the linear combination.

□

The differentiation of the cdf integral representations from Theorem 4 using the Leibniz' integral rule yields to the following corollary. Here, we prove a representation for the pdf of $L(X) = \alpha_1 X_1 + \alpha_2 X_2$ in case of a simplicial and a spherical sample distribution, respectively.

Corollary 5. (a)If $X \sim \Phi_{g,1}$, then the pdf of $L(X)$ is

$$f_L(t) = \begin{cases} \frac{1}{2(\alpha_1^2-\alpha_2^2)I_{2,g,1}} \int\limits_{|t|}^{\infty} \left[g\left(\frac{r}{\alpha_1}\right) - g\left(\frac{r}{\alpha_2}\right)\right] dr & , \alpha_1 \neq \alpha_2 \\ \frac{1}{4\alpha^2 I_{2,g,1}} \left[\int\limits_{|t|}^{\infty} g\left(\frac{r}{\alpha}\right) dr + |t| g\left(\frac{|t|}{\alpha}\right)\right] & , \alpha_1 = \alpha_2 \end{cases}, \quad t \in \mathbb{R}.$$

(b)If $X \sim \Phi_{g,2}$, then the pdf of $L(X)$ is

$$f_L(t) = \frac{1}{\pi\ I_{2,g,2}} \int\limits_{t^*}^{\infty} \frac{r g(r^2)}{\sqrt{(\alpha_1^2+\alpha_2^2)r^2 - t^2}}\ dr,\ t \in \mathbb{R}.$$

Proof. Consider the case $p = 2$. Here, one has

$$\frac{d}{dt} \int\limits_{t^*}^{\infty} \arccos\left(\frac{t^*}{r}\right)\ r g(r^2)\ dr \quad = \quad \begin{cases} -\int\limits_{t^*}^{\infty} \frac{r g(r^2)}{\sqrt{(\alpha_1^2+\alpha_2^2)r^2-t^2}}\ dr & ,\ t > 0 \\ \int\limits_{t^*}^{\infty} \frac{r g(r^2)}{\sqrt{(\alpha_1^2+\alpha_2^2)r^2-t^2}}\ dr & ,\ t < 0 \end{cases} .$$

If $p = 1$ and $|\alpha_1| \neq |\alpha_2|$, the derivative of the functions

$$t \to \int\limits_{t_1}^{t_2} \frac{t \max(|\alpha_1|, |\alpha_2|) + r \max(\alpha_1, \alpha_2)^2}{|\alpha_1^2 - \alpha_2^2|} g(r)dr + \int\limits_{t_2}^{\infty} \left(r + \frac{t}{|\alpha_1| + |\alpha_2|}\right) g(r)dr,\ t < 0,$$

and

$$t \to \int\limits_{0}^{t_1} r g(r)dr + \int\limits_{t_1}^{t_2} \frac{t \max(|\alpha_1|, |\alpha_2|) - r \min(\alpha_1, \alpha_2)^2}{|\alpha_1^2 - \alpha_2^2|} g(r)dr + \int\limits_{t_2}^{\infty} \frac{t}{|\alpha_1| + |\alpha_2|} g(r)dr,\ t \geq 0,$$

is

$$t \to \frac{1}{\alpha_1^2 - \alpha_2^2} \int\limits_{|t|}^{\infty} \left[g\left(\frac{r}{|\alpha_1|}\right) - g\left(\frac{r}{|\alpha_2|}\right)\right] dr\,.$$

In the case $p = 1$ and $|\alpha_1| = |\alpha_2| = \alpha$, the derivative of the functions

$$t \to \int\limits_{\frac{|t|}{\alpha}}^{\infty} \left(r + \frac{t}{2\alpha}\right) g(r)dr,\ t < 0,$$

and

$$t \to \int\limits_{0}^{\frac{t}{\alpha}} r g(r)dr + \frac{t}{2\alpha} \int\limits_{\frac{t}{\alpha}}^{\infty} g(r)dr,\ t \geq 0,$$

is

$$t \to \frac{1}{2\alpha^2} \int\limits_{\frac{|t|}{\alpha}}^{\infty} g(r)dr + \frac{|t|}{2\alpha^2} g\left(\frac{|t|}{\alpha}\right)\ .$$

