Ball numbers of platonic bodies

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Abstract

The ball number function is extended here to platonic bodies and corresponding generalized uniform distributions on platonic spheres are considered. A platonic disintegration formula of the Lebesgue measure is proved as well as a thin layers property.

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1. Introduction

The mathematical research which leads finally to a generalization of the circle number π was more or less implicitly started in [20], [14] or even in an unpublished 1796-work of Gauss [13] and has been further continued by many authors. We refer in this respect to [2], [8], [11], [12], [19], [24], [26], [27], [38], [39], [42] and [43]. The actual notion of the $l_{2,p}$-generalized circle number was introduced in [35] for convex $l_{2,p}$-circles and extended later to non-convex $l_{2,p}$-circles, ellipses and star discs, including bounded and unbounded ones. A certain survey of this development was given in [29]. A suitable quantity for navigation and mapping was considered in [18]. A multivariate generalization of the notion of circle number has been introduced first in [30] and the notion of ball number function was established in [36].
There are essentially two basic notions needed to define the ball number function. The first one generalizes what we call a ball and its radius. For symmetric balls, a closely related discussion of some positively homogeneous functions on $\mathbb{R}^n$ can be found in [23]. From this article, it turns out that the Minkowski functional of a star body can be used for defining the radius of convex and even non-convex balls. The present paper will deal, however, with non-symmetric cases, too. The second main notion for establishing ball numbers is that of a suitably defined non-Euclidean metric surface content. Two ways of introducing this notion in the case of $l_{n,p}$-spheres are discussed in [30]. One of these ways consists in replacing the Euclidean norm of the vector normal to the sphere in the defining integral of the Euclidean surface content by a suitably chosen non-Euclidean norm. While this way of generalizing the notion of surface content deals

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with integration, the other way deals with taking derivatives of volumes of suitably defined ball sectors with respect to the generalized radius of the generalized ball. This local way to define a suitable non-Euclidean surface content was introduced first in [33] and proved to be equivalent to the integral way in [30]. In the latter paper it was further outlined that this surface measure allows also a cone measure interpretation. The equivalence of the local and integral approaches to a generalized notion of surface content has been proved later for ellipsoids in [34]. It becomes obvious from the cited literature that the local definition of generalized surface content may be introduced also for more general balls as far as the Minkowski functionals of these balls are known. However, it will need much more efforts to establish always the corresponding integral approach.

Notice that the notion of generalized surface content is related to the non-Euclidean generalization of the method of indivisibles and to the notion of a generalized uniform distribution on a generalized sphere being the topological boundary of a generalized ball. This method was developed in several articles of the first author and discussed to some extent recently in [32]. For an introduction to the topic of Cavalieri's classical method of indivisibles and the related so called Cavalieri integration, we refer to [3] and to [1] and [21], respectively.

While our survey deals so far with $l_{n,p}$-balls and ellipsoids, in Section 5 of [36] the question of how to extend the ball number function to further classes of balls is discussed. The present article contributes to this research area. Notice that the $l_{3,p}$-ball dealt with in [30] as a special case is the octahedron if $p = 1$ and the cube in the limit case as $p \to \infty$. Here, we study all the platonic bodies. Furthermore, in this paper, we point out applications of platonically generalized uniform distributions in the fields of processing times optimization and graphical visualization. To this aim, we present first an example in which we make use of a tetrahedral-generalized uniform distribution for testing whether processing times of a certain product are optimal under given optimality criteria. Moreover, we will show the applicability of platonically generalized uniform distributions for graphical visualization purposes in another example. For more details, we refer to Section 3.

The paper is organized as follows. We define a generalized radius for each platonic body in Section 2, a corresponding notion of generalized surface content and corresponding generalized uniform distribution in Section 3 and prove a disintegration formula for the Lebesgue measure of platonic bodies in Section 4. Section 5 deals with the extension of the ball number function in $\mathbb{R}^3$ over the platonic bodies. Finally, in Section 6 we describe the asymptotic behaviour of the Lebesgue measure of a thin layer including the boundary of a platonic body.

2. Generalized ball radius

One of the basic steps of generalizing the circle number π in [35] was to change the understanding of the notion of radius. While usually this notion is associated with the idea of possible addition of lengths, this understanding was changed in [35] with the idea of multiplying a generalized reference circle by a positive number for defining the set of all points having the same distance from a given point. It turned out that the Minkowski functional of this generalized reference circle can be interpreted as its generalized radius functional. Here, we would like to note that one can find different definitions of the notion Minkowski functional in the literature. In [17], the functional $p_E(x) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$p_E(x) = \inf\{r > 0 : x/r \in E\}, \tag{1}$$

for every n-dimensional convex body E with the origin in its interior is called Minkowski functional of E if it is subadditive and satisfies the condition

$$p_E(\alpha x) = \alpha p_E(x), \text{ for all } \alpha > 0 \text{ and } x \in \mathbb{R}. \tag{2}$$

Originally, this definition was introduced by Hermann Minkowski in [22]. Furthermore, there exists a definition of the Minkowski functional in the literature which assumes E to be centrally symmetric, so that (2) has to be replaced by $p_E(\alpha x) = |\alpha| p_E(x)$, for all $\alpha \in \mathbb{R}$. For this definition and the area of Minkowski geometry and Minkowski spaces in general, we refer to [42]. In this paper, we follow the more general approach that is given in [17], too, by calling, for every absorbing set E, p_E in (1) the Minkowski functional of E.
The Minkowski functionals h_C of the unit cube C and h_O of the unit octahedron O are well known special cases of the much more general l_p-ball functionals which were considered, e.g. in [17]. In case of the bodies C and O, these functionals are

$$h_C(x) = max\{|x_1|, |x_2|, |x_3|\} \text{ and } h_O(x) = |x_1| + |x_2| + |x_3|, \text{ for } x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

respectively. For some basic knowledge on platonic bodies, we refer to [6], [5], [41] and [4]. The following theorem deals with the Minkowski functionals h_T, h_D and h_I of the unit tetrahedron T, unit dodecahedron D and unit icosahedron I, respectively.

