Geometric and stochastic representations for elliptically contoured distributions

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Abstract. A non-Euclidean geometric measure representation for elliptically contoured distributions and a stochastic representation for corresponding random vectors are derived in a similar way as analogous representations were derived in (Richter, 2007, 2009) for $l_{n,p}$-symmetric distributions. The ball number function and the indivisiblen method of Cavalieri and Torricelli are extended to ellipsoids.

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1 Introduction

It is proved in (Cambanis, Huang and Simons, 1981) that a random vector X has the elliptically contoured density

$$f_{\mu,\Sigma}(x) = |\Sigma|^{-1/2}\tilde{g}((x-\mu)^T\Sigma^{-1}(x-\mu)), x \in \mathbb{R}^n$$

with form matrix Σ and a location vector μ and where the density generator

$$\tilde{g} : [0,\infty) \to [0,\infty)$$

is non-increasing if and only if there exist a positive definite matrix C such that

$$CC^T = \Sigma,$$

and a nonnegative random variable $\tilde{R}^2$ such that X and $CZ + \mu$ are identically distributed,

$$X \stackrel{d}{=} CZ + \mu,$$

where for given $\tilde{R}^2 = r^2$, Z is uniformly distributed on the Euclidean sphere of radius r.
A characterization of the multivariate normal distribution following from this result by specifying the distribution of $\tilde{R}^2$ as a χ^2-distribution with n d.f., $\tilde{R}^2 \sim \chi^2_n$, is given in (Tong, 1990), Theorem 4.1.1.
Here, we derive an alternative stochastic representation

$$X - \mu \stackrel{d}{=} R \cdot U$$

where R and U are independent, R^2 follows a g-generalized χ^2-distribution which is defined in (Richter, 1991) for arbitrary density generating function g, and U follows a certain generalized uniform distribution $\mathfrak{U}_E$ on the σ-algebra $\mathfrak{B}(E)$ of Borel subsets of the ellipsoid E,

$$E = \{x \in \mathbb{R}^n : x^T \Sigma^{-1} x = 1\},$$

i.e.,

$$P\left(\frac{1}{R}(X - \mu) \in A\right) = \mathfrak{U}_E(A), A \in \mathfrak{B}(E).$$

To be more concrete, the distribution $\mathfrak{U}_E$ is a geometric probability measure on $\mathfrak{B}(E)$,

$$\mathfrak{U}_E(A) = \frac{\mathfrak{O}_E(A)}{\mathfrak{O}_E(E)}, A \in \mathfrak{B}(E)$$

where $\mathfrak{O}_E$, in general, denotes a non-Euclidean surface measure having special properties on ellipsoids. For how to distinguish between the notions density generator and density generating function, we refer to Section 7.
Our basic assumption is that an orthogonal matrix $O, O : \mathbb{R}^n \to \mathbb{R}^n$ can be chosen such that the diagonal matrix

$$O \Sigma O^T = diag(a_1^2, ..., a_n^2)$$

is based upon positive numbers $a_1, ..., a_n$. The form matrix of the random vector OX is then a regular matrix, $diag(a_1^2, ..., a_n^2)$, and OE is an axes-aligned ellipsoid. Hence, for simplicity, we may assume in what follows that

$$\Sigma = diag(a_1^2, ..., a_n^2) \text{ and } E = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} (\frac{x_i}{a_i})^2 = 1\}.$$

The surface measure $\mathfrak{O}_E$ is an arc-length measure in the two-dimensional case. The mathematical nature of this arc-length measure has been studied recently in (Richter, in print) based upon a suitably chosen geometry which is a non-Euclidean geometry from the point of view of metric geometry. This geometry is closely connected with the solution to the iso-perimetric problem in the Minkowski plane in (Busemann, 1947). In the present paper, we shall suitably define the surface measure $\mathfrak{O}_E$ for the multivariate case. Doing this, we shall follow the general plan of studying local and global properties of surface measures as it was realized in (Richter, 2009) for the case of $l_{n,p}$-symmetric distributions. Let us start with considering the axes-aligned elliptically contoured Gaussian distribution Φ_a. Its density is

$$\varphi_a(x) = \frac{1}{(2\pi)^{\frac{n}{2}} a_1...a_n} \exp\{-\frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{a_i^2}\},\ x = (x_1, ..., x_n)^T \in \mathbb{R}^n$$

and corresponds to the density generator $\tilde{g}(v) = \frac{1}{(2\pi)^{n/2}} e^{-v/2} I_{(0,\infty)}(v)$. In the special case that all variances are equal to 1, the density φ_a is that of the standard Gaussian distribution. Geometric and stochastic representations follow for this case from (Richter, 2009) and several earlier papers and monographs mentioned therein. The main aim of the present paper is to extend such results to the heteroscedastic case. We shall derive results being analogous to those in (Richter, 2007, 2009) which deal with $l_{n,p}$-symmetric distributions. According to the general method of analyzing the non-Euclidean geometry underlying a multivariate probability distribution which was developed in (Richter, 2007, 2009), we shall study measuring ellipsoids in Section 2, extend the ball number function to a range of definition including ellipsoids in Section 3 and define generalized ellipsoidal coordinates in Section 4. Section 5 deals with the connection between the global and the local approaches to the non-Euclidean surface content upon which our geometric measure representations are based. These representations itself will be considered for the Gaussian case in Section 6 and for the case of general density generating functions in Section 7. The final Section 8 presents some examples for how to apply the general geometric measure representation and gives some outlook onto further studies.

2 Measuring ellipsoids

The main result of this section is a generalized surface content formula for ellipsoids in Lemma 2.

Let $a = (a_1, ..., a_n)^T \in \mathbb{R}^n_+$ be an arbitrary vector having nonnegative components, $|.|_a : \mathbb{R}^n \to [0, \infty)$ the norm defined by

$$|x|_a = (\sum_1^n (\frac{x_i}{a_i})^2)^{1/2}, x \in \mathbb{R}^n$$

and

$$B_a = \{x \in \mathbb{R}^n : |x|_a \leq 1\}$$

the corresponding unit-ball. Its topological boundary, $\partial B_a = E_a$, is an axis-aligned ellipsoid or elliptical sphere. For arbitrary $R > 0$ and $M \subset \mathbb{R}^n$, put

$$RM = \{(Rx_1, ..., Rx_n)^T : (x_1, ..., x_n)^T \in M\}.$$

The evaluation of the volume of $B_a(R) = RB_a$ may be easily reduced to the evaluation of the volume of an Euclidean ball having a suitable radius. To this end, denote the Euclidean ball of radius R by $K_n(R) = RK_n$ where $K_n = B_{\mathbb{1}}, \mathbb{1} = (1, ..., 1)^T \in \mathbb{R}^n$, and its topological boundary, the sphere of radius R, by $S_n(R) = \partial K_n(R) = R\partial K_n$. If

$$a_i^* = \prod_{j=1, j\neq i}^n a_j, i = 1, ..., n \quad \text{and} \quad \text{diag}(a_1^*, \ldots, a_n^*) = \begin{pmatrix} a_2...a_n & & \\ & \ldots & \\ & & a_1...a_{n-1} \end{pmatrix}$$

then

$$diag(a_1^*, \ldots, a_n^*)B_a(R) = K_n(a_1...a_nR).$$

Let λ denote the Lebesgue measure in $\mathbb{R}^n$. Changing variables $u = diag(a_1^*, \ldots, a_n^*)x$ in the integral

$$\lambda(B_a(R)) = \int\limits_{\{x\in R^n: |x|_a \le R\}} dx$$

gives

$$\lambda(B_a(R)) = \int\limits_{K_n(Ra_1...a_n)} \frac{du}{(a_1...a_n)^{n-1}}$$

since

$$\frac{d(u_1, ..., u_n)}{d(x_1, ..., x_n)} = \prod_1^n a_i^* = (\prod_1^n a_i)^{n-1}.$$

Hence,

$$\lambda(B_a(R)) = \frac{\lambda(K_n(Ra_1...a_n))}{(a_1...a_n)^{n-1}} = a_1...a_n \frac{\omega_n}{n} R^n$$

where

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} = O(S_n)$$

is the surface content of the Euclidean unit-sphere $S_n = S_n(1)$.

