Maximum distributions for $l_{2,p}$-symmetric vectors are skewed $l_{1,p}$-symmetric distributions

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Summary: On exploiting a geometric measure representation for the maximum distribution, we show that the density of the maximum of the components of a $l_{2,p}$-symmetrically distributed random vector satisfies the definition of a skewed $l_{1,p}$-symmetric distribution.

Keywords: $l_{2,p}$-symmetric distributions, skewed $l_{1,p}$-symmetric distributions, maximum distribution, extreme values, order statistics.

MSC: 60E99, 62G30

1 Introduction

Relationships between order statistics and skewed distributions are studied by several authors. The density of the maximum of a vector of dependent or elliptically contoured random variables was considered in Arellano-Valle and Genton(2008). Exchangeable vectors are dealt with in Arellano-Valle and Genton(2007) and Loperfido et al.(2007). Restricted to dimension 2, we generalize here the result that the maximum of a normal vector follows a skewed normal distribution to the case of a more general vector distribution.

To be specific, we consider continuous $l_{2,p}$ -symmetrically distributed random vectors X, $X \sim \Phi_{g,p}$, with a density generating function (dgf) g satisfying

$$0 < I_{2,g,p} := \int_0^\infty r g(r^p) dr < \infty$$

and the probability density function (pdf)

$$\phi_{g,p}(x) = C_{2,g,p}\, g\left(|x|_p^p\right),\ x = (x_1, x_2) \in \mathbb{R}^2, |(x_1, x_2)|_p^p := |x_1|^p + |x_2|^p \qquad (1)$$

where $C_{2,g,p}$ denotes according to Richter(2009) a normalizing constant. The function $x \to |x|_p$ is the $l_{2,p}$-norm if $p \geq 1$ and is the $l_{2,p}$-antinorm if $p \in (0,1)$, see Moszyńska and Richter(2012) for more information on antinorms.

The class of $l_{2,p}$-symmetric distributions includes both examples which model heavy tails and those which model light tails. Heavy tails occur often in insurance and financial mathematics while light tails are of special interest,e.g. in reliability theory. We recall moreover that many classical statistical distributions were first derived under the assumption that the sample distribution is a Gaussian one. However, not as many cases are dealt with under the assumption of an Laplace sample distribution, elliptically contoured sample distribution or another one. It is indicated in Richter(2009) that and in which way many exact distributional results from classical statistics may be generalized under the assumption of an $l_{n,p}$-symmetric sample distribution. For just mentioning a few of them, the χ^2-, Student- and Fisher- distributions have been p-generalized in a suitably defined sense.

If $g = g_p$ with $g_p(c) = e^{-\frac{c}{p}}, c > 0$ is the dgf of the p-power exponential or p-generalized Gaussian distribution then the normalizing constant is $C_{2,g,p} = \left(\frac{p^{1-1/p}}{2\Gamma(1/p)}\right)^2$. In this case the random vector $X = (X_1, X_2)^T$ has the pdf

$$\phi_{g_p,p}(x) = \left(\frac{p^{1-\frac{1}{p}}}{2\Gamma\left(\frac{1}{p}\right)}\right)^2 \exp\left\{-\frac{|x_1|^p + |x_2|^p}{p}\right\} \quad , \quad x = (x_1, x_2) \in \mathbb{R}^2$$

with $p > 0$. The cases $p = 2$ and $p = 1$ correspond to the Gaussian and the Laplace distribution, respectively. If g is chosen in such way that $C_{2,g,p} = 1$ then g is called a density generator. The exact cumulative distribution functions (cdf's) and pdf's of certain functionals of X were studied in Kalke et al.(2013) for $p \in \{1, 2\}$ and in Müller and Richter(2013) for arbitrary $p > 0$.

The aim of the present paper is to bring together these results and to study in Section 2 the exact distribution of the maximum statistic $M(X) = max\{X_1, X_2\}$ in terms of skewed $l_{1,p}$-symmetric distributions.

The paper is structured as follows. First, we present in Section 2 some results from the literature based upon which we will be able to derive our new results and to interpret them in the announced way in later sections. In Section 3, we prove the main result of the paper concerning the density of the maximum for $l_{2,p}$-symmetric vector distribution. Section 4 deals with examples and the final Section 5 provides applications and an outlook.

2 Preliminaries

In this section, we first introduce the class of (one-dimensional) skewed $l_{1,p}$-symmetric distributions with (additional) dimensionality parameter 1 which was defined recently in Arellano-Valle and Richter(2012) for arbitrary dimensions. Then we cite from Müller and Richter(2013) an integral representation of the cdf of the maximum statistic in case of an $l_{2,p}$-symmetric sample distribution. The latter will be the starting point in the next section for deriving a representation of the corresponding density which can afterwards be identified to be a special member of the distribution class introduced in Arellano-Valle and Richter(2012).