Theorem 1 *Let g be the Golden Ratio:* $g = \frac{1+\sqrt{5}}{2}$. *Then, for every point* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$h_T(x) = max\{|x_1 + x_3| + x_2,\ |x_1 - x_3| - x_2\},$$
$$h_D(x) = max\left\{\frac{1}{\sqrt{5}}|x_1| + \frac{2}{5+\sqrt{5}}|x_2|,\ \frac{1}{\sqrt{5}}|x_2| + \frac{2}{5+\sqrt{5}}|x_3|,\ \frac{1}{\sqrt{5}}|x_3| + \frac{2}{5+\sqrt{5}}|x_1|\right\} \text{ and}$$
$$h_I(x) = max\{(2-g)(|x_1| + |x_2| + |x_3|),\ (\sqrt{5}-2)|x_1| + (g-1)|x_2|,\ (\sqrt{5}-2)|x_2| + (g-1)|x_3|,$$
$$(\sqrt{5}-2)|x_3| + (g-1)|x_1|\}.$$

Proof Let the unit cube be given by its vertices $(1,-1,1), (1,-1,-1), (1,1,-1), (1,1,1),$ $(-1,1,1), (-1,-1,1), (-1,-1,-1), (-1,1,-1)$ and the unit tetrahedron with its vertices $(1,-1,1),$ $(1,1,-1), (-1,1,1), (-1,-1,-1)$. Note that T is a subset of C, see Figure 1. Every lateral face of the tetrahedron is a subset of a plane in $\mathbb{R}^3$. Hence, T can be considered as the intersection of the suitably chosen corresponding half spaces as follows:

$$T = \{v \in \mathbb{R}^3 : A_T v \le b_T\}, \text{ with } A_T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},\ b_T = \mathbb{1}_4 \in \mathbb{R}^4,$$

where the sign $\le$ means componentwise inequality. An equivalent representation of T is given by

$$T = \left\{(x,y,z) \in \mathbb{R}^3 : max\{x+y+z, x-y-z, -x+y-z, -x-y+z\} \le 1\right\},$$

from which the following equivalent representation of T can be derived:

$$T = \left\{(x,y,z) \in \mathbb{R}^3 : max\{|x+z| + y, |x-z| - y\} \le 1\right\}.$$

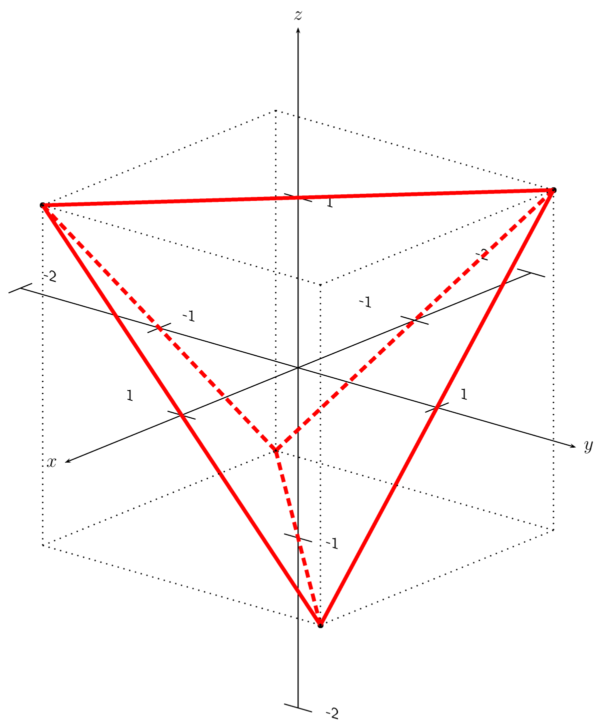

Figure 1: The unit tetrahedron within the unit cube

Let us define the convex function $f : \mathbb{R}^3 \to \mathbb{R}^+$ by

$$f(v) = max\{|x+z| + y, |x-z| - y\}, \text{ for } v = (x, y, z) \in \mathbb{R}^3.$$

The Minkowski functional h_T of the tetrahedron T is

$$\begin{aligned} h_T(v) &= \inf\{r > 0 : v/r \in T\} = \inf\{r > 0 : f(v/r) \leq 1\} \\ &= \inf\{r > 0 : max\{|x+z| + y, |x-z| - y\} \leq r\} \\ &= f(v), \text{ for } v = (x, y, z) \in \mathbb{R}^3. \end{aligned}$$

The first assertion of Theorem 1 has thus been proved. Let us consider now the unit icosahedron with the vertices

$$\begin{aligned} &a_1 = (g, 1, 0),\ b_1 = (-g, 1, 0),\ c_1 = (-g, -1, 0),\ d_1 = (g, -1, 0), \\ &a_2 = (1, 0, g),\ b_2 = (1, 0, -g),\ c_2 = (-1, 0, -g),\ d_2 = (-1, 0, g), \\ &a_3 = (0, g, 1),\ b_3 = (0, -g, 1),\ c_3 = (0, -g, -1),\ d_3 = (0, g, -1), \end{aligned}$$

see Figure 2. The main steps of proof will be done in this case analogously to the previous case. First of all, we determine the equations of the twenty planes each of which contains one lateral face of the unit icosahedron as a subset. The result can be found in the Appendix in Table 1. We consider now the intersection of the suitably chosen half spaces to establish a representation for

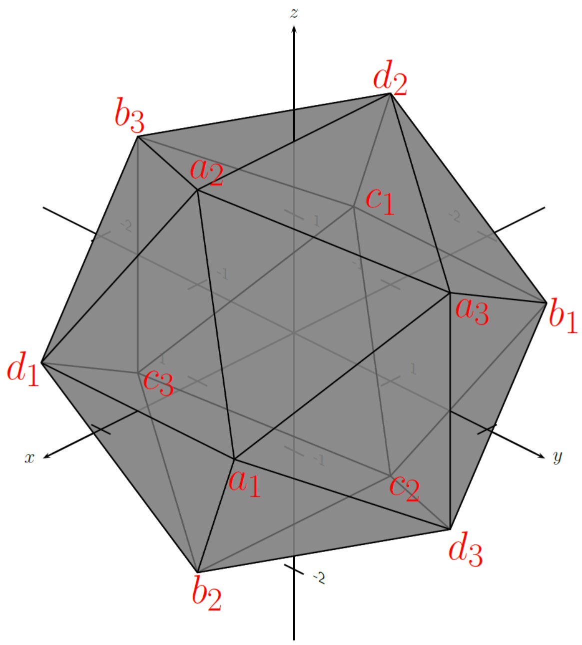

Figure 2: Unit icosahedron with vertices $a_i, b_i, c_i, d_i,\ i = 1, ..3$

the unit icosahedron as follows:

