# CLASSES OF STANDARD GAUSSIAN RANDOM VARIABLES AND THEIR GENERALIZATIONS

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## 1 Introduction

Numerous results on the skewed normal distribution and its generalizations were derived during the last decade. Already the authors in Genton (2004) list various representations of random variables following such a distribution. In the case of univariate skewed distributions, these representations typically make use of bivariate random vectors. A unified geometric approach to different such representations of the one-dimensional skewed normal distribution and its generalizations is given in Günzel, Richter, Scheutzow, Schicker & Venz (2012). This approach is based on a representation of the Gaussian law which was originally derived in Richter (1985) and several subsequent papers for the purposes of large deviation theory. This geometric measure representation was extended in Richter (1991) to spherical distributions. Basic introductions into the area of geometric measure theory and related fields are given in Federer (1969), Nachbin (1976), Morgan (1984), Wijsman (1984), Barndorff-Nielsen, Blaesild & Eriksen (1989), Schindler (2003), Kallenberg (2005), Krantz & Parks (2008), and in Muirhead (1982), Eaton (1983), Farrel (1985), and Richter (2009). A certain uniquely defined measure on the Borel $\sigma$-field $\mathcal{B}$ on the Euclidean sphere, the so called uniform distribution on $\mathcal{B}$, plays a fundamental role for these considerations. Several authors have exploited properties of this distribution for different purposes. Seppo Pynnönen (2013) demonstrates how to use the uniform distribution on a Stiefel manifold for dealing with a fundamental problem in statistics. He derives the distributions of linear combinations of internally studentized ordinary least squares residuals of multivariate regression analysis.

The geometric representation of the Gaussian law in Richter (1985) exploits this uniform distribution with the help of the so called intersection percentage function(ipf). The idea behind the definition of this function stems from the very old method of measuring the content of an area or a body by comparing it with a well studied one. Analyzing the ipf, it was shown in Günzel et al. (2012) that the univariate skewed normal distribution and any of its spherical generalizations is closely connected with measuring intersections of two half planes with the help of a bivariate normal or spherical distribution, respectively. This result was extended in

Richter & Venz (2014) to the higher dimensional situation. There are other $k$-variate distributions which are connected with measuring other types of subsets of $R^n$ with $n > k$. As just to mention a few of such multivariate cases, we recall that Student and Fisher distributions and their generalizations may be studied from a geometric measure theoretical point of view by measuring one- and two-sided cones having their apex in the origin, see Richter (1991, 1995, 2007 and 2009). The Student distribution is also connected with considerations on non-linearly transformed cone type sets, see Richter (1995) and Ittrich & Richter (2005), and noncentral $\chi^2$- and Fisher-distributions are connected with balls and cones having their center or apex outside the origin, respectively, see Ittrich, Krause & Richter (2000) and Krause & Richter (2004). These and several other examples show that geometric measure representations apply in a great variety of situations. For some more two-dimensional results, we refer to Kalke, Richter & Thauer (2013) and Müller & Richter (2014).

The bivariate Gaussian measure geometric representation will be used in the present paper to unify and to extend the proofs of two seemingly different results in Shepp (1964) on normal functions of normal random variables, see also Cohen (1981), Baringhaus, Henze & Morgenstern (1988) and Bansal, Hamedani, Key, Volkmer, Zhang & Behboodian (1999) . It will turn out from Theorem 3 that both results are just special cases of a more general representation formula for the univariate standard Gaussian law. Actually, we construct a class of standard Gaussian distributed random variables including known cases as special cases. This class will be essentially enlarged in Theorem 4 to the class $\Phi(\Phi_{0_2,I_2})$ of univariate standard Gaussian random variables which are derived from bivariate standard Gaussian vectors. The spherical extensions of these results in Theorems 5 and 6 can be viewed as well as generalizations of a result derived in Arellano-Valle (2001).