Let $X_1 \in \mathbb{R}^1$ and $X_2 \in \mathbb{R}^1$ be two random variables following a $l_{2,p}$-symmetric joint distribution with density generator $g = g^{(2)}$, i.e., they have according to Richter(2009) the joint density

$$f_{X_1,X_2}(x_1,x_2) = g^{(2)}(|x_1|^p + |x_2|^p),\ (x_1,x_2) \in \mathbb{R}^2.$$

If $g^{(1)}$ denotes the density generator of the marginal distribution of X_1 then the density of X_1 is $f_{X_1}(z) = g^{(1)}(|z|^p), z \in \mathbb{R}^1$. The following result characterizes the density of a particular so called selection distribution.

Lemma 1. *(Arellano-Valle and Richter(2012)) The conditional density of X_1 under the assumption that $X_2 < \Lambda X_1$ $\Lambda \in \mathbb{R}$ is*

$$f_{X_1|X_2<\Lambda X_1}(z) = 2\, f_{X_1}(z) F^{(1)}_{1,p}\left(\Lambda z; g^{(1)}_{[|z|^p]}\right),\ z \in \mathbb{R}^1$$

where

$$F^{(1)}_{1,p}(a;g) = \int\limits_{\mathbb{R}^1_+} g(|a-u|^p)du.$$

Further, the additional notion $g^{(1)}_{[a]}(|x_1|^p) = f_{X_1|X_2=x_2}(x_1)$ with $a = |x_2|^p$ is used for the conditional density of X_1 under $X_2 = x_2$ which depends actually on x_2 only through $|x_2|^p$.

The distribution of a random variable Z with density of the form

$$f_Z(z) = 2\, g^{(1)}(|z|^p)\, F^{(1)}_{1,p}(\Lambda z; g^{(1)}_{[|z|^p]}),\ z \in \mathbb{R}^1,$$

is called therefore in Arellano-Valle and Richter(2012) a skewed $l_{1,p}$-symmetric distribution with (additional) dimensionality parameter 1, density generator g and skewing parameter Λ, and the notation $SS_{1,1,p}(\Lambda, g)$ is used for this distribution.

An important special case is the one-dimensional skewed $N_{1,p}$ distribution with (additional) dimensionality parameter 1, density generator g and skewing parameter $\Lambda \in \mathbb{R}^1$, denoted in Arellano-Valle and Richter(2012) by $SN_{1,1,p}(\Lambda)$ and having the density

$$f(z) = 2\, \phi_{1,p}(z)\, \Phi^{(1)}_{1,p}(\Lambda z),\ z \in \mathbb{R}^1.$$

We turn over now to the announced integral representation of the cdf of the maximum statistic if the two-dimensional sample distribution is an $l_{2,p}$-symmetric one.

Lemma 2. *(Müller and Richter(2013)) Let $X \sim \Phi_{g,p}$ for arbitrary dgf g and $p > 0$. Then*

$$F_{M,g,p}(t) = \frac{1}{I_{2,g,p}} \begin{cases} \int\limits_{-\sqrt[p]{2}t}^{\infty} \mathfrak{G}_p\left(\pi+\alpha, \frac{3\pi}{2}-\alpha\right) rg(r^p)\, dr, & t \le 0 \\ \int\limits_0^t rg(r^p)\, dr + \int\limits_t^{\sqrt[p]{2}t} \left[\mathfrak{G}_p\left(\pi-\beta, \frac{3\pi}{2}+\beta\right) + \mathfrak{G}_p\left(\frac{\pi}{2}-\beta, \beta\right)\right] rg(r^p)\, dr & \\ + \int\limits_{\sqrt[p]{2}t}^{\infty} \mathfrak{G}_p\left(\pi-\beta, \frac{3\pi}{2}+\beta\right) rg(r^p)\, dr, & t > 0 \end{cases}$$

with $\alpha = \alpha(r,t) = \arctan\left(\dfrac{-t}{\sqrt[p]{r^p - |t|^p}}\right)$, $\beta = \beta(r,t) = \arctan\left(\dfrac{t}{\sqrt[p]{r^p - t^p}}\right)$ *and*

$$\mathfrak{G}_p(\gamma,\delta) := \int\limits_\gamma^\delta \frac{1}{N_p^2(\varphi)}\, d\varphi \cdot \left(\int\limits_0^{2\pi} \frac{1}{N_p^2(\varphi)}\, d\varphi\right)^{-1}, [\gamma,\delta] \subseteq [0,2\pi), \gamma \le \delta,$$

where $N_p(\varphi) := (|\sin(\varphi)|^p + |\cos(\varphi)|^p)^{\frac{1}{p}}$.