$$I = \{v \in \mathbb{R}^3 : A_I v \leq b_I\},$$

$$\text{with } A_I = \begin{pmatrix} A_{I_1} \\ A_{I_2} \end{pmatrix},\ b_I = \mathbb{1}_{20} \in \mathbb{R}^{20},$$

$$A_{I_1} = \begin{pmatrix} 2-g & 2-g & 2-g \\ -(2-g) & 2-g & 2-g \\ 2-g & -(2-g) & 2-g \\ 2-g & 2-g & -(2-g) \\ -(2-g) & -(2-g) & 2-g \\ 2-g & -(2-g) & -(2-g) \\ -(2-g) & 2-g & -(2-g) \\ -(2-g) & -(2-g) & -(2-g) \\ 2-\sqrt{5} & g-1 & 0 \\ -(2-\sqrt{5}) & g-1 & 0 \end{pmatrix} \text{ and } A_{I_2} = \begin{pmatrix} 2-\sqrt{5} & -(g-1) & 0 \\ -(2-\sqrt{5}) & -(g-1) & 0 \\ 0 & 2-\sqrt{5} & g-1 \\ 0 & -(2-\sqrt{5}) & g-1 \\ 0 & 2-\sqrt{5} & -(g-1) \\ 0 & -(2-\sqrt{5}) & -(g-1) \\ g-1 & 0 & 2-\sqrt{5} \\ -(g-1) & 0 & 2-\sqrt{5} \\ g-1 & 0 & -(2-\sqrt{5}) \\ -(g-1) & 0 & -(2-\sqrt{5}) \end{pmatrix}.$$

Finally, we simplify this representation of the boundary of I using the equalities

$$max\{A_I[i]v, i = 1, ..., 8\} = (2 - g)(|x| + |y| + |z|),$$
$$max\{A_I[i]v, i = 9, ..., 12\} = (\sqrt{5} - 2)|x| + (g - 1)|y|,$$
$$max\{A_I[i]v, i = 13, ..., 16\} = (\sqrt{5} - 2)|y| + (g - 1)|z|$$

and

$$max\{A_I[i]v, i = 17, ..., 20\} = (\sqrt{5} - 2)|z| + (g - 1)|x|,$$

where $A_I[i]$ denotes the i-th row of the matrix A_I for $i = 1, ..., 20$.
To check the Minkowski functional of the dodecahedron, we notice first that the unit dodecahedron is the outer dual dodecahedron of the unit icosahedron. It follows that the vectors connecting the origin and the vertices of the unit icosahedron are vectors normal to the lateral faces of the dodecahedron. This enables us to determine the twelve equations of the planes where the lateral faces of the unit dodecahedron belong to. The results are given in the Appendix in Table 2. To generate now the unit dodecahedron, we consider again the intersection of the suitably chosen half spaces. The resulting representation for the unit dodecahedron is

$$D = \{v \in \mathbb{R}^3 : A_D v \leq b_D\},$$
$$\text{where } b_D = \mathbb{1}_{12} \in \mathbb{R}^{12\times 1}, A_D = \begin{pmatrix} A_{D_1} \\ A_{D_2} \\ A_{D_3} \end{pmatrix},$$

$$A_{D_1} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{5+\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{5+\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & -\frac{2}{5+\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{5+\sqrt{5}} & 0 \end{pmatrix}, A_{D_2} = \begin{pmatrix} \frac{2}{5+\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{2}{5+\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{2}{5+\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{2}{5+\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix} \text{ and } A_{D_3} = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{5+\sqrt{5}} \\ 0 & -\frac{1}{\sqrt{5}} & \frac{2}{5+\sqrt{5}} \\ 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{5+\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{5+\sqrt{5}} \end{pmatrix}.$$

Finally, we simplify this representation of the unit dodecahedron D. To this end, we determine explicit expressions for $max\{A_D[i]v, i = 1, ..., 4\}$, $max\{A_D[i]v, i = 5, ..., 8\}$ and $max\{A_D[i]v, i = 9, ..., 12\}$. ■

Let $Plat \in \{T, C, O, D, I\}$ be one of the five platonic unit bodies. The boundary S_{Plat} of $Plat$ allows the representation $S_{Plat} = \{x \in \mathbb{R}^3 : h_{Plat}(x) = 1\}$. Because the five Minkowski functionals satisfy condition (2), we have

$$r \cdot S_{Plat} = \{x \in \mathbb{R}^3 : h_{Plat}(x) = r\}, \ r > 0.$$

This motivates us to say that the formula $S_{Plat}(r) = r \cdot S_{Plat}$ defines the platonic sphere of platonic radius $r = r_{Plat}$. The set $K_{Plat}(r) = r \cdot Plat$ will be called the platonic ball of platonic radius r.

3. Generalized surface content

The notion of generalized surface content of $l_{n,p}$-balls and ellipsoids has been dealt with in [30] and [34], respectively. In both situations, integral and local definitions are given and their equivalence is proved. Note that this notion goes back already to [33] where only its local definition was introduced to obtain several new probabilistic and statistical results. Here again,

we introduce just a local definition of the generalized surface measure and leave it as an open problem to describe an equivalent integral approach.
Moreover, let us consider a set $S \subset \mathbb{R}^3$ and define the Borel-σ-field on S by $\mathfrak{B}^3 \cap S = \{B \cap S : B \in \mathfrak{B}^3\}$. Then $A \in \mathfrak{B}^3 \cap S_{Plat}$ is a Borel measurable subset of a platonic unit sphere. The central projection cone generated by A is defined as

$$CPC_{Plat}(A) = \left\{ x \in \mathbb{R}^3 : \frac{x}{h_{Plat}(x)} \in A \right\}$$

and the set

$$sec_{Plat}(A, r) = CPC_{Plat}(A) \cap K_{Plat}(r)$$

will be called a sector of the platonic ball $K_{Plat}(r)$ with platonic radius $r > 0$.

Definition 1 *The finite measure $\mathfrak{O}_{Plat} : \mathfrak{B}^3 \cap S_{Plat}(r) \to \mathbb{R}^+$ which is defined by*

$$\mathfrak{O}_{Plat}(A) = \frac{d}{d\rho}\lambda(sec_{Plat}(\frac{1}{r}A, \rho))|_{\rho=r},$$

where $\lambda(\cdot)$ is the Lebesgue measure and $A \in \mathfrak{B}^3 \cap S_{Plat}(r)$, will be called the platonically generalized surface measure on the platonic sphere $S_{Plat}(r)$.