Measuring the Euclidean surface content of $E_a(R) = RE_a$ necessarily involves elliptical integrals of different types. In this paper, however, we shall turn over to another definition of surface content which avoids elliptical integrals. To this end, we shall consider the ellipsoid $E_a(R)$ as a subset of the Minkowski space $(\mathbb{R}^n, |.|_{\frac{1}{a}})$, $\frac{1}{a} = (\frac{1}{a_1}, ..., \frac{1}{a_n})^T$. We will introduce in this section the notion of the $|.|_{\frac{1}{a}}$-surface content of $E_a(R)$ in a similar way as the notion of the $l_{n,q}$-surface content was introduced in (Richter, 2009) for $l_{n,p}$-spheres where p and q are connected with each other by the equation $\frac{1}{p} + \frac{1}{q} = 1$.

Let y be defined as the positive solution of

$$\sum_{i=1}^{n-1} (x_i/a_i)^2 + (y/a_n)^2 = R^2.$$

At the point $(x_1, ..., x_{n-1}, y)^T$, the normal vector to the upper half of the ellipsoid $E_a(R)$ is

$$N(x_1, ..., x_{n-1}) = (-1)^n (\sum_{i=1}^{n-1} \frac{\partial y}{\partial x_j} e_j - e_n).$$

Since it always will become clear how to deal with the case $y < 0$, we will not further mention this case.

Definition 1 *Let $\mathfrak{B}_n$ denote the Borel σ-field in $\mathbb{R}^n$ and let $A \subset E_a(R) \cap \mathfrak{B}_n$. The $|.|_{\frac{1}{a}}$-surface content of A is defined by*

$$O_E(A) = \int\limits_{G(A)} |N(x_1, ..., x_{n-1})|_{\frac{1}{a}} dx_1 ... dx_{n-1}$$

where

$$G(A) = \{(x_1, ..., x_{n-1})^T : (x_1, ..., x_n)^T \in A\}..$$

Lemma 2 *The $|.|_{\frac{1}{a}}$-surface content of $E_a(R)$ is*

$$O_E(E_a(R)) = a_1 ... a_n \omega_n R^{n-1}.$$

Proof :

It follows from

$$\frac{\partial y}{\partial x_j} = -\frac{a_n x_j}{a_j^2 (R^2 - \sum\limits_{i=1}^{n-1} (\frac{x_i}{a_i})^2)^{1/2}}, \; j = 1, ..., n-1$$

that

$$|N(x_1, ..., x_{n-1})|_{\frac{1}{a}}^2 = \sum_{j=1}^{n-1} \frac{a_n^2 x_j^2}{a_j^2 (R^2 - \sum\limits_{i=1}^{n-1} (\frac{x_i}{a_i})^2)} + a_n^2 = \frac{a_n^2 R^2}{R^2 - \sum\limits_{i=1}^{n-1} (\frac{x_i}{a_i})^2}.$$

Hence, because of symmetry,

$$O_E(E_a(R)) = 2 a_n R \int\limits_{\sum\limits_{i=1}^{n-1} (\frac{x_i}{a_i})^2 \leq R^2} \frac{dx_1 ... dx_{n-1}}{(R^2 - \sum\limits_{i=1}^{n-1} (\frac{x_i}{a_i})^2)^{1/2}}.$$

Let the $(n-1)$-dimensional standard ellipsoidal coordinate transformation

$$T_a : M_{n-1} \to \mathbb{R}^{n-1}, M_{n-1} = [0, \infty) \times M_{n-1}^*, M_{n-1}^* = [0, \pi)^{\times (n-3)} \times [0, 2\pi)$$

be defined by $x_1 = a_1 r \cos\phi_1, x_2 = a_2 r \sin\phi_1 \cos\phi_2, ..., x_{n-2} = a_{n-2} r \sin\phi_1 ... \sin\phi_{n-3} \cos\phi_{n-2}$, $x_{n-1} = a_{n-1} r \sin\phi_1 ... \sin\phi_{n-3} \sin\phi_{n-2}$. If $a = \mathbb{1} \in \mathbb{R}^{n-1}$ then this transformation coincides with the $(n-1)$-dimensional spherical coordinate transformation $SPH^{(n-1)}$ the Jacobian of which is well known. If we write $J(T)$ for the Jacobian of the coordinate transformation T then

$$J(T_a)(r,\phi) = |\frac{d(x_1,...,x_{n-1})}{d(r,\phi_1,...,\phi_{n-2})}| = a_1 \cdot ... \cdot a_{n-1} J(SPH^{(n-1)})(r,\phi)\,.$$

Let $J^*(SPH^{(n-1)})(\phi) = J(SPH^{(n-1)})(1,\phi)$ be the Jacobian of the restriction of $SPH^{(n-1)}$ to the sphere defined by $r = 1$. Changing Cartesian with standard ellipsoidal coordinates gives

$$O_E(E_a(R)) = 2a_n R \int_0^R \frac{r^{n-2}}{\sqrt{R^2 - r^2}} dr$$

$$\times \int_0^\pi ... \int_0^\pi \int_0^{2\pi} a_1...a_{n-1} J^*(SPH^{(n-2)})(\phi_1,...,\phi_{n-2}) d\phi_{n-2}...d\phi_1.$$

Because of

$$\int_0^R \frac{r^{n-2}dr}{\sqrt{R^2-r^2}} = R^{n-2}\int_0^1 \frac{t^{n-2}dt}{\sqrt{1-t^2}} = \frac{1}{2}R^{n-2}B(\frac{1}{2},\frac{n-1}{2})$$

and

$$B(\frac{1}{2},\frac{n-1}{2})\omega_{n-1} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}\frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} = \omega_n$$

it follows

$$O_E(E_a(R)) = a_1...a_n\omega_n R^{n-1}$$

□

3 The ellipsoid-number function and the generalized indivisiblen method

It is known from (Richter, 2009) that the geometric analysis of $l_{n,p}$-symmetric densities is closely connected with the definition of $l_{n,p}$-ball numbers. Similar circumstances in the analysis of elliptically contoured distributions allow us to extend here the range of definition of the ball-number function in such a way that ellipsoids will be included.

It is well known that

$$\frac{\lambda(K_n(R))}{R^n} = \frac{O(S_n(R))}{nR^{n-1}} = \frac{\omega_n}{n},\ R > 0.$$

This was the motivation to introduce the ball-number function

$$n \to \pi_n(2) = \frac{\omega_n}{n}$$

and further to extend it to a function $(n, p) \to \pi_n(p)$ which assigns a ball number $\pi_n(p)$ to every $l_{n,p}$-ball, $n = 2, 3, ..., p > 0$.

It is obvious that

$$O(S_n(R)) = \frac{d}{dR}\lambda(K_n(R)), R > 0$$

and hence,

$$\lambda(K_n(R)) = \int_0^R O(S_n(r))dr.$$

This formula reflects the indivisiblen method of Cavalieri and Torricelli in the sense that the indivisiblen are the spheres $S_n(r)$ and measuring them is due to the Euclidean surface content. A similar formula was derived in (Richter, 2009) where the indivisiblen are $l_{n,p}$-spheres and measuring them is due to a suitably defined $l_{n,q}$-surface measure. In this section, we prove an extension of this method such that from now on also ellipsoids may play the role of the indivisiblen. From the statistical point of view, this enables us to deal with multivariate sampling densities reflecting the heteroscedastic case.

It follows from what is shown in the preceding section that

$$\frac{\lambda(B_a(R))}{R^n} = \frac{O_E(E_a(R))}{nR^{n-1}} = a_1...a_n\pi_n(2),\ R > 0.$$

The following definition is thus well motivated.

Definition 3 *The ellipsoid-number function $(a, n) \to \pi_n^E(a)$ is defined by*

$$\pi_n^E(a) = a_1...a_n\frac{\omega_n}{n}, a \in R_+^n, n \in \{2, 3, ...\}..$$

It is obvious that

$$O_E(E_a(R)) = \frac{d}{dR}\lambda(B_a(R)), R > 0$$

and hence,

$$\lambda(B_a(R)) = \int\limits_0^R O_E(E_a(r))dr.$$

On the one hand, it is already this formula which may be considered as a generalization of the indivisiblen method of Cavalieri and Torricelli. On the other hand, this formula will be considerably generalized itself in Theorem 5. We shall make use of the following notions, there.