Remark 1. *Let Pol_p denote the p-generalized coordinate transformation introduced in Richter(2007) and Pol_p^* its restriction to the case that the p-radius equals 1. Let further denote by $\mathfrak{U}_p$ the p-generalized uniform distribution on the p-generalized unit circle which was also introduced in Richter(2007) and discussed in more detail in Richter(2008, 2009), then $\mathfrak{G}_p(\gamma, \delta) = \mathfrak{U}_p(Pol_p^*(\gamma, \delta))$.*

3 Main result

The aim of this section is to show that the density corresponding to the cdf in Lemma 2 may be represented in terms of the conditional density from Lemma 1.

In what follows, let $F_{M,g,p}$ denote the cdf of $M(X)$ if $X \sim \Phi_{g,p}$ for a given dgf g and a given value of p, $F_{M,g,p}(t) = P(M(X) < t)$ for $t \in \mathbb{R}$. Correspondingly, let $f_{M,g,p}$ denote in the same case the pdf of $M(X)$.

Theorem 1. *If $X \sim \Phi_{g,p}$ for an arbitrary $p > 0$ then*

$$f_{M,g,p}(t) = 2 f_{X^{(1)}}(t) F_{1,p}^{(1)}(t; g_{[|t|^p]}^{(1)}), t \in \mathbb{R}.$$

Proof. Let $t < 0$. If g is a density generator then according to Richter(2009),

$$\frac{1}{I_{2,g,p}} = 2\pi(p)$$

where $\pi(p)$ denotes the p-generalized circle number introduced in Richter(2008). It follows from Lemma 2, where w.l.g. g is now assumed to be a density generator, that

$$\frac{d}{dt} F_{M,g,p}(t) = 2\pi(p)(\int_0^{2\pi} \frac{1}{N_p^2(\varphi)} d\varphi)^{-1} \frac{d}{dt} \int_{-\sqrt[p]{2}t}^{\infty} (\int_{\alpha+\pi}^{-\alpha+3\pi/2} \frac{1}{N_p^2(\varphi)} d\varphi r g(r^p)) dr$$

where $\alpha = \alpha(r,t)$ is defined in Lemma 2. The function $f(r) := \int_{\alpha(r,t)+\pi}^{-\alpha(r,t)+3\pi/2} \frac{1}{N_p^2(\varphi)} d\varphi$ has the property $f(-\sqrt[p]{2}t) = \int_{\frac{5}{4}\pi}^{\frac{5}{4}\pi} \frac{1}{N_p^2(\varphi)} d\varphi = 0$. Note that

$$\frac{1}{2}\int_0^{2\pi} \frac{1}{N_p^2(\varphi)} d\varphi = \pi(p) = \frac{2\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}.$$

Making use of all these quantities, it follows from the Leibniz integral rule that

$$\frac{d}{dt} F_{M,g,p}(t) = \int_{-\sqrt[p]{2}t}^{\infty} (\frac{d}{dt} \int_{\alpha(r,t)+\pi}^{-\alpha(r,t)+3\pi/2} \frac{1}{N_p^2(\varphi)} d\varphi r g(r^p)) dr.$$

Because of the symmetry properties of the $l_{2,p}$-circle it holds $N_p(-\alpha + 3\pi/2) = N_p(\alpha + \pi) = N_p(\alpha)$. Hence,

$$\frac{d}{dt} \int_{\alpha(r,t)+\pi}^{-\alpha(r,t)+3\pi/2} \frac{1}{N_p^2(\varphi)} d\varphi = \frac{-2}{N_p^2(\alpha(r,t))} \frac{d}{dt} \alpha(r,t).$$

It follows from $\cos\arctan x = 1/\sqrt{1+x^2}$ and $\sin\arctan x = x/\sqrt{1+x^2}$ that $N_p^p(\arctan x) = \frac{1+|x|^p}{(1+x^2)^{p/2}}$. Hence, $N_p^2(\arctan x) = \frac{(1+|x|^p)^{2/p}}{1+x^2}$ and with $x = |t|/\sqrt[p]{r^p - |t|^p}$, we get

$$\frac{1}{N_p^2(\alpha(r,t))} = \frac{(r^p - |t|^p)^{2/p} + |t|^2}{r^2}.$$

Because of $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$, we have

$$\frac{d}{dt}\alpha(r,t) = \frac{1}{1+\frac{t^2}{(r^p-|t|^p)^{2/p}}}\frac{d}{dt}\frac{|t|}{\sqrt[p]{r^p-|t|^p}} = \frac{1}{1+\frac{t^2}{(r^p-|t|^p)^{2/p}}}\frac{-r^p}{(r^p-|t|^p)^{(1+p)/p}}$$

$$= \frac{-r^p(r^p-|t|^p)^{2/p}}{((r^p-|t|^p)^{2/p}+t^2)(r^p-|t|^p)^{(1+p)/p}}.$$