$\square$

In many practical and theoretical situations it is important to know the cdf of a linear combination $a_{(d)}^T X_{(d)}$, where $X_{(d)}$ is a d-dimensional random vector and $a_{(d)} \in \mathbb{R}^d$. For example, the Central Limit Theorem deals with the case $a_{(d)}^T = 1/\sqrt{d} \cdot (1, \ldots, 1)$. A multivariate generalization of (2) and (3) leads to the problem of measuring the d-dimensional half-spheres $A(t) = \{x_{(d)} \in \mathbb{R}^d : \sum_{i=1}^d d_i\, x_i < t\}$, $t \in \mathbb{R}^d$, and especially to determine the ipf of $A(t)$. For certain considerations into this direction, in the case of spherical distributions, we refer to (Richter and Steinebach, 1994) where several properties of the ipf of a half space are exploited.

5 Ratio

In this section, we provide representations for the cdf and the pdf of $R(X) = X_1/X_2$, where $X = (X_1, X_2) \sim \Phi_{g,1}$. In the case $X \sim \Phi_{g,2}$, it was already shown in (Arnold and Brockett, 1992) that $R(X)$ follows a Standard Cauchy distribution, i.e.

$$F_R(t) := P(R(X) < t) = \frac{1}{2} + \frac{1}{\pi} \arctan(t)\,,\ t \in \mathbb{R}, \tag{11}$$

and

$$f_R(t) := \frac{d}{dt} F_R(t) = \frac{1}{\pi}\ \frac{1}{1+t^2}\,,\ t \in \mathbb{R}. \tag{12}$$

Thus, the statistic $R(X)$ does not depend on g in this case and is robust w.r.t. a spherical sample distribution. In (Press, 1969) and (Nadarajah and Gupta, 2005), representations for the pdf of $R(X)$ were proved in the case that X has a bivariate t-distribution. If the components of X are uncorrelated and $\mathbb{E}X = (0,0)^T$, this is a special case of the $l_{2,2}$-symmetric distribution $\Phi_{g,2}$, where g is from Pearson VII type and satisfies

$$g(r) = \left(1 + \frac{r}{m}\right)^{-(m+1)/2},\ r > 0,\ m > 1\,.$$

In this case, the pdf representations from (Press, 1969) and (Nadarajah and Gupta, 2005) are equal to that from equation (12). As we will show, for fixed $p > 0$, the statistic $R(X)$ is robust in the greater class of $l_{2,p}$-symmetric sample distributions and especially w.r.t. a simplicial sample distribution. To this end, the ipf of

$$A(t) = \left\{(x_1, x_2) \in \mathbb{R}^2 : \frac{x_1}{x_2} < t, x_2 \neq 0\right\},\ t \in \mathbb{R},$$

(see Figure 9) will be determined in Theorem 6 to prove analogue representations to that from (11) and (12) for the cdf and the pdf of $R(X)$ in the case $X \sim \Phi_{g,1}$.

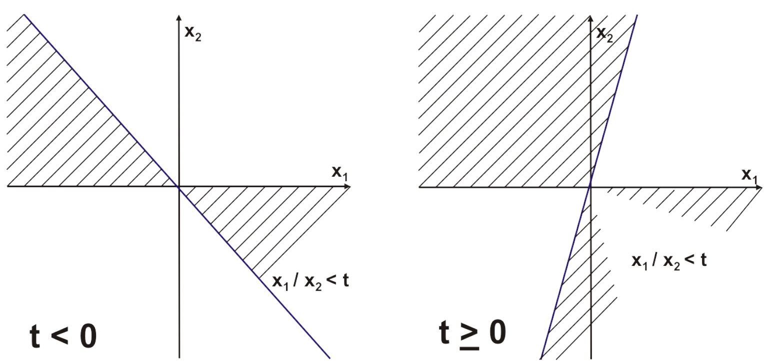

Figure 9: The set $A(t)$ in the case of the ratio of X_1 and X_2.