Definition 4 *(a) The geometric probability distribution*

$$U_a^E(D) = \frac{O_E(D)}{O_E(E_a)},\ D \in \mathfrak{B}_a^E = \mathfrak{B}_n \cap E_a$$

is called the $|.|_{\frac{1}{a}}$*-generalized uniform distribution on* $\mathfrak{B}_a^E$.
(b) For an arbitrary Borel subset A *of* $\mathbb{R}^n$*, the* E_a*-intersection-percentage function (i.p.f.)* $r \to \mathfrak{F}_a^E(A, r)$ *is defined as*

$$\mathfrak{F}_a^E(A, r) = U_a^E([r^{-1}A] \cap E_a), r > 0.$$

Theorem 5 *Let* $A \in \mathfrak{B}_n$ *satisfy* $\lambda(A) < \infty$*, then*

$$\lambda(A) = n\pi_n^E(a) \int\limits_0^\infty \mathfrak{F}_a^E(A, r)\, r^{n-1}\, dr = \int\limits_0^\infty O_E(A \cap E_a(r))dr\,.$$

Proof :

The central projection cone corresponding to the set $D \in \mathfrak{B}_a^E$ is defined as

$$CPC_a(D) = \{x \in \mathbb{R}^n : \frac{x}{|x|_a} \in D\},$$

and the ellipsoidal sector of radius $\rho > 0$ generated by D is

$$sector_a(D, \rho) = CPC_a(D) \cap B_a(\rho).$$

We consider now Borel sets of the type

$$A_a(D,\rho_1,\rho_2) = sector_a(D,\rho_1) \setminus sector_a(D,\rho_2), D \in \mathfrak{B}_a^E$$

for arbitrary $\rho_1 < \rho_2$ from $[0,\infty)$. The collection of all such sets is a semi-algebra on $\mathbb{R}^n$, say $\mathfrak{S}_a$. The smallest algebra including $\mathfrak{S}_a$, i.e. the collection of finite unions of elements from $\mathfrak{S}_a$, will be denoted by $\mathfrak{A}_a$. Let us further consider the finitely additive function λ^* on $\mathfrak{A}_a$ which is already defined by its values for elements from $\mathfrak{S}_a$ as

$$\lambda^*(A) = n\pi_n^E(a)\int\limits_{\rho_1}^{\rho_2} \mathfrak{F}_a^E(A,r)r^{n-1}dr, A = A_a(D,\rho_1,\rho_2), 0 \le \rho_1 < \rho_2 < \infty, D \in \mathfrak{B}_a^E.$$

Note that $\lambda^*(A_n)$ tends to zero whenever $(A_n)_{n\in N}$ is a decreasing sequence of sets from $\mathfrak{A}_a$ satisfying $\bigcap\limits_n A_n = \emptyset$ for the empty set $\emptyset$. This means that λ^* is continuous at $\emptyset$ and therefore a countable additive function on $\mathfrak{A}_a$. Now, it remains to show that λ^* coincides with λ on $\mathfrak{A}_a$. Then, by measure extension theorem, λ^* coincides with λ on the whole σ-algebra $\mathfrak{B}_n$. For doing the latter it suffices to show that $\lambda^*(A)$ is the same as $\lambda(A)$ for sets of the type

$$A = A_a(D,\varrho_1,\varrho_2),\ 0 \le \varrho_1 < \varrho_2 < \infty, D \in \mathfrak{B}_a^E..$$

Due to the product structure of the set $A_a(D,\varrho_1,\varrho_2) = [\varrho_1,\varrho_2]\cdot D$, the function $r \to \mathfrak{F}_a^E(A_a(D,\varrho_1,\varrho_2),r)$ is constant with the constant being $U_a^E(D)$. Hence,

$$\lambda^*(A_a(D,\varrho_1,\varrho_2)) = \pi_n^E(a)U_a^E(D)(\rho_2^n - \rho_1^n).$$

Let us consider now λ. For every $R > 0$, we define the uniform probability measure μ_R on the $|.|_{\frac{1}{a}}$-ball of radius R, $B_a(R)$, by

$$\mu_R(A) = \frac{\lambda(A \cap B_a(R))}{\lambda(B_a(R))}, R > 0, A \in \mathfrak{B}_n$$

and assume that the random vector $Y(R)$ follows the distribution μ_R. Further, we define the random elements

$$\xi(R) = |Y(R)|_a \text{ and } U(R) = \frac{1}{\xi(R)}Y(R).$$

The c.d.f. of $\xi(R)$ is

$$P(\xi(R) < t) = P(Y(R) \in B_a(t)) = \frac{\lambda(B_a(t))}{\lambda(B_a(R))}, 0 < t \leq R,$$

and the probability distribution of $U(R)$ allows the representation

$$P(U(R) \in D) = P(Y(R) \in sector_a(D, R)) = \frac{\lambda(sector_a(D, R))}{\lambda(B_a(R))},\ D \in \mathfrak{B}_a^E.$$

Since

$$\frac{\lambda(sector_a(D, R))}{\lambda(B_a(R))} = \frac{\lambda(sector_a(D, t))}{\lambda(B_a(t))},$$

it follows

$$\begin{aligned} P\left(\xi(R) < t\right) P\left(U(R) \in D\right) &= \frac{\lambda(sector_a(D, t))}{\lambda(B_a(R))} = P\left(Y(R) \in sector(D, t)\right) \\ &= P\left(\xi(R) < t,\, U(R) \in D\right) . \end{aligned}$$

Hence $\xi(R)$ and $U(R)$ are independent.

Lemma 6 *The distribution $P^{U(R)}$ induced by $U(R)$ on $\mathfrak{B}_a^E$ coincides with the generalized uniform surface measure U_a^E.*

For the proof of this lemma, we refer to the end of Section 5.

We are now in a position to finish the proof of Theorem 5.

Due to the product structure of the set $sector_p(D, \varrho) = [0, \varrho] \cdot D$, it follows

$$\lambda(sector_a(D, \varrho)) = \lambda(B_a(R))\mu_R(sector_a(D, \varrho)) = \lambda(B_a(R))P^{\xi(R)}([0, \varrho])P^{U(R)}(D)$$

$$= \pi_n^E(a)\rho^n U_a^E(D), \varrho > 0, D \in \mathfrak{B}_a^E.$$

Hence

$$\lambda(A_a(D, \varrho_1, \varrho_2)) = \pi_n^E(a)U_a^E(D)(\rho_2^n - \rho_1^n), 0 \leq \varrho_1 < \varrho_2 < \infty,\ D \in \mathfrak{B}_a.$$

The other equation in the statement of the theorem follows now from

$$O_E(A \cap E_a(r)) = r^{n-1}\frac{O_E([r^{-1}A] \cap E_a)}{O_E(E_a)}n\pi_n^E(a) = r^{n-1}U_a^E([r^{-1}A] \cap E_a)n\pi_n^E(a)$$

$\square$

The integral representation of the Lebesgue measure in Theorem 5 may be considered as reflecting a non-Euclidean generalization of the indivisiblen method of Cavalieri and Torricelli in the sense that the indivisiblen of a set A are the intersections of A with the ellipsoids $E_a(r), 0 < r < \infty$, $A \cap E_a(r)$, and measuring the latter is due to the non-Euclidean surface measure O_E. Hence, the generalized indivisiblen method proved in (Richter, 2009) for $l_{n,p}$-spheres is extended here to ellipsoids. Theorem 5 may also be considered as presenting a certain so called disintegration formula for the Lebesgue measure.

It follows from the co-area formula of measure theory (see, e.g., in (Evans and Gariepy, 1992)) that the equation in Theorem 5 does not hold, in general, if the surface measure O_E is changed with the usual Euclidean surface content O. To be more specific, let

$$f_E(x) = \inf\{\lambda > 0 : x \in \lambda E^*\}, x \in \mathbb{R}^n,$$

denote the norm generated by the convex body E^* having the boundary E. Then the function $f = f_E | \mathbb{R}^n \to \mathbb{R}^1, n \geq 1$ is Lipschitzian,

$$|f(x) - f(y)| \leq C|x - y|, x, y \in A, A \in \mathfrak{B}_n,$$

and the co-area formula asserts that

$$\int_A J(f)(x)dx = \int_0^\infty O(A \cap f^{-1}(\{r\}))dr\,.$$

To illustrate this, let us consider for simplicity the two-dimensional case with $E = E_{(a,b)^T}, b < 1 < a$. Then the latter formula reads as

$$\nu(A) := \int\limits_A \left(\frac{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}}{\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}}\right)^{1/2} dx = \int\limits_0^\infty O(A \cap (rE))dr.$$

Notice that

$$\nu(M) = \lambda(M) \qquad \text{iff} \qquad a = 1 = b,$$

hence, there is no extension of the classical indivisiblen method to the case $b < 1 < a$ on the basis of Euclidean metric geometry.