Thus,

$$\frac{d}{dt}\int\limits_{\alpha(r,t)+\pi}^{-\alpha(r,t)+3\pi/2}\frac{1}{N_p^2(\varphi)}d\varphi$$

$$= -2\frac{(r^p-|t|^p)^{2/p}+|t|^2}{r^2}\cdot\frac{-r^p(r^p-|t|^p)^{2/p}}{((r^p-|t|^p)^{2/p}+t^2)(r^p-|t|^p)^{(1+p)/p}}$$

$$= 2r^{p-2}(r^p-|t|^p)^{(1-p)/p}.$$

Finally, we arrived at

$$\frac{d}{dt}F_{M,g,p}(t) = 2\int\limits_{-\sqrt[p]{2}t}^{\infty} r^{p-1}(r^p-|t|^p)^{(1-p)/p}g(r^p)dr.$$

We recall that the dgf g of the distribution of X is chosen as a density generator. Hence, it follows that

$$g(|x^{(1)}|_p^p + |x^{(2)}|_p^p) = g^{(1)}(|x^{(2)}|_p^p)g^{(1)}_{[|x^{(2)}|_p^p]}(|x^{(1)}|_p^p)$$

where $g^{(1)}$ is the density generator of the 1-dimensional marginal distribution of X. Changing variables $r^p = s^p + |t|^p$ yields

$$\frac{d}{dt}F_{M,g,p}(t) = 2\int\limits_{|t|}^{\infty}(s^p+|t|^p)^{(p-1)/p}s^{1-p}g(s^p+|t|^p)(\frac{s}{(s^p+|t|^p)^{1/p}})^{p-1}ds$$

$$= 2\int\limits_{|t|}^{\infty} g(s^p+|t|^p)ds.$$

Hence, with $s = |x^{(1)}|_p$ and $|t| = |x^{(2)}|_p$, it follows

$$f_{M,g,p}(t) = 2g^{(1)}(|t|^p)\int\limits_{|t|}^{\infty} g^{(1)}_{[|t|^p]}(s^p)ds$$

$$= 2f_{X^{(1)}}(t)\int_0^{\infty} g^{(1)}_{[|t|^p]}((u-t)^p)du$$

$$= 2f_{X^{(1)}}(t)F^{(1)}_{1,p}(t; g^{(1)}_{[|t|^p]}).$$

Let now $t \geq 0$. From Lemma 2, where g is now a density generator, the distribution function of the maximum M is

$$\begin{aligned} F_{M,g,p}(t) \;&= 2\pi(p) \;\; \Bigg[\int_0^t rg(r^p)dr \\ &+ \int_t^{\sqrt[p]{2}t}\left[\mathfrak{G}_p(\pi-\beta,\frac{3\pi}{2}+\beta)+\mathfrak{G}_p(\frac{\pi}{2}-\beta,\beta)\right]rg(r^p)dr \\ &+ \int_{\sqrt[p]{2}t}^{\infty}\mathfrak{G}_p(\pi-\beta,\frac{3\pi}{2}+\beta)rg(r^p)dr\Bigg]. \end{aligned} \tag{2}$$

The proof of $\frac{d}{dt}F_{M,g,p}(t) = 2f_{X^{(1)}}(t)F^{(1)}_{1,p}(t; g^{(1)})_{[|t|^p]}$ is given by calculating the derivative of the right hand terms in (2), as follows:

$$\frac{d}{dt}\int_0^t rg(r^p)dr = tg(t^p). \tag{3}$$

For $0 \leq \pi - \beta(r,t) \leq \frac{3\pi}{2} + \beta(r,t) < 2\pi$, let

$$f(r,t) = rg(r^p)\int_{\pi-\beta(r,t)}^{\frac{3\pi}{2}+\beta(r,t)} \frac{1}{N_p^2(\phi)}d\phi.$$

Then

$$\begin{aligned} \frac{d}{dt}\int_t^{\sqrt[p]{2}t}\mathfrak{G}_p(\pi-\beta,\frac{3\pi}{2}+\beta)rg(r^p)dr \;&=\; \frac{1}{2\pi(p)}\frac{d}{dt}\int_t^{\sqrt[p]{2}t}\left[\int_{\pi-\beta}^{\frac{3\pi}{2}+\beta}\frac{1}{N_p^2(\phi)}d\phi\right]rg(r^p)dr \\ &=\; \frac{1}{2\pi(p)}\left[\int_t^{\sqrt[p]{2}t}\frac{d}{dt}f(r,t)dr + \sqrt[p]{2}f(\sqrt[p]{2}t,t) - f(t,t)\right] \end{aligned}$$

where the last equation follows from the Leibniz integral rule. Note that $\beta(\sqrt[p]{2}t,t) = \arctan(\frac{t}{\sqrt[p]{2t^p-t^p}}) = \arctan(1) = \frac{\pi}{4}$ and $\beta(t,t) = \lim_{x\to\infty}\arctan(x) = \frac{\pi}{2}$, $f(t,t) = tg(t^p)\int_{\frac{\pi}{2}}^{2\pi}\frac{1}{N_p^2(\phi)}d\phi$ and $f(\sqrt[p]{2}t,t) = \sqrt[p]{2}tg(2t^p)\int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}}\frac{1}{N_p^2(\phi)}d\phi$.