Theorem 6. If $X \sim \Phi_{g,1}$, then

$$F_R(t) = \frac{1}{2} + \frac{t}{2(1+|t|)}\,,\ t \in \mathbb{R}.$$

Proof. The ipf of $A(t)$ satisfies the representation

$$\mathfrak{F}_1(A(t), r) = \frac{1}{2} + \frac{t}{2\,(1+|t|)}\,, t \in \mathbb{R},$$

see Figure 10. The assertion follows from the fact, that $\Phi_{g,1}(A) = \mathfrak{F}_1(A, r)$, because the ipf $\mathfrak{F}_1(A, r)$ does not depend on $r > 0$.

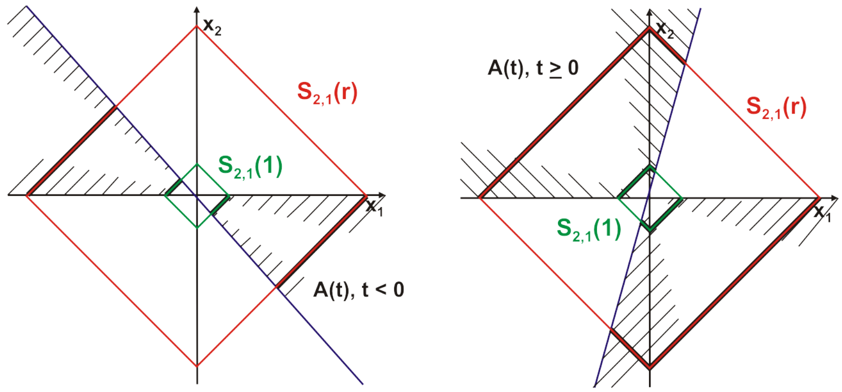

Figure 10: The set $\frac{1}{r}A(t) \cap S_{2,1}$ in the case of the ratio.

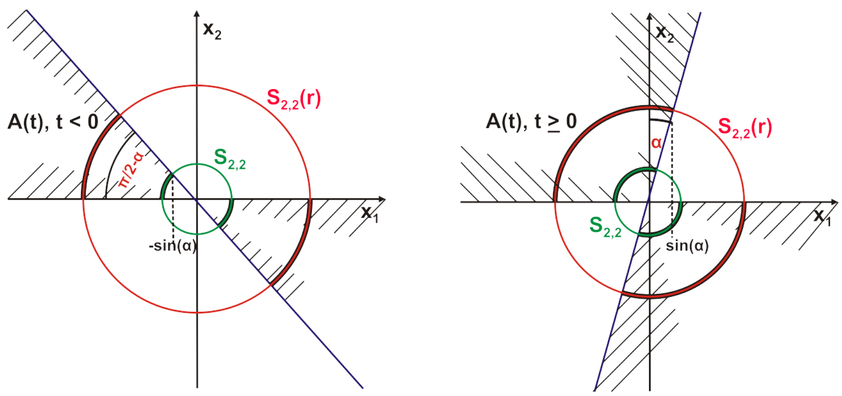

Figure 11: The set $\frac{1}{r}A(t) \cap S_{2,2}$ in the case of the ratio.

$\square$

Corollary 7. If $X \sim \Phi_{g,1}$, then the pdf of $R(X)$ is

$$f_R(t) = \frac{1}{2(1+|t|)^2}\,,\ t \in \mathbb{R}.$$

As can be seen in Theorem 6 and Corollary 7, the ratio $R(X)$ is not only robust w.r.t. a spherical sample distribution but also w.r.t. a simplicial sample distribution. Furthermore it was shown in (Richter, 2009), that $X = (X_1, X_2) \sim \Phi_{g,p}$ satisfies for every $p > 0$ the representation (1). Thus, $R(X) = X_1/X_2$ is independent from $|X|_p$ and consequently the ipf $\mathfrak{F}_p(A(t), r)$ does not depend on $r > 0$. Therefore, for fixed $p > 0$, the ratio $R(X)$ is robust w.r.t. a continuous $l_{2,p}$-symmetric sample distribution $\Phi_{g,p}$ and can be understood as a p-generalization of a Cauchy distributed random variable.