Let $(X,Y)$ denote a random vector taking its values in $R^2$. If $(X,Y)$ follows the two-dimensional standard Gaussian law, we shall write

$$(X,Y) \sim \Phi_{0_2,I_2}. \tag{1}$$

The cumulative distribution function (cdf) of the standard Gaussian law on the real line will be denoted by $\Phi$. The following theorem was in part repeatedly proved in Shepp (1964), Cohen (1981), Baringhaus et al. (1988) and Bansal et al. (1999) by exploiting stable distribution theory, proving McLaurin series expansions, using coordinate transformation or Laplace transformation, exploiting a representation of the densities of chi-distributed random variables and various other techniques.

**Theorem 1** *[A] If the random vector $(X, Y)$ satisfies assumption $(1)$ then*

$$P(\frac{2XY}{\sqrt{X^2 + Y^2}} < w) = \Phi(w),\ w \in R. \tag{2}$$

*[B] If relation $(2)$ holds then assumption $(1)$ is fulfilled.*

Methods from geometric measure theory will be used in Section 3 to reprove part *[A]* of this theorem and a slightly adapted version of this new proof will enable us to even generalize part *[A]* of the theorem for large classes of random variables. In words close to those in Silvermann (2000), this demonstrates the actually given value of reproving. For the corresponding results we refer to Theorems 3 and 4.

If $(X, Y)$ follows a two-dimensional spherical distribution with density generator (dg) $h$, i.e. if the probability density function (pdf) $f_{(X,Y)}$ of $(X, Y)$ is

$$f_{(X,Y)}(x, y) = h(||(x, y)||^2),\ (x, y) \in R^2$$

where $||.||$ denotes the Euclidean norm in $R^2$, we shall write

$$(X, Y) \sim \Phi_{h;0_2,I_2}. \tag{3}$$

The cdf of any marginal distribution of $\Phi_{h;0_2,I_2}$ is an univariate spherical distribution and is denoted throughout this note by $\Phi_h$. According to Fang, Kotz & Ng (1990), its density $\varphi_h$ satisfies the representation

$$\varphi_h(w) = 2\int_0^\infty h(z^2 + w^2)dz,\ w \in (-\infty,\ \infty),$$

and the cdf itself will be called the spherical marginal $h$-generalization $\Phi_h$ of $\Phi$. The next theorem is a generalization of the first one and was proved in Arellano-Valle (2001) using analytical methods which are based upon a stochastic representation for spherically distributed random vectors. Such representations go back to Kelker (1970), Johnson & Kotz (1970), Cambanis, Huang & Simons (1981), Anderson & Fang (1990), Fang et al. (1990), and Fang & Shang (1990). To be more concrete, properties of uniformly distributed vectors are combined in Arellano-Valle (2001) with certain relations from trigonometry.

**Theorem 2** *[A] If $(X, Y)$ satisfies assumption $(3)$ then*

$$P(\frac{2XY}{\sqrt{X^2 + Y^2}} < w) = \Phi_h(w), w \in R. \tag{4}$$

*[B] If relation* $(4)$ *holds then assumption* $(3)$ *is fulfilled.*

The rest of this note is organized as follows. We state generalizations of Theorem 1 dealing with Gaussian distributions in Section 2.1 and generalizations of Theorem 2 dealing with spherical distributions in Section 2.2. Theorem 6 will be the main result of this note. In Section 3 we provide geometric measure theoretical reproofs of known results from Section 1. The proofs of the results in Section 2 will be based upon the reproofs outlined in Section 3 and will be given in the final Section 4.

# 2 Main results

## *2.1 Classes of standard normally distributed functions of bivariate standard Gauss vectors*

Let us consider the random variable $S(X,Y) = \frac{\lambda_1 X^2+\lambda_2 XY+\lambda_3 Y^2}{\sqrt{\mu_1 X^2+\mu_2 XY+\mu_3 Y^2}}$ which is a measurable function of the random vector $(X,Y)$.

**Theorem 3** *If (1) holds then*

$$P(S(X,Y) < w) = \Phi(w),\ w \in R \tag{5}$$

*for all coefficients* $\lambda_i, \mu_i,\ i \in \{1,2,3\}$, *satisfying*

$$\lambda_1 = 2a_{11}a_{21},\ \lambda_2 = 2(a_{11}a_{22} + a_{12}a_{21}),\ \lambda_3 = 2a_{12}a_{22} \tag{6}$$

*and*

$$\mu_1 = a_{11}^2 + a_{21}^2,\ \mu_2 = 2(a_{11}a_{12} + a_{21}a_{22}),\ \mu_3 = a_{12}^2 + a_{22}^2 \tag{7}$$

*where* $O = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *is an orthogonal matrix.*

The following two examples describe those random variables which were considered under the normality assumption for $(X,Y)$ already in Shepp (1964), Cohen (1981), Baringhaus et al. (1988) and Bansal et al. (1999).