Since $N_p(\pi-\beta) = N_p(\frac{3\pi}{2}+\beta) = N_p(\beta)$, then

$$\frac{d}{dt}\int_{\pi-\beta(r,t)}^{\frac{3\pi}{2}+\beta(r,t)}\frac{1}{N_p^2(\phi)}d\phi = \left[\frac{2}{N_p^2(\beta(r,t))}\right]\frac{d}{dt}\beta(r,t)$$

with

$$\frac{d}{dt}\beta(r,t) = \frac{\sqrt[p]{r^p-t^p} + t^p(r^p-t^p)^{\frac{1-p}{p}}}{(r^p-t^p)^{2/p}+t^2}.$$

Thus

$$\frac{d}{dt}\int_{\pi-\beta(r,t)}^{\frac{3\pi}{2}+\beta(r,t)}\frac{1}{N_p^2(\phi)}d\phi = 2\frac{\sqrt[p]{r^p-t^p}+t^p(r^p-t^p)^{\frac{1-p}{p}}}{r^2}.$$

Using the change of variable $r^p = s^p + t^p$ we obtain

$$\int_t^{\sqrt[p]{2}t} \frac{\sqrt[p]{r^p - t^p} + t^p (r^p - t^p)^{\frac{1-p}{p}}}{r} g(r^p) dr = \int_0^t g(s^p + t^p) ds$$

Finally, we obtain

$$\frac{d}{dt} \int_t^{\sqrt[p]{2}t} \mathfrak{G}_p(\pi - \beta, \frac{3\pi}{2} + \beta) r g(r^p) dr = \frac{1}{2\pi(p)} \left[\sqrt[p]{2} f(\sqrt[p]{2}t, t) - f(t,t) + 2 \int_0^t g(s^p + t^p) ds \right] \tag{4}$$

Now,

$$\begin{aligned} \frac{d}{dt} \int_t^{\sqrt[p]{2}t} \mathfrak{G}_p(\frac{\pi}{2} - \beta, \beta) r g(r^p) dr &= \frac{1}{2\pi(p)} \frac{d}{dt} \int_t^{\sqrt[p]{2}t} \left[\int_{\frac{\pi}{2} - \beta}^{\beta} \frac{1}{N_p^2(\phi)} d\phi \right] r g(r^p) dr \\ &= \frac{1}{2\pi(p)} \left[\int_t^{\sqrt[p]{2}t} \frac{d}{dt} h(r,t) dr + \sqrt[p]{2} h(\sqrt[p]{2}t, t) - h(t,t) \right] \end{aligned}$$

with

$$h(r,t) = r g(r^p) \int_{\frac{\pi}{2} - \beta(r,t)}^{\beta(r,t)} \frac{1}{N_p^2(\phi)} d\phi, \; h(\sqrt[p]{2}t, t) = \sqrt[p]{2} t g(2t^p) \int_{\pi/4}^{\pi/4} \frac{1}{N_p^2(\phi)} d\phi = 0$$

and

$$h(t,t) = t g(t^p) \int_0^{\pi/2} \frac{1}{N_p^2(\phi)} d\phi.$$

Since

$$\begin{aligned} \frac{d}{dt} \int_{\frac{\pi}{2} - \beta(r,t)}^{\beta(r,t)} \frac{1}{N_p^2(\phi)} d\phi &= \frac{2}{N_p^2(\beta(r,t)} \frac{d}{dt} \beta(r,t) \\ &= \frac{2(t^2 + (r^p - t^p)^{2/p})}{r^2} \frac{\sqrt[p]{r^p - t^p} + t^p (r^p - t^p)^{\frac{1-p}{p}}}{(r^p - t^p)^{2/p} + t^2} \\ &= \frac{2}{r^2} (\sqrt[p]{r^p - t^p} + t^p (r^p - t^p)^{\frac{1-p}{p}}), \end{aligned}$$

then

$$\begin{aligned} &\frac{d}{dt} \int_t^{\sqrt[p]{2}t} \mathfrak{G}_p(\frac{\pi}{2} - \beta, \beta) r g(r^p) dr \\ &= \frac{1}{2\pi(p)} \left[\int_t^{\sqrt[p]{2}t} \frac{2g(r^p)}{r} (\sqrt[p]{r^p - t^p} + t^p (r^p - t^p)^{\frac{1-p}{p}}) dr - h(t,t) \right] \\ &\frac{1}{2\pi(p)} \left[2 \int_0^t g(s^p + t^p) ds - h(t,t) \right]. \end{aligned}$$