Definition 8. Let $X = (X_1, X_2) \sim \Phi_{g,p}$, $p > 0$. The random variable Y is called p-generalized Cauchy distributed iff

$$Y \stackrel{d}{=} \frac{X_1}{X_2}\,.$$

Corollary 9. If $p \in \{1,2\}$, then the pdf $\phi_p(t)$ of the p-generalized Cauchy distribution satisfies

$$\phi_p(t) = \frac{p}{2\Gamma\left(\frac{1}{p}\right)^2} \left(1+|x|^p\right)^{-2/p} \quad , t \in \mathbb{R}.$$

6 Quantiles

In this section, we present tables of quantiles for the statistics $P(X) = X_1 \cdot X_2$ and $L(X) = X_1 + X_2$, where $X \sim \Phi_{g,p}$, $p \in \{1,2\}$, and g is a Kotz type density generating function or a Pearson type VII density generating function. Notice, that quantiles $\xi_q = F_R^{-1}(q)$ of the statistic $R(X)$ do not depend on the density generating function g, where q is a given quantile order and

$$F_R^{-1}(x) = \begin{cases} \tan\left(\pi\left(x-\frac{1}{2}\right)\right) & \text{, if } p=2 \wedge x \in (0,1) \\ 1-\frac{1}{2x} & \text{, if } p=1 \wedge x \in (0,\frac{1}{2}] \\ \frac{1-2x}{2x-2} & \text{, if } p=1 \wedge x \in (\frac{1}{2},1) \end{cases} \quad . \tag{13}$$

Because of the explicit representation of the inverse distribution function in (13), there is no need to table the quantiles of $R(X)$. In case of the product and the linear combination, the distribution function F of $P(X)$ or $L(X)$ depends on g and can not be inverted easily in an explicit way. A q-quantile $\xi_q = F^{-1}(q)$ of F will therefore be approximated by numerically solving

$$\tilde{F}(\xi_q) - q = 0 \tag{14}$$

with the matlab routine fzero, see (Schweizer, 2009), where $\tilde{F}$ is an approximation of the cdf F satisfying $|\tilde{F}(x) - F(x)| < \varepsilon$, $\forall x \in \mathbb{R}$. Here, the distribution functions of $P(X)$ and $L(X)$ given by the integral representations in the Theorems 2 and 4 will be approximated by functions derived with the help of the matlab routine quadgk (see (Schweizer, 2009)), which is known to evaluate integrals with an accuracy ε smaller than 10^{-12}.

Remark 10. Let F be a continuous and strictly monotonic increasing distribution function and $\xi_q = F^{-1}(q)$ be the quantile of order $q \in (0,1)$ of F. Assume further that $[x]$ denotes the largest integer less than or equal to x, $\tilde{\xi}_q$ is an approximation of ξ_q , $\tilde{\xi}^-_{q,k} = \frac{[10^k \tilde{\xi}_q]}{10^k}$ is the rounded value of $\tilde{\xi}_q$ after cutting off all decimals after the k-th one and $\tilde{\xi}^+_{q,k} = \tilde{\xi}^-_{q,k} + \frac{1}{10^k}$. The approximation $\tilde{\xi}_q$ of ξ_q is thus correct to the first k decimals if

$$F\left(\tilde{\xi}^-_{q,k}\right) \le q \tag{15}$$

and

$$F\left(\tilde{\xi}^+_{q,k}\right) > q \quad . \tag{16}$$

If $\tilde{\xi}_q$ satisfies (15) and (16), then $\tilde{\xi}^-_{q,k}$ is the largest real number with k decimals less than or equal to the exact quantile ξ_q. In the case that F is unknown but an approximation $\tilde{F}$ is available with $|F(x) - \tilde{F}(x)| < \varepsilon$, $\forall x \in \mathbb{R}$, $\tilde{\xi}_q$ is correct to the first k decimals if

$$\tilde{F}\left(\tilde{\xi}^-_{q,k}\right) \le q - \varepsilon \tag{17}$$

and

$$\tilde{F}\left(\tilde{\xi}^+_{q,k}\right) > q + \varepsilon \quad , \tag{18}$$

see Figure 12.