**Example 1** *If* $O = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *then* $S(X,Y) = \frac{2XY}{\sqrt{X^2+Y^2}}$.

**Example 2** *If* $O = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ *then* $S(X,Y) = \frac{X^2-Y^2}{\sqrt{X^2+Y^2}}$. *Notice that the orthogonal matrix O used here describes an anticlockwise rotation around the origin of* $R^2$ *through the angle of 45 degrees.*

Let us recall that it was shown already in Shepp (1964) by exploiting stable distribution theory and in Cohen (1981) by using McLaurin expansion that under (1) the random variable $\frac{2XY}{\sqrt{X^2+Y^2}}$ follows the standard Gaussian law. Some elementary proofs of this result were given in Baringhaus et al. (1988) and a characterization of the standard Gaussian law by this property was derived in Bansal et al. (1999). The proof of Theorem 3 will be based upon an invariance property of the bivariate Gaussian law and will be given in Section 4.

The aim of our following consideration is to significantly enlarge the class of univariate random variables being standard Gaussian distributed if assumption (1) is fulfilled. To this end, we denote the Euclidean circle of radius $r$ and having its center in the origin of $R^2$ by

$$C(r) = \{(x,y) \in R^2 : x^2 + y^2 = r^2\},\ r > 0.$$

**Lemma 1** *Let the Borel sets* $A$ *and* $B$ *from* $R^2$ *satisfy the equation*

$$B \cap C(r) = O(r)[A \cap C(r)] \textit{ for almost all } r > 0$$

*where each* $O(r)$ *is an orthogonal matrix. Then*

$$\Phi_{0_2,I_2}(B) = \Phi_{0_2,I_2}(A).$$

The method of measuring subsets of $R^2$ resulting from this lemma will be called the bivariate standard Gaussian measure indivisiblen method. This method reflects in a generalized sense the ancient ideas of Cavalieri and Torricelli and was basically established in Richter (1985) and some subsequent papers within certain considerations on multivariate large deviation probabilities. Moreover, it was exploited in several papers for studying the Gaussian measure of sets from certain statistically well motivated classes of sets, e.g., in Richter (1995), Ittrich et al. (2000), Krause & Richter (2004) and Ittrich & Richter (2005) as just to mention a few of them. The next definition is in the spirit of such work. It makes use of the notion of the Euclidean circle ipf of a Borel set $A$ from $R^2$ which is defined as

$$\mathcal{F}(A,r) = \frac{1}{2\pi r} l(A \cap C(r)),\ r > 0$$

where $l$ means the Euclidean arc-length.

**Definition 1** *A random variable $T : R^2 \to R$ belongs to the class IpfRep of random variables if the Euclidean circle ipfs of its sublevel sets*

$$A(w) = \{(x, y) \in R^2 : T(x, y) < w\},\ w \in R,$$

*allow the joint representation*

$$\mathcal{F}(A(w), r) = I_{[0,w]}(r) I_{[0,\infty)}(w) + I_{[|w|,\infty)}(r)[\frac{1}{2} + \frac{sign(w)}{2\pi} \arccos(1 - \frac{2w^2}{r^2})].$$

**Example 3** *It follows immediately from the proof in Section 3.1 that, under the assumptions (6) and (7), $S(X, Y) \in$ IpfRep.*

**Definition 2** *If a bivariate random vector $(X, Y)$ satisfies the assumption (1) and, for a function $T : R^2 \to R$, the random variable $T(X, Y)$ follows the univariate standard Gaussian distribution then we say that $T(X, Y)$ belongs to the class $\Phi(\Phi_{0_2,I_2})$ of univariate standard Gaussian random variables derived from a bivariate standard Gaussian vector.*