For the last term in (2) we have

$$\frac{d}{dt} \int_{\sqrt[p]{2}t}^{\infty} \mathfrak{G}_p(\pi - \beta, \frac{3\pi}{2} + \beta) r g(r^p) dr = \frac{1}{2\pi(p)} \left[\int_{\sqrt[p]{2}t}^{\infty} \frac{d}{dt} k(r,t) dr - \sqrt[p]{2} k(\sqrt[p]{2}t, t) \right]$$

with

$$k(r,t) = rg(r^p)\int_{\pi-\beta(r,t)}^{\frac{3\pi}{2}+\beta(r,t)} \frac{1}{N_p^2(\phi)}d\phi.$$

Since

$$\begin{aligned}\frac{d}{dt}k(r,t) &= rg(r^p)\frac{2}{N_p^2(\beta(r,t)}\frac{d}{dt}\beta(r,t)\\ &= 2g(r^p)\frac{\sqrt[p]{r^p-t^p}+t^p(r^p-t^p)^{\frac{1-p}{p}}}{r}\end{aligned}$$

and

$$\begin{aligned}\int_{\sqrt[p]{2}t}^{\infty}\frac{d}{dt}k(r,t)dr &= \int_{\sqrt[p]{2}t}^{\infty}2g(r^p)\frac{\sqrt[p]{r^p-t^p}+t^p(r^p-t^p)^{\frac{1-p}{p}}}{r}\\ &= 2\int_t^{\infty}g(s^p+t^p)ds\end{aligned}$$

then

$$\frac{d}{dt}\int_{\sqrt[p]{2}t}^{\infty}\mathfrak{G}_p(\pi-\beta,\frac{3\pi}{2}+\beta)rg(r^p)dr = \frac{1}{2\pi(p)}\left[2\int_t^{\infty}g(s^p+t^p)ds-\sqrt[p]{2}k(\sqrt[p]{2}t,t)\right]. \tag{5}$$

From (3), (4), (3) and (5) we have

$$\begin{aligned}\frac{d}{dt}F_{M,g_p,p}(t) &= 2\pi(p)tg(t^p)+\Big[\sqrt[p]{2}f(\sqrt[p]{2}t,t)-f(t,t)+2\int_0^t g(s^p+t^p)ds\\ &+ 2\int_0^t g(s^p+t^p)ds-h(t,t)+2\int_t^{\infty}g(s^p+t^p)ds-\sqrt[p]{2}k(\sqrt[p]{2}t,t)\Big].\end{aligned}$$

Proceeding now as in the case $t<0$ yields

$$\frac{d}{dt}F_{M,g,p}(t) = 2f_{X^{(1)}}(t)F_{1,p}^{(1)}(t;g_{[|t|^p]}^{(1)})$$

□

4 Examples and applications

The first example of this section deals with arbitrary dgf's and may be derived directly from Lemma 2 by specialization of the parameter p, i.e. by choosing $p\in\{1,2\}$, and slightly reformulating the respective results. Originally, however, it was derived in Müller and Richter(2013) making use of the case specific geometric properties when applying a geometric measure representation.

Example 1. *Let $X\sim\Phi_{g,p}$ for an arbitrary dgf g and parameter $p\in\{1,2\}$. The cdf of $M(X)$ satisfies the representations*

$$F_{M,g,1}(t) = \frac{1}{I_{2,g,1}}\begin{cases}\int\limits_{-2t}^{\infty}\left(\frac{r}{4}+\frac{t}{2}\right)g(r)\,dr & ,\ t\le 0\\ \frac{1}{4}I_{2,g,1}+\frac{3}{4}\int\limits_0^t rg(r)\,dr+\frac{1}{4}\int\limits_t^{2t}(4t-r)\,g(r)\,dr+\frac{1}{2}\int\limits_{2t}^{\infty}tg(r)\,dr & ,\ t>0\end{cases}$$

and

$$F_{M,g,2}(t) = \frac{1}{I_{2,g,2}} \begin{cases} \int\limits_{-\sqrt{2}t}^{\infty} \left(\frac{1}{4} + \frac{\alpha}{\pi}\right) rg(r^2)\, dr & , t \leq 0 \\ \frac{1}{4} I_{2,g,2} + \frac{3}{4} \int\limits_{0}^{t} rg(r^2)\, dr \; + \frac{1}{2\pi} \int\limits_{t}^{\sqrt{2}t} \left(4\beta - \frac{\pi}{2}\right) rg(r^2)\, dr & \\ \qquad + \frac{1}{\pi} \int\limits_{\sqrt{2}t}^{\infty} \beta rg(r^2)\, dr & , t > 0 \end{cases}$$