We finish this section with an asymptotic comparison of $|.|_{\frac{1}{a}}$-norms, $a \in \mathbb{R}^n_+$. It is well known that all norms defined in $\mathbb{R}^n$ are equivalent. From an asymptotic point of view, however, it is possible to distinguish between different $|.|_{\frac{1}{a}}$-norms in the following way.

Corollary 7 *The asymptotic relation*

$$\lambda(\{x \in \mathbb{R}^n : r < |x|_a < r + \varepsilon\}) \sim n\pi_n^E(a) r^{n-1}\varepsilon, \varepsilon \to 0$$

follows immediately from Theorem 5.

Proof :
The i.p.f. of the set $A = \{x \in \mathbb{R}^n : r < |x|_a < r + \varepsilon\}$ is

$$\mathfrak{F}_a^E(A, \rho) = I_{(r,r+\varepsilon)}(\rho),\ \rho > 0.$$

Theorem 5 applies,

$$\lambda(A) = n\pi_n^E(a) \int_0^\infty \rho^{n-1} I_{(r,r+\varepsilon)}(\rho) d\rho.$$

Hence, $\lambda(A) = \pi_n^E(a)[(r + \varepsilon)^n - r^n]$ □

Definition 8 *The asymptotic relation in Corollary 7 is called the ellipsoidal thin-layers property of the Lebesgue measure.*

4 Generalized ellipsoidal coordinates

We recall that $l_{n,p}$-generalized trigonometric functions and coordinates have been proved in (Richter, 2007, 2009) to be powerful tools for studying $l_{n,p}$-symmetric distributions. The coordinates which we define in this section are used, e.g., in the next section for showing the equivalence of two approaches to the surface measure O_E.
The $E_{(a,b)}$-generalized trigonometric functions $\sin_{(a,b)}$ and $\cos_{(a,b)}$ were recently introduced in (Richter, submitted). On using them, let the generalized ellipsoidal coordinate transformation

$$T_a^E : M_n \to \mathbb{R}^n, M_n = [0, \infty) \times M_n^*, M_n^* = [0, \pi)^{\times(n-2)} \times [0, 2\pi)$$

be defined by

$$x_1 = a_1 r \cos_{(a_1,a_2)}(\phi_1), x_2 = a_2 r \sin_{(a_1,a_2)}(\phi_1)\cos_{(a_2,a_3)}(\phi_2), ...,$$

$$x_{n-1} = a_{n-1} r \sin_{(a_1,a_2)}(\phi_1)...\sin_{(a_{n-2},a_{n-1})}(\phi_{n-2})\cos_{(a_{n-1},a_n)}(\phi_{n-1}),$$

$$x_n = a_n r \sin_{(a_1,a_2)}(\phi_1)...\sin_{(a_{n-2},a_{n-1})}(\phi_{n-2})\sin_{(a_{n-1},a_n)}(\phi_{n-1}).$$

Theorem 9 *The map T_a^E is almost one-to-one, its inverse $(T_a^E)^{-1}$ is given by*

$$r = (\sum_1^n (\frac{x_i}{a_i})^2)^{1/2}, \phi_j = \arccos_{(a_j,a_{j+1})} \frac{x_j/a_j}{(\sum\limits_{i=j}^n (x_i/a_i)^2)^{1/2}}, \phi_{n-1} = \arctan \frac{x_n}{x_{n-1}}$$

where $\arccos_{(a_j,a_{j+1})}$ *denotes the inverse function to* $\cos_{(a_j,a_{j+1})}$.

Proof :

The proof of this theorem is quite similar to that of Theorem 1 in (Richter, 2007) and will therefore be omitted, here. □

Theorem 10 *The Jacobian of the coordinate transformation T_a^E satisfies the representation*

$$J(T_a^E)(r,\phi_1,...,\phi_{n-1}) = r^{n-1} J^*(T_a^E)(\phi_1,...,\phi_{n-1})$$

where

$$J^*(T_a^E)(\phi_1,...,\phi_{n-1}) = a_n \prod_{i=1}^{n-1} \frac{(\sin_{(a_i,a_{i+1})}(\phi_i))^{n-1-i}}{a_{i+1} N^2_{(a_i,a_{i+1})}(\phi_i)}.$$

For the proof if this theorem, see the Appendix.

Corollary 11 *It follows from Theorem 10 that if $n = 2$ then*

$$J(T_a^E)(r,\phi) = \frac{r}{N^2_{(a_1,a_2)}(\phi)}$$

and if $n = 3$ then

$$J(T_a^E)(r,\phi_1,\phi_2) = \frac{r^2 \sin\phi_1}{a_2^2 N^2_{(a_2,a_3)}(\phi_2) N^3_{(a_1,a_2)}(\phi_1)}.$$

The first assertion in Corollary 11 is known from (Richter, 2011) and the second one follows alternatively by straightforward calculations from

$$J(T_a^E)(r,\phi_1,\phi_2) = |\frac{D(x_1,x_2,x_3)}{D(r,\phi_1,\phi_2)}| = | \begin{vmatrix} \frac{x_1}{r} & -x_1 l(\phi_1) - \frac{r\sin\phi_1}{N(\phi_1)} & 0 \\ \frac{x_2}{r} & -x_2 l(\phi_1) + \frac{x_1\cos\phi_2}{a_2 N(\phi_2)} & -x_2 l(\phi_2) - x_3 \\ \frac{x_3}{r} & \frac{x_1\sin\phi_2}{a_2 N(\phi_2)} - x_3 l(\phi_1) & -x_3 l(\phi_2) + x_3 \end{vmatrix} |$$

where $l(\phi_i) = N'_{(a_i,a_{i+1})}(\phi_i)/N_{(a_i,a_{i+1})}(\phi_i)$.

5 The local approach to the $|.|_{\frac{1}{a}}$-surface measure

The surface measure O_E was introduced in §2 following a differential geometrical, integral or global approach. In the present section, we deal with an alternative local approach using derivatives. In this sense, we continue to follow the general method of analyzing the non-Euclidean geometry underlying a multivariate probability distribution which was developed in (Richter, 2007, 2009). To start with, we consider for the set $A \in \mathfrak{B}_a^E$ the volume of the ellipsoidal sector of radius $\rho > 0$, $f(\rho) = \lambda(sector_a(A,\rho))$.

Definition 12 *The measure $\mathfrak{O}_E : \mathfrak{B}_a^E \to r^+$ defined by*

$$\mathfrak{O}_E(A) = f'(1)$$

is called the E_a-generalized surface measure.

The following theorem says that the E_a-generalized surface measure coincides with the $|.|_{\frac{1}{a}}$-surface measure. For a comparison of both these surface measures, it is sufficient to consider them for sets $A \in \mathfrak{B}_a^E$.

Theorem 13 $\mathfrak{O}_E(A) = O_E(A), \forall A \in \mathfrak{B}_a^E$.