**Theorem 4** *If the random variable $T$ satisfies the assumption $T \in$ IpfRep then $T(X, Y) \in \Phi(\Phi_{0_2,I_2})$.*

In other words, if there holds (1) and $T \in IpfRep$ then

$$P(T(X, Y) < w) = \Phi(w),\ w \in R.$$

**Example 4** *It is well known that the random variables $T(X, Y) = X$ and $T(X, Y) = Y$ belong to the class $\Phi(\Phi_{0_2,I_2})$ and that their cdfs may be determined by measuring half planes. Figure 2 indicates that the $\Phi_{0_2,I_2}$-value of a half plane might be composed by the $\Phi_{0_2,I_2}$-values of two quadrants from $R^2$ arising in the proof of Theorem 3. This provides formally a reproof of the well known result on marginal distributions, using the method of Section 3.*

**Remark 1** *An example of a certain non-linear transformation of a cone which has the same ipf as the cone itself was introduced in Richter (1995) and further developed in Ittrich & Richter (2005) for the purposes of non-linear regression analysis. In such cases, Lemma 1 applies. Notice that one can construct different types of statistics from the class IpfRep in an analogous way. For more examples of how this method works, we refer to Richter (1995).*

According to Lemma 1, the Euclidean circle ipfs of the random variable's $T$ sub-level sets became the main tool of investigation in this note. Making use of it, we are now in a position to further generalize the results of Theorems 3 and 4. This will be done in the next section.

### 2.2 *Classes of functions of two-dimensional spherical random vectors following the univariate spherical marginal $h$-generalization $\Phi_h$ of $\Phi$*

In the present section we state far-reaching generalizations of the results presented in the previous section. To be concrete, results will be given under the assumption (3) upon the random vector $(X, Y)$ being much more general than those under assumption (1). Assumption (3) allows both heavy and light distribution tails of the bivariate vector distribution. Examples of different types of density generating functions can be found, e.g., in Fang et al. (1990) and in Kalke et al. (2013). First, we give a generalization of Theorem 3.

**Theorem 5** *If $(X, Y)$ satisfies (3) then $P(S(X, Y) < w) = \Phi_h(w), w \in R$ for all coefficients $\lambda_i, \mu_i, i \in \{1, 2, 3\}$ fulfilling conditions (6) and (7), respectively.*

At the same time as this theorem generalizes Theorem 3, it generalizes part A of Theorem 2. The following definition extends Definition 2.

**Definition 3** *If a bivariate random vector $(X, Y)$ satisfies assumption (3) and, for a function $T : R^2 \to R$, the random variable $T(X, Y)$ follows the univariate spherical marginal $h$-generalization $\Phi_h$ of $\Phi$ then we say that $T(X, Y)$ belongs to the class $\Phi_h(\Phi_{h;0_2,I_2})$ of univariate random variables following a spherical marginal $h$-generalization $\Phi_h$ of $\Phi$ derived from a bivariate spherical distribution.*

The final and main result of this note follows immediately the line of the previous results and their proofs in Sections 3 and 4.

**Theorem 6** *If $T \in IpfRep$ then $T(X, Y) \in \Phi_h(\Phi_{h;0_2,I_2})$.*

That is, if (3) and $T \in IpfRep$ then $P(T < w) = \Phi_h(w), w \in R$.

# 3 Reproofs

### *3.1 Geometric measure theoretical proof of Theorem 1 [A]*

In this section, we shall give a geometric measure theoretical reproof of Theorem 1*[A]*. Reproofs are of special interest in the area of didactics of mathematics. They help to make mathematical relationships more clear. For a discussion of the general value of reproofs, see Silvermann (2000). Sometimes, reproofs open new perspectives for proving either sharper or more general versions of known results. Actually, the latter is the case in the present note. To be more concrete, our reproofs of known results in Section 3 will be the basic parts in Section 4 of the proofs of the main results from Section 2. Let