with $\alpha = \arcsin\left(\frac{|t|}{r}\right)$ *and* $\beta = \arcsin\left(\frac{t}{r}\right)$ *and the corresponding pdf's are*

$$f_{M,g,1}(t) = \frac{1}{I_{2,g,1}} \begin{cases} \frac{1}{2} \int\limits_{-2t}^{\infty} g(r)\, dr & , t < 0 \\ \int\limits_{t}^{2t} g(r)\, dr + \frac{1}{2} \int\limits_{2t}^{\infty} g(r)\, dr & , t > 0 \end{cases}$$

and

$$f_{M,g,2}(t) = \frac{1}{I_{2,g,2}} \begin{cases} \frac{1}{\pi} \int\limits_{-\sqrt{2}t}^{\infty} \frac{g(r^2)}{\sqrt{1-\frac{t^2}{r^2}}}\, dr & , t < 0 \\ \frac{2}{\pi} \int\limits_{t}^{\sqrt{2}t} \frac{g(r^2)}{\sqrt{1-\frac{t^2}{r^2}}}\, dr + \frac{1}{\pi} \int\limits_{\sqrt{2}t}^{\infty} \frac{g(r^2)}{\sqrt{1-\frac{t^2}{r^2}}}\, dr & , t > 0, \end{cases}$$

respectively.

The following example deals with a coupled choice of the dgf g and the parameter p.

Example 2. *Let* $\phi_p(t) = \frac{p^{1-\frac{1}{p}}}{2\Gamma(\frac{1}{p})} e^{-\frac{|t|^p}{p}}$ *and* $\Phi_p(t) = \int\limits_{-\infty}^{t} \phi_p(s)ds$, *then*

$$f_{M,g_p,p}(t) = 2\phi_p(t)\Phi_p(t), t \in \mathbb{R},\ p > 0.$$

Note that ϕ_1, Φ_1 *and* ϕ_2, Φ_2 *are the pdf and the cdf of the univariate Laplace and standard Gauss distribution, respectively.*

The result of Example 2 is easily interpreted for $p = 2$. The pdf $f_{M,g_2,2}$ has the structure of the so called skew-normal distribution. For this distribution we refer to Genton(ed.)(2004) and the numerous papers cited therein. Many authors have been dealt with generalizations of the skew-normal distribution. For only to mention some of them, we refer to Arellano-Valle and Azzalini(2006), Arellano-Valle and Genton(2005), Arellano-Valle and Genton(2010) and Dominguez-Molina et al.(2007). For $p = 1$, however, is the result of Example 2 possibly best understood to be a special case of the class of skewed $l_{1,p}$-symmetric distributions considered in Arellano-Valle and Richter(2012). To be more specific, it has the structure of the skewed p-power exponential distribution for the case of dimension 2 and parameter $p = 1$. Moreover, a specific result concerning just the case $p = 1$ was also derived in Nekoukhou and Alamatsaz(2011).

In the final example, we restrict our consideration to the dgf of the p-generalized Student type.

Example 3. *Let*

$$g(t) = D_{2,p,\nu}\left(1+\frac{t}{\nu}\right)^{-(\nu+2)/p}, t > 0$$

denote the density generator of the two-dimensional p-generalized Student distribution with ν d.f. where $D_{2,p,\nu}$ denotes the suitably normalizing constant. This distribution was introduced in Richter(2007) for the one-dimensional case, and the n-dimensional case was dealt with in Arellano-Valle and Richter(2012). We consider now $M(X)$ and assume that the vector X is distributed according to the density $f_X(x) = g(|x|_p^p), x \in \mathbb{R}^2$. It follows from Theorem 1 that $f_{M,g,p}(t) = 2f_{X^{(1)}}(t)F_{1,p}^{(1)}(t; g_{[|t|^p]}^{(1)}), t \in \mathbb{R}$. Due to Lemma 3.1 in Arellano-Valle and Richter(2012), the density generator of the marginal distribution of X_1 is

$$g^{(1)}(u) = \frac{2}{p}\int_u^\infty g(y)(y-u)^{1/p-1}dy$$

$$= \frac{2}{p}D_{2,p,\nu}\int_u^\infty (1+\frac{y}{\nu})^{-(\nu+2)/p}(y-u)^{1/p-1}dy.$$

Changing variables $x = y-u$ and $t = \frac{x}{\nu+u}$, it follows

$$g^{(1)}(u) = \frac{C(p,\nu)}{(1+\frac{u}{\nu})^{(\nu+1)/p}}$$

where

$$C(p,\nu) = \frac{2}{p}\nu^{1/p}D_{2,p,\nu}\int_0^\infty t^{1/p-1}(1+t)^{-(\nu+2)/p}dt.$$