Let $V_{Plat}(r) = \lambda(K_{Plat}(r))$. The equation

$$\lambda(K_{Plat}(r)) = \int_0^r \mathfrak{O}_{Plat}(S_{Plat}(\rho))d\rho \tag{3}$$

reflects a certain generalization of the method of indivisibles of Cavalieri and Torricelli. We observe that the ratios in

$$\frac{V_{Plat}(r)}{r^3} = \frac{\mathfrak{O}_{Plat}(S_{Plat}(r))}{3r^2} \tag{4}$$

do not only coincide but are even independent of r. The common constant value of the left and right hand side of (4) may be considered as $V_{Plat}(1) = \frac{1}{3}\mathfrak{O}_{Plat}(S_{Plat})$. These circumstances will motivate our basic definition in Section 5. The notion of platonically generalized surface content, moreover, motivates the following definition which is closely connected with the corresponding definitions in cases of $l_{n,p}$-spheres in [30] and ellipsoids in [34]. Note that the distribution P^U induced by a random vector U on the Borel σ-field $\mathfrak{B}^3$ has the property $P^U(A) = P(U \in A)$ for every Borel measurable set A.

Definition 2 *Let B be a Borel subset of S_{Plat} and U a random vector taking its values in S_{Plat}. The platonically generalized uniform distribution of U is defined by*

$$P(U \in B) = \frac{\mathfrak{O}_{Plat}(B)}{\mathfrak{O}_{Plat}(S_{Plat})}.$$

Let a random vector X follow the uniform probability distribution on the platonic body $Plat$, i.e. $P(X \in A) = \lambda(A)/\lambda(Plat)$, $A \in \mathfrak{B}^3 \cap Plat$ and put $Y = X/h_{Plat}(X)$ where division is defined componentwise.

Theorem 2 *The random vector Y follows the platonically generalized uniform distribution on* S_{Plat}.

Proof It follows from the definition of $\mathfrak{O}_{Plat}$ that equation (3) may be generalized as

$$\lambda(sec_{Plat}(A,r)) = \int_0^r \mathfrak{O}_{Plat}(S_{Plat}(\rho) \cap [\rho A])d\rho,\ A \in \mathfrak{B}^3 \cap S_{Plat}. \tag{5}$$

Accordingly, (4) extends to all sets $A \in \mathfrak{B}^3 \cap S_{Plat}(r)$,

$$\frac{V_A(r)}{r^3} = \lambda\left(sec_{plat}\left(\frac{1}{r}A, 1\right)\right) = \frac{\mathfrak{O}_{Plat}(A)}{3r^2},$$

where $V_A(r) = \lambda\left(sec_{Plat}\left(\frac{1}{r}A, r\right)\right)$. Hence, for $A \in \mathfrak{B}^3 \cap S_{Plat}$,

$$\begin{aligned} P(Y \in A) &= P(X \in CPC_{Plat}(A) \cap K_{Plat}) = P(X \in sec_{Plat}(A,1)) \\ &= \frac{\lambda(sec_{plat}(A,1))}{\lambda(Plat)} = \frac{\frac{d}{d\rho}\lambda(sec_{plat}(A,\rho))}{\frac{d}{d\rho}\lambda(K_{plat}(\rho))}\bigg|_{\rho=1} = \frac{\mathfrak{O}_{Plat}(A)}{\mathfrak{O}_{Plat}(S_{Plat})}. \end{aligned}$$ ■

Remark 1 *It follows from the consideration in the last line of the preceding proof that the platonically generalized uniform distribution on* $\mathfrak{B}^3 \cap S_{Plat}$ *coincides with the so called cone measure, i.e.*

$$\frac{\mathfrak{O}_{Plat}(A)}{\mathfrak{O}_{Plat}(S_{Plat})} = \frac{\lambda(sec_{plat}(A,1))}{\lambda(Plat)},\ A \in \mathfrak{B}^3 \cap S_{Plat}.$$

For more details on the cone measure, we refer to [40], [25] and the discussion in [30].

It was shown in [30] in accordance with earlier results in [10], [9] and [15] that the n-dimensional 1-generalized surface content measure on the $l_{n,1}$-sphere satisfies the equation $\mathfrak{O}_{n,1}(\cdot) = \mathcal{O}(\cdot)/\sqrt{n}$ where $\mathcal{O}$ is the Euclidean surface content measure. In the present notation, this means $\mathfrak{O}_O(A) = \mathcal{O}(A)/\sqrt{3},\ A \in \mathfrak{B}^3 \cap S_O$. Note that $1/\sqrt{3}$ is the Euclidean distance between the origin and the boundary of the platonic body O. The following Theorem 3 generalizes this result. To this end, let l_{Plat} denote the Euclidean distance between the origin and one of the planes each point of A belongs to. It is well known that $\lambda(sec_{Plat}(A,1)) = l_{Plat}\mathcal{O}(A)/3,\ A \in \mathfrak{B}^3 \cap S_{Plat}$. Hence,

$$\begin{aligned} \mathfrak{O}_{Plat}(A) &= \frac{1}{3} \cdot l_{Plat} \cdot \mathfrak{O}_{Plat}(S_{Plat}) \cdot \frac{\mathcal{O}(A)}{\lambda(Plat)} = \frac{1}{3} \cdot l_{Plat} \cdot 3 \cdot \lambda(Plat) \cdot \frac{\mathcal{O}(A)}{\lambda(Plat)} \\ &= l_{Plat} \cdot \mathcal{O}(A),\ A \in \mathfrak{B}^3 \cap S_{Plat}. \end{aligned}$$

Theorem 3 *Let* $A \in \mathfrak{B}^3 \cap S_{Plat}$. *Then*

$$\mathfrak{O}_{Plat}(A) = l_{Plat} \cdot \mathcal{O}(A),$$

where

$$l_T = 1/\sqrt{3},\ l_C = 1,\ l_O = 1/\sqrt{3},\ l_D = \sqrt{g^2+1},\ l_I = (g+1)/\sqrt{3}.$$

Proof It remained only to prove the formulae for l_{Plat}. Note that l_{Plat} coincides with the inscribed Euclidean ball radius R_{plat} of a platonic unit body. For formulae connecting the edge length of $Plat$ and the radius R_{Plat}, we refer to [41]. It can be shown that the unit platonic bodies $T,\ C,\ O,\ D$ and I posses the edge lengths $a_T = \sqrt{8}$, $a_C = 2$, $a_O = \sqrt{2}$, $a_D = 3\sqrt{5} - 5$ and $a_I = 2$, correspondingly. To verify this for $T,\ C,\ D$ and I, we refer to the proof of Theorem 5. For O, which has got the vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(-1, 0, 0)$, $D(0, -1, 0)$, $E(0, 0, 1)$, $F(0, 0, -1)$, the quantity a_O can be calculated in an analogous way. Finally, l_T, l_C, l_O, l_D and l_I are calculated as claimed in the theorem. ■