$$W = \frac{2XY}{\sqrt{X^2+Y^2}}.$$

We consider now the two-dimensional Borel set

$$A(w) = \{(x,y) \in R^2 : \frac{2xy}{\sqrt{x^2+y^2}} < w\},\ w \in R.$$

There holds

$$P(W < w) = \Phi_{0_2,I_2}(A(w)), w \in R.$$

The probability $P(W < w)$ can be splitted as

$$P(W < w) = I_{(-\infty,0)}(w)P(sign(X) \neq sign(Y),\ W < w)$$

$$+I_{[0,\infty)}(w)[\frac{1}{2} + P(sign(X) = sign(Y),\ W < w)],\ w \in R.$$

The shaded areas in Figures 1-3 illustrate the sets to be measured in $R^2$ with $\Phi_{0_2,I_2}$ or $\Phi_{h;0_2,I_2}$ for representing the cdf of $W$ under the assumption (1) or (3), respectively.

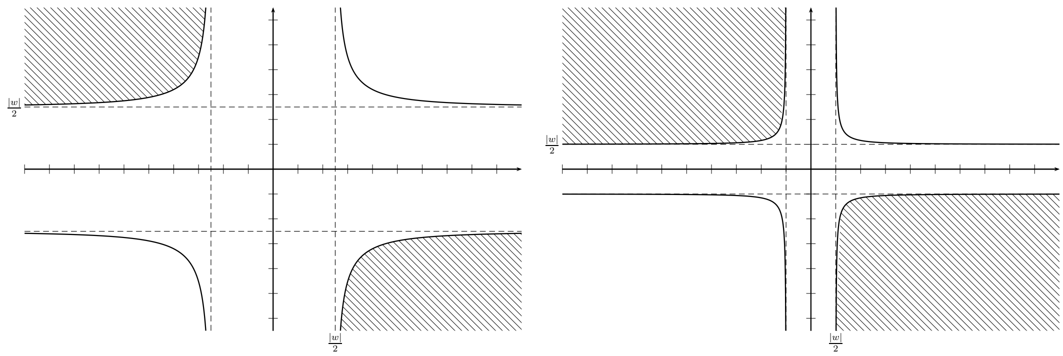
The $p$-functional $|.|_p : R^2 \to [0,\infty)$ which is defined by

$$|(x,y)|_p := (|x|^p + |y|^p)^{1/p},\ (x,y) \in R^2$$

is a norm if $p \geq 1$ and, according to Moszyńska & Richter (2012), an antinorm if $p \in (0,1)$ and a semi-antinorm if $p < 0$. Making use of this functional, we get

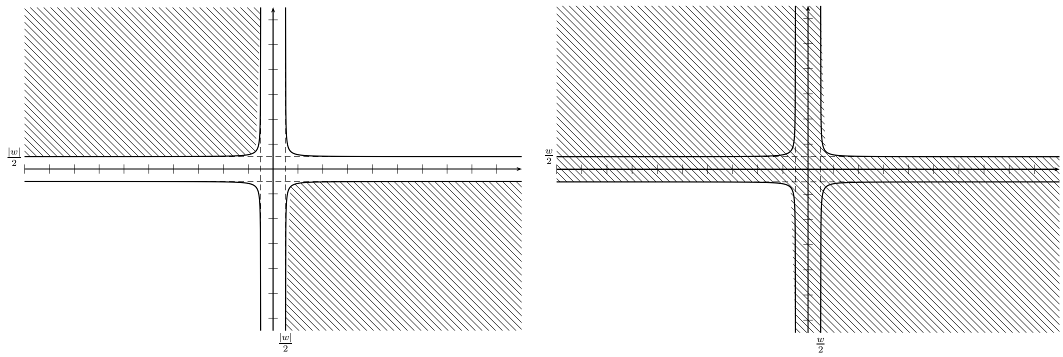
$$P(W < w) = I_{(-\infty,0)}(w)P(sign(X) \neq sign(Y),\ |(X,Y)|_{-2} > -\frac{w}{2})$$

$$+I_{[0,\infty)}(w)[\,\frac{1}{2} + P(sign(X) = sign(Y),\ |(X,Y)|_{-2} < \frac{w}{2})\,].$$