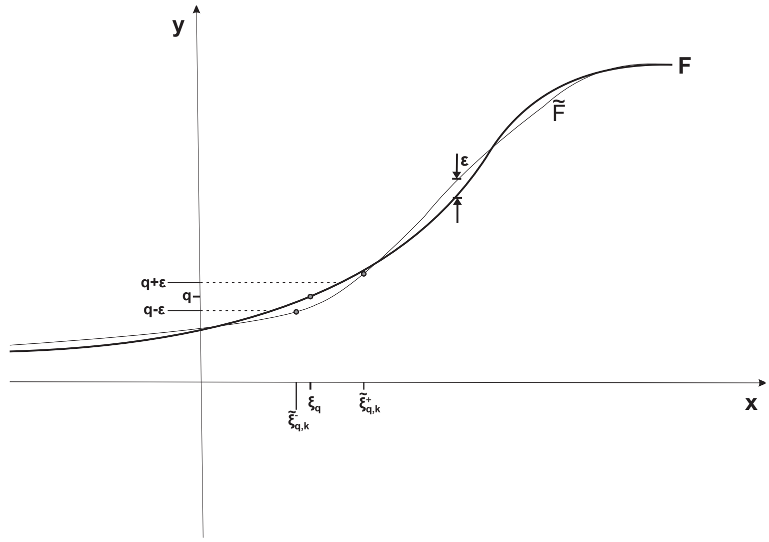

Figure 12: The q-order quantile ξ_q and it's approximations $\tilde{\xi}^-_{q,k}$ and $\tilde{\xi}^+_{q,k}$

In Tables 5 to 8, we give quantile approximations which are according to (17) and (18) at least correct to the first 6 decimals. Here, $F_{P,p}$ and $F_{L,p}$ denote the cdf's of the statistics $P(X)$ and $L(X)$ if the underlying sample distribution is $\Phi_{g,p}$, $p \in \{1,2\}$, and the density generating function g is chosen according to the Pearson VII type, see (4), and according to the Kotz type, i.e.

$$g(r) = r^{M-1} \exp\left(-\beta r^{\gamma}\right) \quad , r > 0,\, M > 0,\, \beta > 0,\, \gamma > 0 \quad .$$

M	m	$F_{P,1}^{-1}(0.95)$	$F_{P,1}^{-1}(0.99)$	$F_{P,1}^{-1}(0.999)$	$F_{P,2}^{-1}(0.95)$	$F_{P,2}^{-1}(0.99)$	$F_{P,2}^{-1}(0.999)$
3.0	1	52.031663	1492.470774	153712.913519	0.705356	2.080341	7.482711
3.0	2	208.126654	5969.883099	614851.654076	1.410712	4.160682	14.965422
3.0	3	468.284973	13432.236973	1383416.221673	2.116069	6.241023	22.448133
3.0	5	1300.791592	37311.769371	3842822.837980	3.526782	10.401706	37.413556
3.0	10	5203.166371	149247.077487	15371291.351921	7.053564	20.803412	74.827112
5.0	1	0.727305	3.526132	22.200496	0.261284	0.598148	1.402708
5.0	2	2.909223	14.104529	0.088.801984	0.522568	1.196297	2.805417
5.0	3	6.545751	31.735191	199.804465	0.783852	1.794445	4.208126
5.0	5	18.182644	88.153309	555.012403	1.306420	2.990742	7.013544
5.0	10	72.730576	352.613238	2220.049614	2.612840	5.981485	14.027088
10	1	0.056723	0.175123	0.551348	0.099563	0.202424	0.391474
10	2	0.226893	0.700495	2.205394	0.199126	0.404848	0.782949
10	3	0.510510	1.576114	4.962137	0.298689	0.607273	1.174424
10	5	1.418083	4.378094	13.783716	0.497816	1.012121	1.957373
10	10	5.672334	17.512378	55.134865	0.995633	2.024243	3.914747

Table 5: Quantiles of the product statistic $P(X)$ in case of a Pearson VII type d.g.f. and $p \in \{1,2\}$.