Hence,

$$f_{X^{(1)}}(t) = \frac{C(p,\nu)}{(1+\frac{|t|^p}{\nu})^{(\nu+1)/p}}, t \in \mathbb{R}.$$

Due to Lemma 3.2 in Arellano-Valle and Richter(2012), the conditional density generator of the cdf of X_1 under $X_2 = x_2$ is

$$g_{[|x_2|^p]}^{(1)}(u) = \frac{\frac{p}{2}g^{(1)}(u+|x_2|^p)}{\int_0^\infty g^{(1)}(z+|x_2|^p)z^{1/p-1}dz}$$

$$= \frac{\frac{p}{2}(1+(u+|x_2|^p)/\nu)^{(\nu+1)/p}}{\int_0^\infty (1+(u+|x_2|^p)/\nu)^{(\nu+1)/p}z^{1/p-1}dz}.$$

Hence,

$$F_{1,p}^{(1)}(t; g_{[|t|^p]}^{(1)}) = \frac{\frac{p}{2}\int_0^\infty (1+(|t-u|^p+|t|^p)/\nu)^{(\nu+1)/p}}{\int_0^\infty (1+(z+|t|^p)/\nu)^{(\nu+1)/p}z^{1/p-1}dz}, t \in \mathbb{R}$$

which means that $M(X)$ follows the one-dimensional skewed p-generalized Student distribution with ν d.f.; in other words, $M(X)$ follows the skew-$t_{1,p}(\nu)$-distribution.

5 Applications and outlook

The results of the present note may find various applications. First of all, let us remark that, because of symmetry reasons, Theorem 1 applies to the minimum statistic. Since

$$P(\min(X_1, X_2) < t) = 1 - P(M(X) < -t),$$

the density function of the mininum $m(X) = \min(X_1, X_2)$ is

$$f_{m,g_p,p}(t) = 2 f_{X^{(1)}}(t) F_{1,p}^{(1)}(-t; g_{[|t|^p]}^{(1)}),$$

which is a skewed density with parameter $\lambda = -1$.

The distributions of the maximum and the minimum of random variables are very important for their statistical applications. For example, in reliability analysis it is useful to know whether the distributions of the order statistics have increasing failure rates (IFR) or decreasing ones.

In the case $p = 2$ the maximum $M(X)$ is a particular case of the Generalized Skew Normal Model, given in Gupta and Gupta(2004), and it is IFR.

For $p = 1$, from Theorem 1,a), the cdf of $M(X)$ is

$$F_{M,g,1}(t) = \begin{cases} \frac{1}{4}e^{2t} & t \leq 0 \\ 1 + \frac{1}{4}e^{-2t} - e^{-t} & t > 0. \end{cases}$$

and the hazard function is

$$r_{M,g,1}(t) = \begin{cases} \frac{\frac{1}{2}e^{2t}}{1-\frac{1}{4}e^{2t}} & t \leq 0 \\ \frac{e^{-t}(1-\frac{1}{2}e^{-t})}{e^{-t}-\frac{1}{4}e^{-2t}} & t > 0. \end{cases}$$

Since $r'(t) > 0$ for all t, then $M(X)$ has increasing failure rate (IFR).

It has been proved in Gupta and Gupta(2001) within a more general setting than that considered here that the maximum and the minimum distributions have the IFR property if X follows a Gaussian law. This result has been extended in Arellano-Valle and Genton(2008) to the general case of $p = 2$, i.e. to the spherical case. Moreover, it has been proved in Nekoukhou and Alamatsaz(2011) that the maximum and the minimum distributions have the IFR property if X follows a Laplace law, i.e. in the case $p = 1$. We pose here the problem to check whether the distribution of an extreme value statistic has the IFR property for other choices of p, too.

Note that there are several representations for random variables that are skewed $l_{1,p}$-symmetrically distributed. One of these representations makes use of a two-dimensional $l_{2,p}$-symmetrically distributed random vector and a certain selection rule. One can simulate the two-dimensional p-power exponential distribution using, e.g., the R-modul 'pg-norm' going back to the algorithms discussed in Kalke and Richter(2013). Hence, applying the mentioned selection rule, one may simulate the one-dimensional skewed $l_{1,p}$-symmetrically distribution.

The question concerning the possible extension of the present result to the non-spherical case may be answered as follows. As a first step into this direction, one would need an extension of the geometric measure representation in Richter(2009) to this more general case. Such representation should be a suitable generalization of that recently proved in Richter(2013) for the case $p = 2$. The measure theoretical questions behind such work would be closely connected with a fundamental problem from geometry concerning the so called ball number function. This problem was recently solved in Richter(2011) for dimension 2. Work is currently under progress on all these issues, and the third author hopes to report all findings in a future paper.

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