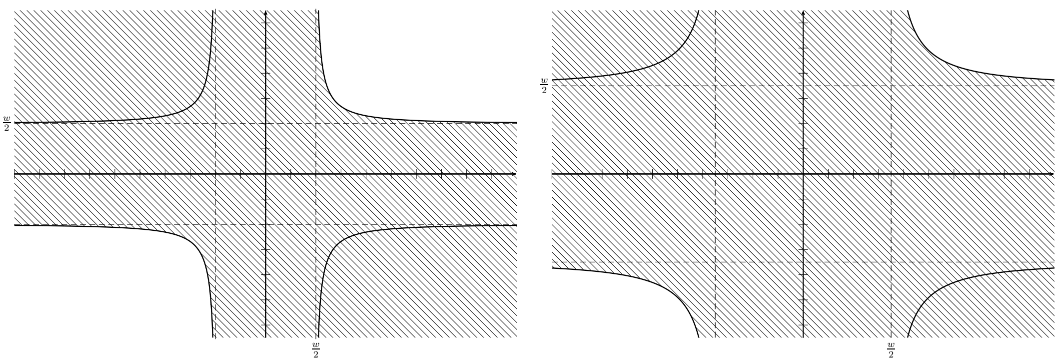
(a) w=-5 (b) w=-2

**Figure 1.** Values from the left tail of the cdf of $T$ are the Gaussian or spherical measure of the shaded areas.



(a) w=-1 (b) w=1

**Figure 2.** Values from the central region of the cdf of $T$. Skip over the case $w = 0$ being closely connected with the case of a half space.



(a) w=4 (b) w=7

**Figure 3.** Values from the right tail of the cdf of $T$ are $\Phi_{0_2,I_2}$ or $\Phi_{h;0_2,I_2}$-measure values of the shaded areas.

Let the sectors of the canonical fan in $R^2$ be denoted according to the anticlockwise enumeration by $C_1, ..., C_4$ and let $\mathcal{F}(A(w), r), r > 0$ be the Euclidean circle ipf of the set $A(w)$. Then

$$\mathcal{F}(A(w), r) = \sum_{i=1}^{4} \mathcal{F}(A(w) \cap C_i, r).$$

In the case $w > 0$,

$$\mathcal{F}(A(w) \cap C_2, r) = \mathcal{F}(A(w) \cap C_4, r) = \frac{1}{4}, \forall r > 0,$$

$$\mathcal{F}(A(w) \cap C_1, r) = \mathcal{F}(A(w) \cap C_3, r) = \frac{1}{4}, 0 \leq r \leq w$$

and

$$\mathcal{F}(A(w), r) = \frac{1}{2} + 2\mathcal{F}(A(w) \cap C_1, r),\ r > w.$$

Moreover,

$$A(w) \cap C_1 \cap C(r) = P(\frac{rw}{2}) \cap C_1 \cap C(r) \qquad (8)$$

where

$$P(t) = \{(x, y) : xy < t\}, t \in R.$$

In a certain sense, calculations needed for considering the ipf of the set $A(w)$ which is generated by the statistic $W$ are transformed into calculations needed for considering the ipf of the set $P(\frac{rw}{2})$ which is generated by the random variable $X \cdot Y$. It follows from the results in Kalke et al. (2013) that the restriction of the ipf of the set $P(t), t > 0$ to the set $P(t) \cap C_1$ allows the representation

$$\mathcal{F}(P(t) \cap C_1, r) = \frac{\alpha}{\pi} I_{(\sqrt{2|t|}, \infty)}(r) + \frac{1}{4} I_{[0, \sqrt{2|t|}\,]}(r)$$

where

$$\alpha = \frac{1}{4} \arccos(1 - \frac{8t^2}{r^4}).$$

With $t = \frac{rw}{2}$ and $\alpha = \frac{1}{4} \arccos(1 - \frac{2w^2}{r^2})$, we have that

$$\mathcal{F}(A(w), r) = (\frac{1}{2} + \frac{1}{2\pi} \arccos(1 - \frac{2w^2}{r^2}))\, I_{[w, \infty)}(r) + I_{[0, w]}(r). \qquad (9)$$