Example 1 *Platonically generalized uniform distributions occur naturally in the field of optimization of production scheduling for testing whether or not a production process fulfills given optimality criteria. This may concern, for example, assumed processing times.*
Consider a production process P which is divided into three sub-processes $P_1,\ P_2$ *and* P_3. *For every sub-process, a processing time of* 45 *hours is planned on average with a deviation of* ± 5 *hours. We assume that the target time of P has to be* 140 *hours. That means, there are* 5 *hours tolerance which can be spread over the processing times* $t_1,\ t_2$ *and* t_3 *of* $P_1,\ P_2$ *and* P_3, $(t_1, t_2, t_3) \in [40, 50]^3$. *This leads to the first criteria for P to be in target if*

$$(t_1 - 45) + (t_2 - 45) + (t_3 - 45) \leq 5.$$

Furthermore, we assume that the time, a sub-process spends more than the minimal amount of 40 *hours, is not allowed to be bigger than the sum of the times the other two sub-processes are spending more than the* 40 *hours. This restriction is motivated to avoid the effect that only one sub-process is spending far too much of the tolerance time, in comparison to the other sub-processes. To model this criteria we have to require that the triple of times* $(t_1, t_2, t_3) \in [40, 50]^3$ *fulfills the three inequalities*

$$\begin{aligned}
&(t_1 - 40) \leq (t_2 - 40) + (t_3 - 40),\\
&(t_2 - 40) \leq (t_1 - 40) + (t_3 - 40) \textit{ and}\\
&(t_3 - 40) \leq (t_1 - 40) + (t_2 - 40),
\end{aligned}$$

which can be equivalently converted to

$$\begin{aligned}
(t_1 - 45) - (t_2 - 45) - (t_3 - 45) &\leq 5,\\
-(t_1 - 45) + (t_2 - 45) - (t_3 - 45) &\leq 5 \textit{ and}\\
-(t_1 - 45) - (t_2 - 45) + (t_3 - 45) &\leq 5.
\end{aligned}$$

Summarizing, we can say, that the processing time for P, which is the sum $t_1 + t_2 + t_3$ *is in target, if*

$$\max(|(t_1 - 45) + (t_3 - 45)| + (t_2 - 45), |(t_1 - 45) - (t_3 - 45)| - (t_2 - 45)) \leq 5.$$

This means that (t_1, t_2, t_3) *has to be a point from the tetrahedron which is centered at* $(45, 45, 45)$ *and has platonic radius* 5. *If we assume that there are no systematic failures in* $P_1,\ P_2$ *and* P_3 *leading to a systematic falsification of the processing times, we can say that the production process P is in target if the vector of processing times belongs to the considered tetrahedron. From a probabilistic point of view, it could be uniformly distributed within the mentioned tetrahedron. For an illustration, we refer to Figure 3 showing* 100 *simulated vectors* *of processing times, being uniformly distributed in the tetrahedron with center* $(45, 45, 45)$ *and generalized radius* 5 *and satisfying the described optimality criteria. The pictures are drawn by using the computer algebra system software Maple* $^{TM}12$*. For the simulation of the pseudo random points in Figure 3, we used the acceptance-rejection sampling method for random samples, by generating a uniform pseudo random vector* (Z_1, Z_2, Z_3) *on* $[40, 50]^3$ *and accepting it if* $\max(|(Z_1 - 45) + (Z_3 - 45)| + (Z_2 - 45), |(Z_1 - 45) - (Z_3 - 45)| - (Z_2 - 45)) \leq 5$*. Note that, for* $C = \{(Z_1, Z_2, Z_3) \in [40, 50]^3\}$*,* $T^* = \{(x, y, z) \in \mathbb{R}^3 : h_T((x - 45, y - 45, z - 45)) \leq 5\}$ *and B being a subset of* T^**,*

$$P((Z_1, Z_2, Z_3) \in B \mid (Z_1, Z_2, Z_3) \in T^*) = \frac{P((Z_1, Z_2, Z_3) \in B)}{P((Z_1, Z_2, Z_3) \in T^*))} = \frac{\lambda(B)/\lambda(C)}{\lambda(T^*)/\lambda(C)} = P(Y \in B),$$

where $\lambda(\cdot)$ *denotes the Lebesgue measure in* $\mathbb{R}^3$ *and* $Y = (Y_1, Y_2, Y_3)$ *is a random vector which follows the uniform distribution on the tetrahedron* T^**. Note that, in accordance with Theorem 2 we can generate a random vector* $U = (U_1, U_2, U_3) = Y/h_T((Y_1 - 45, Y_2 - 45, Y_3 - 45))$ *being generalized uniformly distributed on the surface of the tetrahedron* T^**. For more details concerning the acceptance-rejection sampling method, we refer to [16].*

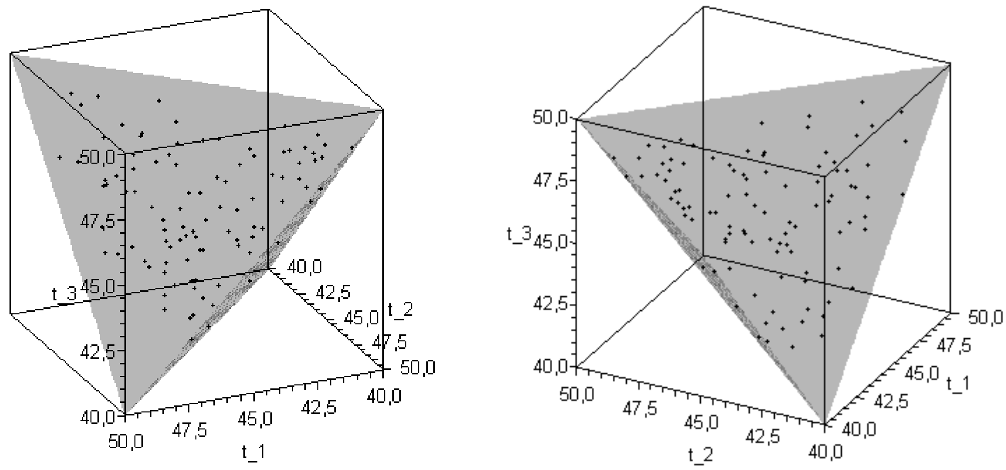

Figure 3: Simulation of 100 processing times according to the optimality problem of Example 1