Proof :

We start from the equation

$$O_E(A) = a_n \int\limits_{G(A)} \frac{d(x_1,...,x_{n-1})}{(1 - \sum\limits_{i=1}^{n-1}(x_i/a_i)^2)^{1/2}}$$

and change Cartesian with generalized ellipsoidal coordinates in $(n-1)$ dimensions,

$$(x_1, ..., x_{n-1}) \overset{T_a^E(n-1)}{\longrightarrow} (r, \phi_1, ..., \phi_{n-2}).$$

Because of

$$\cos^2_{(a_i,a_{i+1})}(\phi_i) + \sin^2_{(a_i,a_{i+1})}(\phi_i) = 1,$$

there holds $\sum\limits_{i=1}^{n-1}(x_i/a_i)^2 = r^2$. Hence,

$$O_E(A) = a_n \int\limits_{(T_a^E(n-1))^{-1}(G(A))} \frac{r^{n-2}}{\sqrt{1-r^2}} J^*(T_a^E(n-1))(\phi_1, ..., \phi_{n-2}) d(r, \phi_1, ..., \phi_{n-2})$$

where according to Theorem 10

$$J^*(T_a^E(n-1))(\phi_1, ..., \phi_{n-2}) = a_{n-1} \prod_{i=1}^{n-2} \frac{(\sin_{(a_i,a_{i+1})}(\phi_i))^{n-1-i}}{a_{i+1} N^2_{(a_i,a_{i+1})}(\phi_i)}.$$

If $A = A(r_1, r_2, M^*)$

$$= \{(y_1, ..., y_{n-1}, (1 - \sum_{i=1}^{n-1}(x_i/a_i)^2)^{1/2})^T : (y_1, ..., y_{n-1})^T = T_a^E(n-1)([r_1, r_2) \times M^*)\},$$

with

$$M^* = \{(\phi_1, ..., \phi_{n-2}) : \phi_{il} \leq \phi_i \leq \phi_{iu}, i = 1, ..., n-2\} \subset [0, \pi)^{\times(n-3)} \times [0, 2\pi) = M^*_{n-1}$$

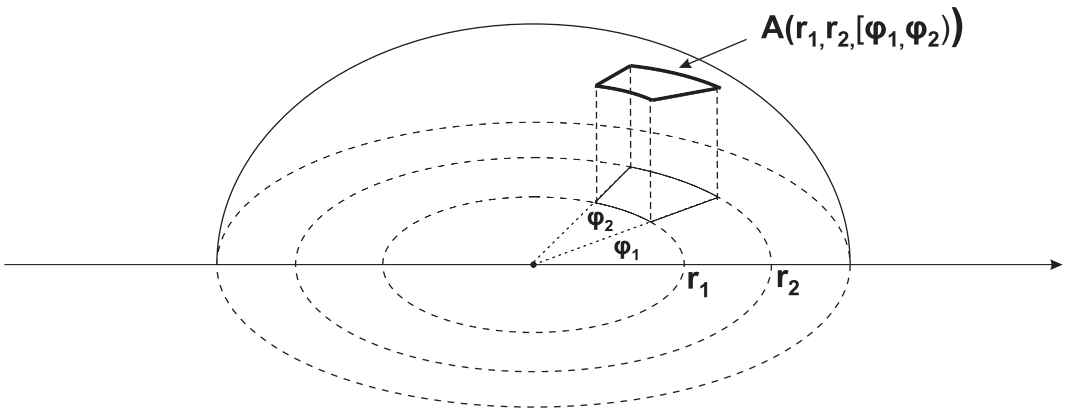

Figure 1: $n = 3$, the set $A(r_1, r_2, [\varphi_1, \varphi_2)) \in \mathfrak{B}_a^E$

then

$$O_E(A) = a_n a_{n-1} \int\limits_{r^1}^{r_2} \frac{r^{n-2}}{\sqrt{1-r^2}} dr \int\limits_{M^*} J^*(T_a^E(n-1))(\phi_1, ..., \phi_{n-2}) d(\phi_1, ..., \phi_{n-2}) \,.$$

In what follows, we shall make use of the coordinate transformation $\tilde{T}_a : (R, r, \phi) \to z[R, r, \phi]$ defined by

$$z_1 = a_1 Rr \cos_{(a_1,a_2)}(\phi_1),\, z_2 = a_2 Rr \sin_{(a_1,a_2)}(\phi_1) \cos_{(a_2,a_3)}(\phi_2), ...,$$

$$z_{n-2} = a_{n-2} Rr \sin_{(a_1,a_2)}(\phi_1) \cdot ... \cdot \sin_{(a_{n-3},a_{n-2})}(\phi_{n-3}) \cos_{(a_{n-2},a_{n-1})}(\phi_{n-2})$$

$$z_{n-1} = a_{n-1} Rr \sin_{(a_1,a_2)}(\phi_1) \cdot ... \cdot \sin_{(a_{n-3},a_{n-2})}(\phi_{n-3}) \sin_{(a_{n-2},a_{n-1})}(\phi_{n-2}),$$

$$z_n = a_n R\sqrt{1-r^2}.$$

It follows from

$$(\frac{z_1}{a_1})^2 + ... + (\frac{z_{n-1}}{a_{n-1}})^2 = R^2 r^2 \text{ and } (\frac{z_n}{a_n})^2 = R^2(1-r^2)$$

that $(\frac{z_1}{a_1})^2 + ... + (\frac{z_n}{a_n})^2 = R^2$.

The coordinate transformation $\tilde{T}_a$ allows the following representations:

$$\begin{aligned} A(r_1, r_2, M^*) &= \tilde{T}_a(1, [r_1, r_2), M^*) \\ &= \{z[R, r, \phi] : R = 1, r \in [r_1, r_2), \phi \in M^*\} \text{ (see Figure 1)} \end{aligned}$$

and

$$sector(A(r_1, r_2, M^*), \rho) = \tilde{T}_a([0, \rho), [r_1, r_2), M^*)$$

$$= \{z[R, r, \phi] : 0 \le R < \rho, r \in [r_1, r_2), \phi \in M^*\}.$$

The quantity

$$\lambda(sector(A(r_1, r_2, M^*), \rho)) = \int\limits_{sector(A(r_1,r_2,M^*),\rho)} dz$$

may therefore be written as

$$\lambda(sector(A(r_1, r_2, M^*), \rho)) = \int\limits_{R=0}^{\rho} \int\limits_{r=r_1}^{r_2} \int\limits_{\phi \in M^*} J(\tilde{T}_a)(R, r, \phi) dR dr d\phi.$$

Here,

$$J(\tilde{T}_a)(R,r,\phi) = \frac{D(z_1,...,z_n)}{D(R,r,\phi_1,...,\phi_{(n-2)})}$$

$$= |z_{nr} \begin{vmatrix} z_{1R} & z_{1\phi_1} & & z_{1\phi_{n-2}} \\ \vdots & \vdots & \dots & \\ z_{n-1R} & z_{n-1\phi_1} & & z_{n-1\phi_{n-2}} \end{vmatrix} - z_{nR} \begin{vmatrix} z_{1r} & z_{1\phi_1} & & z_{1\phi_{n-2}} \\ \vdots & \vdots & \dots & \\ z_{n-1r} & z_{n-1\phi_1} & & z_{n-1\phi_{n-2}} \end{vmatrix} |$$

where the variable in the second index of any function z_i indicates that the derivative of this function is taken w.r.t. the corresponding variable. Notice that the two determinants may be considered as the Jacobians of generalized ellipsoidal coordinate transformations in $(n-1)$-dimensions where respectively the variables r and R are replaced with rR. The derivation w.r.t. R forces an additional factor r in the first term and the derivation w.r.t. r forces an additional factor R in the second term. Hence,

$$J(\tilde{T}_a)(R,r,\phi) = |z_{nr}a_{n-1}(rR)^{n-2}J^*(T_a^E(n-1))(\phi)r - z_{nR}a_{n-1}(rR)^{n-2}J^*(T_a^E(n-1))(\phi)R|$$

$$= |(-\frac{a_nRr}{\sqrt{1-r^2}}(rR)^{n-2}r - a_n\sqrt{1-r^2}(rR)^{n-2}R)J^*(T_a^E(n-1))(\phi)a_{n-1}|$$

$$= a_{n-1}a_nJ^*(T_a^E(n-1))(\phi)R^{n-1}\frac{r^{n-2}}{\sqrt{1-r^2}}.$$

It follows that

$$\lambda(sector(A(r_1,r_2,M^*),\rho)) = a_{n-1}a_n\int_0^\rho R^{n-1}dR\int_{r_1}^{r_2}\frac{r^{n-2}}{\sqrt{1-r^2}}dr\int_{M^*}J^*(T_a^E(n-1))(\phi)d\phi.$$

We observe now

$$\frac{d}{d\rho}\lambda(sector(A(r_1,r_2,M^*),\rho))|_{\rho=1} = a_{n-1}a_n\int_{r_1}^{r_2}\frac{r^{n-2}}{\sqrt{1-r^2}}dr\int_{M^*}J^*(T_a^E(n-1))(\phi)d\phi$$

so that

$$\mathfrak{O}_E(A(r_1,r_2,M^*)) = O_E(A(r_1,r_2,M^*)).$$

The measures $\mathfrak{O}_E$ and O_E coincide on the semi-algebra which is generated by the sets of the type $A(r_1,r_2,M^*)$. It follows from the measure extension theorem that the measures $\mathfrak{O}_E$ and O_E coincide also on the whole Borel-σ-field $\mathfrak{B}_a^E$ on E_a $\square$

We are now in a position to prove Lemma 6 from Section 3.