M	β	γ	$F_{P,1}^{-1}(0.95)$	$F_{P,1}^{-1}(0.99)$	$F_{P,1}^{-1}(0.999)$	$F_{P,2}^{-1}(0.95)$	$F_{P,2}^{-1}(0.99)$	$F_{P,2}^{-1}(0.999)$
0.5	0.25	1	26.214802	70.402609	169.836527	1.775893	3.979741	7.619683
0.5	0.25	2	1.304990	2.580777	4.561397	0.595947	1.072908	1.644503
0.5	0.50	1	6.553700	17.600652	42.459131	0.887946	1.989870	3.809841
0.5	0.50	2	0.652495	1.290388	2.280698	0.421398	0.758660	1.162839
0.5	1.00	1	1.638425	4.400163	10.614782	0.443973	0.994935	1.904920
0.5	1.00	2	0.326247	0.645194	1.140349	0.297973	0.536454	0.822251
1.0	0.25	1	41.327493	100.486698	224.671250	3.190207	5.967622	10.150927
1.0	0.25	2	1.653381	3.055415	5.156822	0.849096	1.328502	1.882640
1.0	0.50	1	10.331873	25.121674	56.167812	1.595103	2.983811	5.075463
1.0	0.50	2	0.826690	1.527707	2.578411	0.600401	0.939393	1.331227
1.0	1.00	1	2.582968	6.280418	14.041953	0.797551	1.491905	2.537731
1.0	1.00	2	0.413345	0.763853	1.289205	0.424548	0.664251	0.941320
2.0	0.25	1	79.351874	170.291892	344.532139	5.660650	9.214272	14.151185
2.0	0.25	2	2.302585	3.912023	6.214608	1.163087	1.644976	2.185124
2.0	0.50	1	19.837968	42.572973	86.133034	2.830325	4.607136	7.075592
2.0	0.50	2	1.151292	1.956011	3.107304	0.822426	1.163173	1.545116
2.0	1.00	1	4.959492	10.643243	21.533258	1.415162	2.303568	3.537796
2.0	1.00	2	0.575646	0.978005	1.553652	0.581543	0.822488	1.092562
5.0	0.25	1	251.565368	451.728690	785.483809	12.345260	17.458520	23.918547
5.0	0.25	2	4.086171	6.161637	8.921513	1.738476	2.228502	2.756061
5.0	0.50	1	62.891342	112.932172	196.370952	6.172630	8.729260	11.959273
5.0	0.50	2	2.043085	3.080818	4.460756	1.229288	1.575789	1.948830
5.0	1.00	1	15.722835	28.233043	49.092738	3.086315	4.364630	5.979636
5.0	1.00	2	1.021542	1.540409	2.230378	0.869238	1.114251	1.378030

Table 6: Quantiles of the product statistic $P(X)$ in case of a Kotz type d.g.f. and $p \in \{1, 2\}$.