Example 2 *The papers [7] and [37] deal with generating random points in a tetrahedron and tetrahedral meshes for graphical purposes, to handle the visualization of unstructured data sets and create three-dimensional pictures on the computer. The more general algorithm mentioned in Example 1 works as follows. For a given center* $C = (x_c, y_c, z_c)$ *and a platonically generalized radius r, platonically generalized uniformly distributed pseudo random points can be generated in the tetrahedron with center C and radius r, using the following acceptance-rejection sampling method:*

1. *Generate independent uniformly distributed pseudo random variables* Z_1 *in* $[x_c - r, x_c + r]$, Z_2 *in* $[y_c - r, y_c + r]$ *and* Z_3 *in* $[z_c, -r, z_c + r]$.
2. *If* $\max(|(Z_1 - x_c) + (Z_3 - z_c)| + (Z_2 - y_c), |(Z_1 - x_c) - (Z_3 - z_c)| - (Z_2 - y_c)) \leq r$*, then accept* (Z_1, Z_2, Z_3).

For an illustration of the tetrahedral case, see Figure 4. Again, the pictures are drawn by using Maple ^{TM}12.

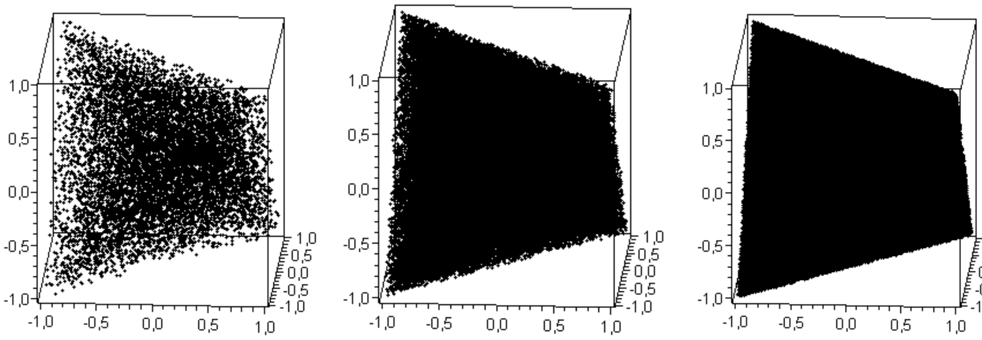

Figure 4: Simulation of 10^4, 10^5 and 10^6 uniformly distributed pseudo random points in the unit tetrahedron T

4. Platonic disintegration of the Lebesgue measure

Platonic disintegration of the Lebesgue measure is closely related to the notion of geometric measure representation that has been dealt with in [33], [30] and [36] for $l_{n,p}$-balls as well as in [34] for ellipsoids. A corresponding survey is given in [32]. In this sense, we will introduce a disintegration formula for $\lambda(B)$, $B \subset \mathbb{R}^3$ Borel measurable, using the platonically generalized surface measure. Further, we will give an example showing how platonic disintegration of the Lebesgue measure works.

Theorem 4 *For all Borel measurable subsets* $B \subset \mathbb{R}^3$,

$$\lambda(B) = \int_0^\infty \mathfrak{O}_{Plat}(B \cap S_{Plat}(r))dr.$$

Proof We follow the line of the proof of Theorem 4 in [33]. The collection of $\mathfrak{G}_{Plat}$ of all sets of the type

$$A_{Plat}(\mathfrak{D}; \rho_1, \rho_2) := sec_{Plat}(\mathfrak{D}, \rho_2) \backslash sec_{Plat}(\mathfrak{D}, \rho_1),\ 0 < \rho_1 < \rho_2 < \infty,\ \mathfrak{D} \in \mathfrak{B}^3 \cap S_{Plat}$$

is a semi-ring. We consider the finite additive set function

$$\lambda^*(A_{Plat}(\mathfrak{D}; \rho_1, \rho_2)) = \int_{\rho_1}^{\rho_2} \mathfrak{O}_{Plat}(A_{Plat}(\mathfrak{D}; \rho_1, \rho_2) \cap S_{Plat}(r))dr,$$

where the integrand does not depend on r. Notice that

$$\lambda^*(A_{Plat}(\mathfrak{D};\rho_1,\rho_2)) = \int\limits_{\rho_1}^{\rho_2} \mathfrak{O}_{Plat}(r\mathfrak{D})dr \ \text{ and } \ \lambda([\rho_1,\rho_2]\mathfrak{D}) = \int\limits_{\rho_1}^{\rho_2} \frac{d}{d\rho}\lambda([0,\rho)\cdot\mathfrak{D})d\rho.$$

Hence, $\lambda^*(\cdot)$ and $\lambda(\cdot)$ coincide on $\mathfrak{G}_{Plat}$. Let us denote the smallest ring including $\mathfrak{G}_{Plat}$ by $\mathfrak{R}_{Plat}$. If (B_k) is a sequence from $\mathfrak{R}_{Plat}$ satisfying $B_{k+1} \subset B_k,\ \forall k$ and $\bigcap\limits_{k=1}^{\infty} B_k = \emptyset$ then $\lambda^*(B_k) \underset{k\to\infty}{\longrightarrow} 0$, i.e. $\lambda^*(\cdot)$ is continuous at $\emptyset$ and therefore countable additive on $\mathfrak{R}_{Plat}$. It follows by measure extension theorem that $\lambda^*(\cdot)$ and $\lambda(\cdot)$ coincide on the σ-algebra of all Borel measurable subsets $B \subset \mathbb{R}^3$. ∎

Remark 2 *In view of Theorem 3, this result can be reformulated as follows:*

$$\lambda(B) = l_{Plat} \int\limits_0^{\infty} \mathcal{O}(B \cap S_{Plat}(r))dr.$$

Corollary 1 *For every Borel measurable subset $B \subset \mathbb{R}^3$,*

$$\lambda(B) = \int\limits_0^{\infty} r^2 \mathfrak{O}_{Plat}\left(\frac{1}{r}B \cap S_{Plat}\right)dr = l_{Plat} \int\limits_0^{\infty} r^2 \mathcal{O}\left(\frac{1}{r}B \cap S_{Plat}\right)dr.$$