Proof :

Proof of Lemma 6 We start from the representation of $P^{U(R)}$ in the proof of Theorem 5. If we write integrals for the Lebesgue measure of the sets $sector_a(D,R)$ and $B_a(R)$, and change Cartesian with generalized ellipsoidal coordinates, then we get according to Theorem 10

$$P^{U(R)}(D) = \frac{\int\limits_{(T_a^{E*})^{-1}(D)} J^*(T_a^E)(\phi)d\phi}{\int\limits_{M_n^*} J^*(T_a^E)(\phi)d\phi},\ D \in \mathfrak{B}_a^E.$$

Here, T_a^{E*} denotes the restriction of the map T_a^E to the case $r = 1, T_a^{E*}(\phi) = T_a^E(1,\phi)$. According to Definition 4, Theorem 13 and Definition 12,

$$U_a^E(D) = \frac{\frac{d}{dr}\lambda(sector_a(D,\rho))|_{\rho=1}}{\frac{d}{dr}\lambda(B_a(\rho))|_{\rho=1}}.$$

If we write now integrals for the Lebesgue measure of the sets $sector_a(D,R)$ and $B_a(R)$ and change variables as above, then it follows from Theorem 10 that $U_a^E(D)$ satisfies the same representation as $P^{U(R)}(D)$. □

Corollary 14 *According to Theorem 13 and Definition 12, it follows from Theorem 10 that the $|.|_{\frac{1}{a}}$-surface content O_E on $\mathfrak{B}_a^E$ allows the representation*

$$O_E(D) = \int\limits_{(T_a^{E*})^{-1}(D)} J^*(T_a^E)(\phi)d\phi.$$

6 Representations for axes-aligned elliptically contoured Gaussian distributions

In this section, we prove that there is a natural stochastic product representation for axes-aligned elliptically contoured Gaussian random vectors in terms of a suitably defined radius variable and a suitably defined uniform basis vector on the ellipsoid E_a. Moreover, both these quantities determine essentially the corresponding geometric measure representation. In the next section, we will consider an analogous stochastic representation, but then as a defining property, and that for a much larger class of probability distributions.

Let a random vector $X = (X_1, ..., X_n)^T$ be distributed according to the density φ_a and put

$$R_a = |X|_a \text{ and } U_a = X/R_a.$$

The random variable R_a may be considered as the a-radius of X. The random vector U_a takes values in the ellipsoid E_a and is called the uniform basis of X. The probability distribution generated by U_a on the Borel σ-field $\mathfrak{B}_a^E$ over E_a is singular w.r.t. the Lebesgue measure λ. We consider now the joint distribution of R_a and U_a,

$$P(R_a < t, U_a \in D) = \int\limits_{sector_a(D,t)} \varphi_a(x)dx,\ t > 0, D \in \mathfrak{B}_a^E.$$

Changing Cartesian with generalized ellipsoidal coordinates yields that, according to Corollary 14,

$$P(R_a < t, U_a \in D) = \frac{\int\limits_0^t r^{n-1}\, e^{-\frac{r^2}{2}}\, dr \quad \int\limits_{\left(T_a^{E^*}\right)^{-1}(D)} J^*\left(T_a^E\right)(\phi)\, d\phi}{(2\pi)^{\frac{n}{2}} a_1 \cdot \ldots \cdot a_n} = \frac{\int\limits_0^t r^{n-1} e^{-r^2/2} dr}{\int\limits_0^\infty r^{n-1} e^{-r^2/2} dr} \cdot \frac{O_E(D)}{O_E(E_a)}$$

because $\int\limits_0^\infty r^{n-1}e^{-r^2/2}dr = 2^{n/2-1}\Gamma(n/2)$ and $\frac{2^{n/2-1}\Gamma(n/2)}{(2\pi)^{n/2}a_1 \cdot \ldots \cdot a_n} = \frac{1}{\omega_n a_1 \cdot \ldots \cdot a_n}$. Hence, the probability distribution of R_a^2 is the well known χ^2-distribution with n d.f., $R_a^2 \sim \chi^2(n)$, and the vector U_a follows the E_a-generalized or $|.|_{\frac{1}{a}}$-uniform distribution U_a^E on $\mathfrak{B}_a^E$. Moreover, the random elements R_a and U_a are stochastically independent. Thus, the stochastic representation $X \overset{d}{=} R_a U_a$ holds with independent factors R_a and U_a. A generalization of this stochastic representation will be the starting point of the next section.

Theorem 15 *If $X \sim \varphi_a$ then the geometric measure representation*

$$P(X \in A) = C_n \int\limits_0^\infty \mathfrak{F}_a^E(A, r) r^{n-1} e^{-r^2/2} dr, A \in \mathfrak{B}_n$$

holds where the normalizing constant is $C_n = 2^{1-n/2}/\Gamma(n/2)$.

Proof :

The proof of this theorem is closely connected with that of Theorem 5 and similar to that of Theorem 4 in (Richter, 2009). Changing Cartesian with generalized ellipsoidal coordinates in the integral $P(X \in A) = \int\limits_A \varphi_a(x)dx$ yields

$$P(X \in A) = \frac{1}{(2\pi)^{n/2} a_1 \cdot ... \cdot a_n} \int\limits_{(T_a^E)^{-1}(A)} r^{n-1} e^{-r^2/2} J^*(T_a^E)(\phi) d(r, \phi).$$

Let us restrict our consideration for a while to sets A of the type $A_a(D, \rho_1, \rho_2)$. Because of the independence of R_a and U_a, and due to the product structure of the set $A_a(D, \rho_1, \rho_2) = [\rho_1, \rho_2] \cdot D$,

$$P(X \in A_a(D, \rho_1, \rho_2)) = P^{R_a}([\rho_1, \rho_2]) P^{U_a}(D) = C_n \int\limits_{\rho_1}^{\rho_2} r^{n-1} e^{-r^2/2} dr U_a^E(D).$$

Let us further consider the finitely additive set function

$$N_a^*(A) = C_n \int\limits_{\rho_1}^{\rho_2} \mathfrak{F}_a^E(A, r) r^{n-1} e^{-r^2/2} dr, A = A_a(D, \rho_1, \rho_2), 0 \leq \rho_1 < \rho_2 < \infty, D \in \mathfrak{B}_a^E.$$

Due to the product structure of the set $A_a(D, \rho_1, \rho_2)$, the function $r \to \mathfrak{F}_a^E(A_a(D; \rho_1, \rho_2), r)$ is constant with the constant being $U_a^E(D)$. Hence,

$$N_a^*(A_a(D; \rho_1, \rho_2)) = C_n \int\limits_{\rho_1}^{\rho_2} r^{n-1} e^{-r^2/2} dr U_a^E(D).$$

Now, the measure extension theorem applies $\square$

7 General density generating functions

In this section, we assume the stochastic representation from the preceding section to be now a defining property, but at the same time for a much larger class of random vectors. The structure of this section follows that of Section 3 in (Richter, 2009). Let $\mathfrak{R}$ denote the set of all nonnegative random variables defined on the same probability space where R_a and U_a are defined, let F be any cumulative distribution function of a positive random variable, and put

$$L_n(F) = \{X : X \stackrel{d}{=} R \cdot U_a, R \in \mathfrak{R} \text{ has c.d.f. } F, R \text{ and } U_a \text{ are independent}\}.$$

Throughout this section, let X be an arbitrary element of $L_n(F)$. The random vector X is called axes-aligned elliptically contoured distributed and the random variable R is called its generating variate. The assumption $X \in L_n(F)$ implies that X has a density iff R has a density. In this case, the density of X is

$$\varphi_{g,a}(x) = C_a(n,g)g(|x|_a^2), x \in \mathbb{R}^n$$

where $g|\mathbb{R}^+ \to \mathbb{R}^+$ is called the density-generating function. It is assumed that g satisfies the assumption $I_{n+2,g} < \infty$. Here, we use the notation $I_{k,g} = \int\limits_0^\infty r^{k-1}g(r^2)dr$, and

$$C_a(n,g) = \frac{1}{n\,\pi_n^E(a)\,I_{n,g}}$$

is the suitably chosen normalizing constant. It follows from

$$P(R<r) = P(X \in B_a(r)) = \int\limits_{B_a(R)} \varphi_{g,a}(x)dx = C_a(n,g)\int\limits_0^r \rho^{n-1}g(\rho^2)d\rho\, O_E(E_a)$$

that the density f of R allows the representation

$$f(r) = I_{n,g}^{-1}r^{n-1}g(r^2)I_{(0,\infty)}(r).$$

We shall use the notation $EC_{g,a}$ for an axes-aligned elliptically contoured distribution defined this way. This distribution is the axes-aligned elliptically contoured Gaussian distribution if its density-generating function g is g_G where $g_G(r) = I_{(0,\infty)}(r)e^{-r/2}$. In this case, we have $I_{n,g}^{-1} = C_n$. The measure $EC_{g_G,a} = \Phi_a$ is a product measure.