Now, the geometric measure representation formula in Richter (1985 and 1995), see also Kalke et al. (2013), applies such that

$$P(W < w) = \Phi_{0_2, I_2}(A(w))$$

$$= \int_0^w re^{-r^2/2}dr + \int_w^\infty re^{-r^2/2}[\frac{1}{2} + \frac{1}{2\pi}\arccos(1-\frac{2w^2}{r^2})]dr$$

$$= 1 - e^{-w^2/2} + \frac{1}{2}\int_w^\infty re^{-r^2/2}dr + \frac{1}{2\pi}\int_w^\infty re^{-r^2/2}\arccos(1-\frac{2w^2}{r^2})dr$$

$$= 1 - \frac{1}{2}e^{-w^2/2} + \frac{1}{2\pi}\int_w^\infty re^{-r^2/2}\arccos(1-\frac{2w^2}{r^2})dr.$$

It follows immediately that

$$\frac{d}{dw}P(W < w)$$

$$= \frac{1}{2}we^{-w^2/2} + \frac{1}{2\pi}\int_w^\infty re^{-r^2/2}(-1)\frac{-4w/r^2}{\sqrt{1-(1-\frac{2w^2}{r^2})^2}}dr - \frac{1}{2\pi}we^{-w^2/2}\arccos(1-\frac{2w^2}{w^2})$$

$$= \frac{1}{\pi}\int_w^\infty re^{-r^2/2}\frac{dr}{\sqrt{r^2-w^2}}.$$

On changing variables $r^2 - w^2 = z^2$, we get $dz = \frac{r\,dr}{\sqrt{r^2-w^2}}$ and

$$\frac{d}{dw}P(W < w) = \phi_{0,1}(w).$$

The following calculations will show that the latter equation is also true if $w < 0$. In this case,

$$\mathcal{F}(A(w) \cap C_1, r) = \mathcal{F}(A(w) \cap C_3, r) = 0, \forall r > 0,$$

$$\mathcal{F}(A(w) \cap C_2, r) = \mathcal{F}(A(w) \cap C_4, r) = 0, 0 \le r \le w$$

and

$$\mathcal{F}(A(w), r) = 2\mathcal{F}(A(w) \cap C_2, r),\ \forall r > w.$$

Further,

$$A(w) \cap C_2 \cap C(r) = P(\frac{wr}{2}) \cap C_2 \cap C(r) \qquad (10)$$

$$= \{(x, y) \in R^2 : x < 0, y > 0, x^2 + y^2 = r^2, \frac{2(-x)y}{r} > -w\}.$$

It follows from the results in Kalke et al. (2013) that the restriction of the ipf of the set $P(t), t < 0$ to the set $P(t) \cap C_2$ allows the representation

$$\mathcal{F}(P(t) \cap C_2, r) = (\frac{1}{4} - \frac{\alpha}{\pi})\, I_{(\sqrt{2|t|},\infty)}(r),\ r > 0.$$

With $t = \frac{rw}{2}$, we conclude that

$$\mathcal{F}(A(w), r) = 2\,(\frac{1}{4} - \frac{\alpha}{\pi})\, I_{[|w|,\infty)}(r). \tag{11}$$

It follows from (9) and (11) that $W \in$ IpfRep. The geometric measure representation applies, hence

$$P(W < w) = \Phi_{0_2,I_2}(A(w))$$
$$= \int\limits_{|w|}^{\infty} re^{-r^2/2} 2[\frac{1}{4} - \frac{1}{4\pi}\arccos(1 - \frac{2w^2}{r^2})]dr$$
$$= \frac{1}{2}e^{-w^2/2} - \frac{1}{2\pi}\int\limits_{|w|}^{\infty} re^{-r^2/2}\arccos(1 - \frac{2w^2}{r^2})dr.$$

This yields

$$\frac{d}{dw}P(W < w) = -\frac{w}{2}e^{-w^2/2} + \frac{1}{2\pi}\int\limits_{|w|}^{\infty} re^{-r^2/2}\frac{4w/r^2}{\sqrt{1 - (1 - \frac{2w^2}{r^2})^2}}dr$$

$$+\frac{1}{2\pi}we^{-w^2/2}\arccos(1 - 2) = \frac{1}{\pi}\int\limits_{|w|}^{\infty} re^{-r^2/2}\frac{dr}{\sqrt{r^2 - w^2}}.$$

Changing variables as in the case $w > 0$, the result follows immediately.