Proof Let $A \in \mathfrak{B}^3 \cap S_{Plat}$. Then

$$\begin{aligned}\mathfrak{O}_{Plat}(rA) &= \frac{d}{d\rho}\lambda(sec_{Plat}(rA/r,\rho))\Big|_{\rho=r} = \frac{d}{d\rho}\rho^3\lambda(sec_{Plat}(A,1))\Big|_{\rho=r}\\ &= 3r^2\lambda(sec_{Plat}(A,1)) = r^2\mathfrak{O}_{Plat}(A).\end{aligned}$$

By Theorem 4 with $rA = B \cap S_{Plat}(r),\ B \subset \mathbb{R}^3$, the corollary follows. ∎

Definition 3 *Let $B \subset \mathbb{R}^3$ be Borel measurable. The function $r \mapsto \mathfrak{F}_{Plat}(B,r)$ defined by*

$$\mathfrak{F}_{Plat}(B,r) = \frac{\mathcal{O}\left(\frac{1}{r}B \cap S_{Plat}\right)}{\mathcal{O}(S_{Plat})},\ r > 0,$$

is called the platonic intersection percentage function (ipf) of the set B.

Let $\omega_{Plat} = \mathfrak{O}_{Plat}(S_{Plat}) = l_{Plat}\mathcal{O}(S_{Plat})$ denote the platonically generalized surface content of the platonic unit sphere. The reformulation of the disintegration formula of the Lebesgue measure $\lambda(\cdot)$ for Borel measurable subsets $B \subset \mathbb{R}^3$

$$\lambda(B) = \omega_{Plat} \int\limits_0^{\infty} r^2 \mathfrak{F}_{Plat}(B,r)dr$$

is called a platonic geometric measure representation formula of the Lebesgue measure. The aim of the following example is to demonstrate that this measure representation applies successfully. To this end, we will re-prove the well known result for the volume of the unit cube C. In this case, the geometric measure representation is based upon the ipf of the tetrahedron.

Example 3 *Let* $B := \{x \in \mathbb{R}^3 : h_C(x) \leq 1\}$. *We have*

$$\omega_T = l_T \mathscr{O}(S_T) = 8$$

and therefore

$$\lambda(B) = \omega_T \int_0^\infty r^2 \mathfrak{F}_T(B,r)dr = 8 \int_0^\infty r^2 \mathfrak{F}_T(B,r)dr.$$

To calculate $\mathfrak{F}_T(B,r)$*, we have to distinguish between three cases.*

1. *If* $0 < r \leq 1$*, then* $S_T(r) \subset B$ *(see Figure 5); hence* $\mathfrak{F}_T(B,r) = 1$.
2. *If* $1 < r \leq 3$*, we consider first the set* $B \cap \tilde{S}_T(r)$*, where*

$$\tilde{S}_T(r) = S_T(r) \cap \left\{(x,y,z) \in \mathbb{R}^3 : x + y + z = 1\right\}.$$

   $B \cap \tilde{S}_T(r)$ *is the intersection of* B *with only one lateral face of the tetrahedron; it is a triangle with vertices* e_1*,* e_2 *and* e_3*, see Figure 5.*
   To calculate $\mathscr{O}\left(\frac{1}{r}B \cap \tilde{S}_T\right)$ *we have to know the edge lengths of the triangle* $\frac{1}{r}B \cap \tilde{S}_T$ *to calculate its Euclidean surface area* $\mathscr{O}\left(\frac{1}{r}B \cap \tilde{S}_T\right)$*. For this, we will first calculate the edge lengths of the triangle* $B \cap \tilde{S}_T(r)$ *and use the definition of* CPC_T *to project the edge lengths onto the unit sphere. Given the Euclidean length of the line segments* $\overline{sw_1} = \overline{sw_2} = \overline{sw_3} = 2$ *as edge lengths of the unit cube,* $\overline{n_2 0} = l_T = 1/\sqrt{3}$ *as radius of the inscribed ball of the unit tetrahedron,* $\overline{n_1 0} = r \cdot l_T = r/\sqrt{3}$ *and* $\overline{s0} = 3l_T = \sqrt{3}$*, it follows by using the first intercept theorem that* $\frac{\overline{sn_1}}{\overline{sn_2}} = \frac{\overline{se_1}}{\overline{sw_1}} = \frac{\overline{se_2}}{\overline{sw_2}} = \frac{\overline{se_3}}{\overline{sw_3}}$*. This is equivalent to* $\frac{3l_t - rl_t}{3l_t - l_t} = \frac{\overline{se_1}}{2} = \frac{\overline{se_2}}{2} = \frac{\overline{se_3}}{2}$ *and* $\overline{se_1} = \overline{se_3} = \overline{se_3} = 3 - r$*. Because* $\overline{se_1}$*,* $\overline{se_2}$ *and* $\overline{se_3}$ *have equal lengths, it follows by Pythagoras' theorem that* $\overline{e_1 e_2}$*,* $\overline{e_2 e_3}$ *and* $\overline{e_1 e_3}$ *have equal lengths, too. Thus,* $B \cap \tilde{S}_T(r)$ *is an equilateral triangle with edge length*

$$a(r) = \sqrt{\overline{se_1}^2 + \overline{se_2}^2} = \sqrt{\overline{se_2}^2 + \overline{se_3}^2} = \sqrt{\overline{se_1}^2 + \overline{se_3}^2} = \sqrt{2(3-r)^2} = \sqrt{2}(3-r).$$

   By the definition of CPC_T *it is* $a(1) = \frac{1}{r}a(r)$ *and*

$$\mathscr{O}\left(\frac{1}{r}B \cap \tilde{S}_T\right) = \left(\frac{a(1)^2}{4}\sqrt{3}\right) = \frac{2(3-r)^2}{4r^2}\sqrt{3}.$$

   Because of the tetrahedron's symmetry, the intersections of every lateral face with B *have the same surface content and it follows that*

$$\mathscr{O}\left(\frac{1}{r}B \cap S_T\right) = 4 \cdot \mathscr{O}\left(\frac{1}{r}B \cap \tilde{S}_T\right) = 2\sqrt{3} \cdot \frac{(3-r)^2}{r^2},$$

   hence

$$\mathfrak{F}_T(B,r) = \frac{1}{8\sqrt{3}} \mathscr{O}\left(\frac{1}{r}B \cap S_T\right) = \frac{1}{4r^2}(3-r)^2.$$

3. *If* $3 \leq r$ *then* $B \cap S_T(r) = \emptyset$*, hence* $\mathfrak{F}_T(B,r) = 0$.