Theorem 16 *The axes-aligned elliptically contoured distribution having density-generating function g satisfies the representation*

$$EC_{g,a}(A) = \frac{1}{I_{n,g}}\int\limits_0^\infty \mathfrak{F}_a^E(A,r)r^{n-1}g(r^2)dr, A \in \mathfrak{B}_n.$$

Proof :

The proof may be formulated similar to that of Theorem 15 and Theorem 4 in (Richter, 2009). It is therefore sufficient to give here a sketch of this proof. We consider

$$EC_{g,a}(A) = \int\limits_A \varphi_{g,a}(x)dx = C_a(n,g) \int\limits_{(T_a^E)^{-1}(A)} r^{n-1} g(r^2) J^*(T_a^E)(\phi) d(r,\phi)$$

for the special sets of the type $A = A_a(D, \rho_1, \rho_2)$. Because of the definition of $L_n(F)$ and the product structure of the set A,

$$P(X \in A_a(D, \rho_1, \rho_2)) = P^R([\rho_1, \rho_2]) P^{U_a}(D) = \int\limits_{\rho_1}^{\rho_2} f(r) dr \, U_a^E(D).$$

Let us introduce now the finitely additive set function

$$EC^*_{g,a}(A) = \frac{1}{I_{n,g}} \int\limits_0^\infty \mathfrak{F}_a^E(A,r) r^{n-1} g(r^2) dr, A = A_a(D, \rho_1, \rho_2), 0 \leq \rho_1 < \rho_2 \leq \infty, D \in \mathfrak{B}_a^E.$$

It follows from the properties of the i.p.f. and the definition of f that

$$EC^*_{g,a}(A_a(D, \rho_1, \rho_2)) = \int\limits_{\rho_1}^{\rho_2} f(r) dr \, U_a^E(D).$$

Now, the measure extension theorem applies $\square$

The geometric measure representation formulae in Theorems 15 and 16 enable one to derive exact distributions of several suitably chosen statistics $T = T(X)$ when the sampling distribution is an elliptically contoured distribution. This was shown in (Richter, 2009) (and several papers mentioned therein) for the analogous situation of an $l_{n,p}$-symmetric sampling distribution.

Remark 17 *a)It follows from the representation*

$$\varphi_{g,a}(x) = C_a(n,g) \, g(|x|_a^2) = \frac{1}{a_1 \cdot \ldots \cdot a_n} \left[\frac{g(|x|_a^2)}{\omega_n I_{n,g}} \right]$$

that the density generator $\tilde{g}$ and the density generating function g of an axes-aligned elliptically contoured distribution are connected by the equation

$$\tilde{g}(|x|_a^2) = \frac{g(|x|_a^2)}{\omega_n \cdot I_{n,g}},\ x \in \mathbb{R}^n\,.$$

b) Consequently, an alternative representation of $EC_{g,a}$ is

$$EC_{g,a}(A) = \int_0^\infty O_E\left(A \cap E_a(r)\right)\, \tilde{g}(r^2)\, dr,\ A \in \mathfrak{B}_n.$$

8 Discussion

Geometric measure representation formulae apply to many statistical and probabilistic problems. Typical applications are only in exceptional cases three-line-examples but in general rather complex calculations. This is analogous to typical applications of characteristic functions. But while the theory of characteristic functions is well developed and the characteristic function exists for any probability distribution, the research area of geometric measure representations in the sense of the present paper is rather new and we are far from knowing finally the general nature of analogous geometric measure representations for arbitrary location-scale models or other models like those in (Lange and Sinsheimer, 1993). An answer to this general question would surely be of high interest, but the work on measure-theoretical results going a certain step into this direction is still under progress. For only to speak about geometric-analytical problems behind this probabilistic-statistical question, they concern, e.g., our understanding of the notion of surface content and the extension of the method of indivisibles wich was initially developed by the pupils of Galilei, Cavalieri and Torricelli, and further developed in the sense of the present paper only during the last 30 years. Both the suitably modified notion of the (non-Euclidean) surface content and the generalization of the method of indivisibles are closely connected with the i.p.f. This function is most frequently known in applications for the geometric representation of the standard Gaussian law. Among these applications are the construction of exact tests and confidence regions in non-linear regression, the derivation of exact probabilities of correct classifications, the evaluation of probabilities and quantiles of noncentral distributions as well as remainder-term estimations in the multivariate central limit theorem and the determination of the asymptotic behavior of large deviation probabilities in the multivariate case. Several references for such applications and a certain survey over this work are given in (Richter, 2009). This research is also directed to generalize some of the classical statistical results in the general linear and the generalized linear model when the normality assumption is replaced by the assumption that the error vector follows, e.g., an $l_{n,p}$-symmetric or another non-normal distribution. The distributions of certain statistics from regression analysis and analysis of variances may be generalized this way. Using geometric measure representations and corresponding stochastic representations, recently in (Kalke et al., 2012) the exact distributions of linear combinations, products and ratios of simplicial or spherical variates are derived, in (Arrellano-Valle and Richter, 2012) the notion of skewed distributions is extended to that of skewed $l_{n,p}$-symmetric distributions and in (Kalke and Richter, 2012) the classical polar and rejecting polar methods were extended to generating p-generalized Gaussian random variables. Finally it will become clear only in the future which kind of all the mentioned possible applications of geometric measure representations will be the most useful application of Theorem 16. The ellipsoid-number function which was introduced in Section 3 may be of interest in different branches of mathematics and science.

In data analysis, one is often interested in parametric density generating functions. Such functions allow easy and fast adaption of the model to a given data set and the parameters can sometimes even be interpreted in a specific way.

Example 18 *The density-generating function*

$$g_K(r) = r^{M-1}e^{-\beta r^\gamma},\ r > 0,\ \beta > 0,\ \gamma > 0,\ 2M + n > 2$$

generates the (heteroscedastic axes-aligned elliptically contoured) Kotz-type distribution in $\mathbb{R}^n$*. This is the multinormal distribution if* $(M, \beta, \gamma) = (1, \frac{1}{2}, 1)$.

Example 19 *The density-generating function*

$$g_P(r) = \frac{1}{\left(1+\frac{r}{m}\right)^{\frac{m+n}{2}}}\,,\; r>0,\; m>0$$

generates the n-dimensional (heteroscedastic axes-aligned elliptically contoured) Student distribution. Note that several characterizations of the Student distribution were given in (Arrellano-Valle and Bolfarine, 1995). The more general density-generating function

$$g_P(r) = \frac{1}{\left(1+\frac{r}{m}\right)^{M}}\,,\; r>0,\; M>\frac{n}{2},\; m>0$$

generates a Pearson-VII-type distribution.

Example 20 *The density-generating function*

$$g(r) = \int_0^1 u^{(k/2)+\nu-1} e^{-ur^2/2} du$$

generates the k-dimensional (heteroscedastic axes-aligned elliptically contoured) Slash distribution. This family is used in (Lange and Sinsheimer, 1993) for adaptive robust regression.

The following example of a set A is just one of the simplest examples where Theorem 16 applies.