M	m	$F_{L,1}^{-1}(0.95)$	$F_{L,1}^{-1}(0.99)$	$F_{L,1}^{-1}(0.999)$	$F_{L,2}^{-1}(0.95)$	$F_{L,2}^{-1}(0.99)$	$F_{L,2}^{-1}(0.999)$
3.0	1	13.658910	73.665171	748.666518	1.507443	2.649491	5.072205
3.0	2	27.317821	147.330343	1497.333036	2.131846	3.746947	7.173182
3.0	3	40.976731	220.995515	2245.999555	2.610968	4.589054	8.785318
3.0	5	68.294552	368.325859	3743.332591	3.370745	5.924444	11.341796
3.0	10	136.589105	736.651718	7486.665183	4.766954	8.378429	16.039723
5.0	1	1.687893	3.781444	9.564249	0.929774	1.448229	2.250395
5.0	2	3.375787	7.562888	19.128499	1.314899	2.048106	3.182539
5.0	3	5.063681	11.344332	28.692748	1.610415	2.508407	3.897799
5.0	5	8.439468	18.907220	47.821247	2.079037	3.238340	5.032037
5.0	10	16.878937	37.814440	95.642495	2.940203	4.579704	7.116375
10	1	0.479562	0.858599	1.536113	0.578021	0.850793	1.203494
10	2	0.959124	1.717198	3.072226	0.817445	1.203203	1.701998
10	3	1.438686	2.575797	4.608340	1.001162	1.473617	2.084514
10	5	2.397811	4.292995	7.680567	1.292494	1.902431	2.691096
10	10	4.795622	8.585991	15.361134	1.827863	2.690444	3.805785

Table 7: Quantiles of the linear combination $L(X)$ in case of a Pearson VII type d.g.f. and $p \in \{1, 2\}$.

M	β	γ	$F_{L,1}^{-1}(0.95)$	$F_{L,1}^{-1}(0.99)$	$F_{L,1}^{-1}(0.999)$	$F_{L,2}^{-1}(0.95)$	$F_{L,2}^{-1}(0.99)$	$F_{L,2}^{-1}(0.999)$
0.5	0.25	1	10.355786	17.338409	27.126329	2.401606	3.757573	5.324911
0.5	0.25	2	2.351913	3.348161	4.449996	1.416572	1.983611	2.505210
0.5	0.50	1	5.177893	8.669204	13.563164	1.698192	2.657005	3.765280
0.5	0.50	2	1.663053	2.367507	3.146622	1.191190	1.668011	2.106622
0.5	1.00	1	2.588946	4.334602	6.781582	1.200803	1.878786	2.662455
0.5	1.00	2	1.175956	1.674080	2.224998	1.001667	1.402625	1.771451
1.0	0.25	1	13.087248	20.767280	31.217119	3.289707	4.652695	6.180464
1.0	0.25	2	2.658695	3.645545	4.729166	1.726109	2.226517	2.690230
1.0	0.50	1	6.543624	10.383640	15.608559	2.326174	3.289952	4.370248
1.0	0.50	2	1.879981	2.577790	3.344025	1.451479	1.872270	2.262204
1.0	1.00	1	3.271812	5.191820	7.804279	1.644853	2.326347	3.090232
1.0	1.00	2	1.329347	1.822772	2.364583	1.220543	1.574385	1.902280
2.0	0.25	1	18.276562	27.112778	38.673711	4.457680	5.836231	7.335275
2.0	0.25	2	3.152056	4.126839	5.186822	2.050184	2.494617	2.907671
2.0	0.50	1	9.138281	13.556389	19.336855	3.152056	4.126839	5.186822
2.0	0.50	2	2.228840	2.918115	3.667637	1.723992	2.097714	2.445050
2.0	1.00	1	4.569140	6.778194	9.668427	2.228840	2.918115	3.667637
2.0	1.00	2	1.576028	2.063419	2.593411	1.449699	1.763961	2.056034
5.0	0.25	1	32.864736	44.290398	58.373461	6.699945	8.117634	9.595839
5.0	0.25	2	4.218838	5.176431	6.201173	2.542040	2.924737	3.278035
5.0	0.50	1	16.432368	22.145199	29.186730	4.737577	5.740034	6.785283
5.0	0.50	2	2.983169	3.660289	4.384891	2.137592	2.459401	2.756488
5.0	1.00	1	8.216184	11.072599	14.593365	3.349972	4.058817	4.797919
5.0	1.00	2	2.109419	2.588215	3.100586	1.797494	2.068101	2.317921

Table 8: Quantiles of the linear combination $L(X)$ in case of a Kotz type d.g.f. and $p \in \{1, 2\}$.

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