### *3.2 Geometric measure theoretical proof of Theorem 2[A]*

Because of the equation

$$P(W < w) = \Phi_{h;0_2,I_2}(A(w))$$

this proof follows basically the line of the preceding one until that point where we proved that $W \in$ IpfRep. Now, the geometric measure representation formula for the spherical distribution with dg $h$ in Richter (1991) applies. Notice that the integral

$$I_h = \int\limits_0^{\infty} r\, h(r^2)dr$$

occuring in that formula for an arbitrary dg $h$ equals $\frac{1}{2\pi}$. Hence, for $w > 0$,

$$P(W < w) = \Phi_{h;0_2,I_2}(A(w))$$

$$= 2\pi \left( \frac{1}{4\pi} + \frac{1}{2} \int_0^w rh(r^2)dr + \frac{1}{2\pi} \int_w^\infty rh(r^2) \arccos(1 - \frac{2w^2}{r^2})dr \right).$$

It follows that

$$\frac{d}{dw} P(W < w) = \pi w h(w^2)$$

$$+ \int_w^\infty rh(r^2) \frac{4w/r^2}{\sqrt{1-(1-\frac{2w^2}{r^2})^2}} dr - wh(w^2) \arccos(-1)$$

$$= 2 \int_w^\infty \frac{r}{\sqrt{r^2-w^2}} h(r^2)dr.$$

If $w < 0$ then

$$P(W < w) = 2\pi \int_{|w|}^\infty rh(r^2) 2[\frac{1}{4} - \frac{1}{4\pi} \arccos(1 - \frac{2w^2}{r^2})]dr$$

$$= \pi \int_{|w|}^\infty rh(r^2)dr - \int_{|w|}^\infty rh(r^2) \arccos(1 - \frac{2w^2}{r^2})dr.$$

Hence,

$$\frac{d}{dw} P(W < w) = -\pi w h(w^2) + \int_{|w|}^\infty rh(r^2) \frac{4w/r^2}{\sqrt{1-(1-\frac{2w^2}{r^2})^2}} dr + wh(w^2) \arccos(1-2)$$

$$= 2 \int_{|w|}^\infty \frac{r}{\sqrt{r^2-w^2}} h(r^2)dr = 2 \int_0^\infty h(z^2+w^2)dz.$$

# 4 Proofs of the main results

The basic idea is to further exploit the geometric measure representations of the two-dimensional Gaussian and spherical distribution laws in the proofs of Theorems 4 up to 6. The proof of Theorem 3, however, more directly refers to known results.

*Proof of Theorem 3* It follows from the well known invariance properties of the bivariate standard Gaussian law that

$$P(\begin{pmatrix} X \\ Y \end{pmatrix} \in A(w)) = P(O \begin{pmatrix} X \\ Y \end{pmatrix} \in A(w))$$

for any orthogonal matrix $O$. Hence,

$$\Phi(w) = P(\frac{2XY}{\sqrt{X^2 + Y^2}} < w)$$

$$= P(\frac{2(a_{11}X + a_{12}Y)(a_{21}X + a_{22}Y)}{\sqrt{(a_{11}X + a_{12}Y)^2 + (a_{21}X + a_{22}Y)^2}} < w)$$

$$= P(\frac{\lambda_1 X^2 + \lambda_2 XY + \lambda_3 Y^2}{\sqrt{\mu_1 X^2 + \mu_2 XY + \mu_3 Y^2}})$$

where the coefficients $\lambda_i$ and $\mu_i, i = 1, 2, 3$ satisfy the equations (6) and (7), respectively.

*Proofs of Theorems 4,5 and 6*

Looking through again the reproofs of Theorems 1 and 2 and the final proof of Theorem 3, Definition 1 and Lemma 1 apply. Theorems 4, 5 and 6 are now immediate conclusions from the consideration in Section 3. These proofs show that if we use geometric measure representations in the proofs of Theorems 1 and 2 then the proofs of the generalizations in Theorems 4, 5 and 6 become quite short. However, some of the earlier proofs of, e.g., Theorem 1 are actually shorter than our reproof in Section 3.1.

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