Example 21 *Let* $A(r) = \{x \in \mathbb{R}^n : |x|_a^2 < r\}$. *Then* $\mathfrak{F}_a^E(A(r),\rho) = I_{(0,\sqrt{r})}(\rho)$. *Hence, the density of the random variable* $|X|_a^2$ *is*

$$\frac{d}{dr}\, EC_{g,a}(A(r)) = \frac{1}{2I_{n,g}}\, r^{\frac{n}{2}-1} g(r)\, I_{(0,\infty)}(r)\,.$$

This density was called in (Richter, 1991), for the special case $a = \mathbb{1} = (1,...,1)$, *the g-generalized* χ^2*-density with* n *d.f..*

Obviously,

$$\frac{1}{I_{n,g}} = C_a(n,g)\, a_1 \cdot ... \cdot a_n\, \omega_n\,.$$

Remark 22 *It is well known that*

$$I_{n,g_K} = \frac{\Gamma(\frac{n-2+2M}{2\gamma})}{2\gamma\beta^{\frac{n-2+2M}{2\gamma}}} \quad \text{and} \quad I_{n,g_P} = \frac{m^{\frac{n}{2}}}{2} B(\frac{n}{2}, M - \frac{n}{2}).$$

Hence,

$$C_a(n, g_K) = \frac{\gamma\beta^{\frac{n-2+2M}{2\gamma}}}{\pi^{\frac{n}{2}} a_1 ... a_n} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-2+2M}{2\gamma})} \quad \text{and} \quad C_a(n, g_P) = \frac{\Gamma(M)}{(m\pi)^{\frac{n}{2}} a_1 ... a_n \Gamma(M - \frac{n}{2})}$$

where $\Gamma(.)$ *and* $B(.,.)$ *denote the Gamma- and the Beta-function, respectively.*

Example 23 *Assume that a statistic* T *generates sets*

$$A(t) = \{x \in \mathbb{R}^n : T(x) < t\}, t \in \mathbb{R}$$

in such a way that the i.p.f. $r \to \mathfrak{F}_a^E(A(t), r)$ *does not depend on* $r > 0$, *i.e. it attains the constant value* $\mathfrak{F}_a^E(A(t))$. *Then, according to Theorem 16,*

$$EC_{g,a}(A(t)) = \mathfrak{F}_a^E(A(t)), t \in \mathbb{R}.$$

Hence, the distribution of T *does not depend on the density generating function* g. *Such statistics will be called robust w.r.t. the density generating function.*

Remark 24 *As can be seen from the proof of Theorem 6 in (Richter, 2009), the derivation of the* t*-dependent value* $\mathfrak{F}_a^E(A(t))$, *i.e. of the function* $t \to \mathfrak{F}_a^E(A(t))$ *will become an extensive work in interesting cases. To realize this work and to draw all the possible conclusions concerning the theory of statistical modeling would go beyond the scope of the present paper. We restrict our consideration in the following two examples therefore to the derivation of a certain robustness property of two statistics which are generalizations of two well known statistics.*

Example 25 Generalized Fisher-statistic *Let us call the statistic*

$$T(X) = T(X_1, ..., X_{m+n}) = \frac{\frac{1}{m}\sum_1^m (\frac{X_i}{\sigma_i})^2}{\frac{1}{n}\sum_1^n (\frac{X_{m+i}}{\sigma_{m+i}})^2}$$

a generalized Fisher-statistic. For each $t > 0$, the subset of the sample space

$$K_t = \{(x_1, ..., x_{m+n}) \in \mathbb{R}^{m+n} : T((x_1, ..., x_{m+n})) < t\}$$

is a cone with vertex in the origin. The intersection of this cone with any ellipsoid $E_a(r), r > 0$ is

$$K_t \cap E_a(r) = \{(x_1, ..., x_{m+n}) \in \mathbb{R}^{m+n} : \frac{\frac{1}{m}\sum_1^m (\frac{x_i}{\sigma_i})^2}{\frac{1}{n}\sum_1^n (\frac{x_{m+i}}{\sigma_{m+i}})^2} < t, \sum_1^m (\frac{x_i}{\sigma_i})^2 + \sum_1^n (\frac{x_{m+i}}{\sigma_{m+i}})^2 = r^2\}.$$

If we introduce generalized ellipsoidal coordinates from Section 4 separately in the two subspaces $\mathfrak{L}_1 = \mathfrak{L}(\{e_1, ..., e_m\})$ and $\mathfrak{L}_2 = \mathfrak{L}(\{e_{m+1}, ..., e_{m+n}\})$ which are spanned up by the vectors from the standard orthonormal basis from $\mathbb{R}^{m+n}$ then we find that

$$K_t \cap E_a(r) = \{(\frac{r_1}{r_2})^2 \leq \frac{mt}{n}, r_1^2 + r_2^2 = r^2\}$$

where r_i describes the radius variable in the subspace $\mathfrak{L}_i, i = 1, 2$. This means that the structure of this intersection does not change if we change r. In other words, the i.p.f. $r \to \mathfrak{F}_a^E(K_t, r)$ of the cone does not depend on $r > 0$. Hence, the generalized Fisher-statistic $T(X)$ is robust w.r.t. the density-generating function.

Example 26 Generalized Student-statistic *In a similar way as in the preceding special case one can prove that the generalized Student-statistic*

$$T(X) = T(X_1, ..., X_n) = \frac{X_1}{\frac{1}{n-1}\sum_2^n (\frac{\sigma_1}{\sigma_i} X_i)^2}$$

is robust w.r.t. the density-generating function.

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Appendix: Proof of Theorem 10

The proof will be given in four steps. First, we change variables $\frac{x_i}{a_i} = y_i, i = 1, ..., n$.The Jacobian of this transformation is

$$|\frac{D(x_1, ..., x_n)}{D(y_1, ..., y_n)}| = a_1 \cdot ... \cdot a_n.$$

Next, we change variables

$$y_1 = \tilde{r}\mu_1, y_2 = \tilde{r}(1 - |\mu_1|^2)^{1/2}\mu_2, ..., y_{n-1} = \tilde{r}(1 - |\mu_1|^2)^{1/2} \cdot ... \cdot (1 - |\mu_{n-2}|^2)^{1/2}\mu_{n-1},$$

$$y_n = +(-)\tilde{r}(1 - |\mu_1|^2)^{1/2} \cdot ... \cdot (1 - |\mu_{n-2}|^2)^{1/2}(1 - |\mu_{n-1}|^2)^{1/2}.$$

As it was shown in the proof of Theorem 2 in (Richter, 2007), the Jacobian of this transformation is

$$|\frac{D(y_1, ..., y_n)}{D(\tilde{r}, \mu_1, ..., \mu_{n-1})}| = \tilde{r}^{n-1} \prod_{i=1}^{n-1} (1 - |\mu_i|^2)^{(n-2-i)/2}.$$

Third, we change variables

$$\tilde{r} = r, \mu_i = \cos_{(a_i, a_{i+1})}(\phi_i), i = 1, ..., n - 1.$$

The Jacobian of this transformation is

$$|\frac{D(\tilde{r}, \mu_1, ..., \mu_{n-1})}{D(r, \phi_1, ..., \phi_{n-1})}| = |\det diag(1, \frac{d}{d\phi_1} \cos_{(a_1,a_2)}(\phi_1), ..., \frac{d}{d\phi_{n-1}} \cos_{(a_{n-1},a_n)}(\phi_{n-1}))|.$$

It is known from (Richter, in print) that

$$\cos'_{(a_i,a_{i+1})}(\phi_i)) = -\frac{\sin \phi_i}{a_i a_{i+1}^2 N^3_{(a_i,a_{i+1})}(\phi_i)}.$$

Hence,

$$|\frac{D(\tilde{r}, \mu_1, ..., \mu_{n-1})}{D(r, \phi_1, ..., \phi_{n-1})}| = \prod_{i=1}^{n-1} \frac{\sin \phi_i}{a_i a_{i+1}^2 N^3_{(a_i,a_{i+1})}(\phi_i)}.$$

On combining all three transformations, we get finally

$$J(T^E_a)(r, \phi_1, ..., \phi_{n-1}) = a_1 \cdot ... \cdot a_n \cdot r^{n-1} \prod_{i=1}^{n-1} (\sin_{(a_i,a_{i+1})}(\phi_i))^{n-2-i} \prod_{i=1}^{n-1} \frac{\sin \phi_i}{a_i a_{i+1}^2 N^3_{(a_i,a_{i+1})}(\phi_i)}.$$