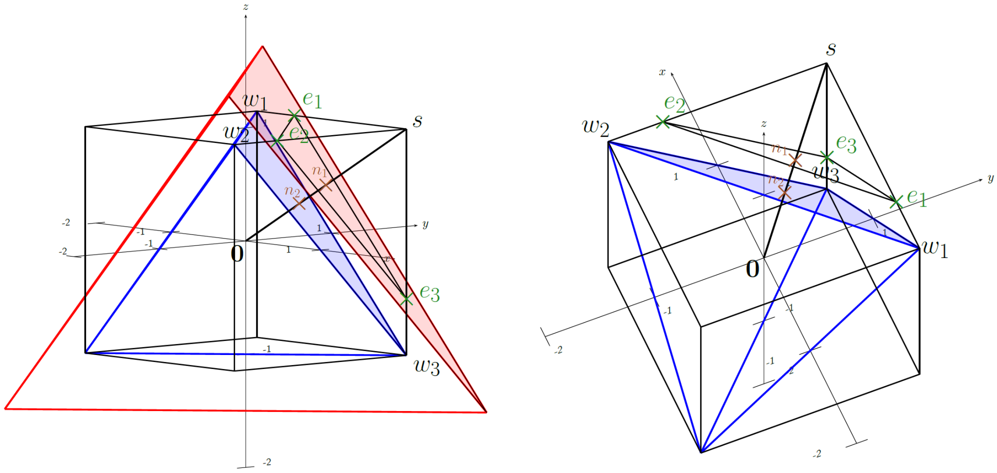

Figure 5: Intersection of unit cube and tetrahedron with generalized radius r

Summarizing the results, it follows

$$\mathfrak{F}_T(B,r) = \begin{cases} 1, & 0 < r \leq 1 \\ \frac{(3-r)^2}{4r^2}, & 1 < r \leq 3 \\ 0, & 3 < r \end{cases}.$$

Hence,

$$\lambda(B) = 8\int\limits_0^\infty r^2 \mathfrak{F}_T(B,r)dr = 8\left[\int\limits_0^1 r^2 dr + \frac{1}{4}\int\limits_1^3 (3-r)^2 dr\right] = \frac{8}{3} + 2\cdot\frac{8}{3} = 8.$$

Remark 3 *The geometric measure representation formula using the ipf of platonic bodies is not only useful for calculating the Lebesgue measure of Borel measurable sets $B \subset \mathbb{R}^3$. Its generalization, involving rather arbitrary density generating functions in the sense of [30] and [32] becomes important to generate new large classes of probability distributions including as well those with light as those with heavy distribution tails which occur in various fields of application. Doing this, however, is postponed to future work.*

5. The platonic ball numbers

Let us recall that the circumference and area content properties of Euclidean circles which motivate the definition of the circle number π and have been discussed to a certain extent in [35] were extended in [30] and [36] to the generalized surface and volume properties of generalized balls. In the present note, we follow this line. In this sense, equation (4) motivates the following definition of platonic ball numbers. Let still $Plat \in \{T, C, O, D, I\}$.

Definition 4 *The platonic ball number π_{Plat} is defined by*

$$\pi_{Plat} = V_{Plat}(1) = \mathfrak{O}_{Plat}(S_{Plat})/3.$$

The volumes of the cube C and octahedron O can be understood as $l_{n,p}$-ball numbers for "$p = \infty$" and $p = 1$, respectively. According to [36],

$$\pi_C = 8 \text{ and } \pi_O = 4/3.$$

As because suitable volume formulae are known for the other three platonic bodies we are now in a position to derive the corresponding three new platonic ball numbers. For the corresponding basic facts on platonic bodies, we refer again to [6], [41] and [4].

Theorem 5 *The platonic ball numbers of the tetrahedron, icosahedron and dodecahedron are*

$$\pi_T = 8/3,\ \pi_I = 10(3+\sqrt{5})/3 \text{ and } \pi_D = 50(3-\sqrt{5}),$$

respectively.

Proof The volume of a tetrahedron $T = T(a_T)$, given its Euclidean edge length a_T, can be expressed as follows:

$$\lambda(T(a_T)) = a_T^3\sqrt{2}/12.$$

The unit tetrahedron has got the edge length $a_T = \sqrt{8}$, see Figure 1. This follows by noting that T is a subset of the unit cube C which has edge length $a_C = 2$, and using Pythagoras' theorem. Thus, we can express π_T in the following way:

$$\pi_T = V_T(1) = \lambda(T(\sqrt{8})) = \sqrt{8}^3\sqrt{2}/12 = 8/3.$$

The volume $\lambda(I(a_I))$ of an icosahedron with edge length a_I can be expressed as

$$\lambda(I(a_I)) = \frac{5}{12}a_I^3(3+\sqrt{5}).$$

The corresponding edge length of the unit icosahedron with vertices as defined in the proof of Theorem 1 has the value $a_I = 2$. Now we can calculate π_I by

$$\pi_I = V_I(1) = \lambda(I(2)) = \frac{5}{12}2^3(3+\sqrt{5}) = 10(3+\sqrt{5})/3.$$

The volume $\lambda(D(a_D))$ of a dodecahedron with edge length a_D allows the representation

$$\lambda(D(a_D)) = \frac{a_D^3}{4}(15+7\sqrt{5}).$$

To calculate the actual value of a_D for the unit dodecahedron, we use the duality property being true for the dodecahedron and the icosahedron. The unit dodecahedron D represents the outer dual dodecahedron to the unit icosahedron I. It is therefore possible to get the inscribed ball radius r_D by calculating the Euclidean length of the line segment from the origin 0 to a vertex of the unit icosahedron, i.e. A_1. It follows that

$$r_D = ||\vec{OA}_1||_2 = ||(g,1,0)||_2 = \sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}.$$

The edge length a_D of D can be calculated based upon the known relation between r_D and a_D,

$$r_D = \frac{a_D}{2}\sqrt{\frac{25+11\sqrt{5}}{10}},$$

and it follows that

$$a_D = 2\cdot\frac{r_D\sqrt{10}}{\sqrt{25+11\sqrt{5}}} = 3\sqrt{5}-5.$$

Hence,

$$\pi_D = V_D(1) = \lambda(D(3\sqrt{5}-5)) = \frac{(3\sqrt{5}-5)^3}{4}(15+7\sqrt{5}) = 50(3-\sqrt{5}). \qquad \blacksquare$$

6. Thin layers property

In [36] as well as in [31] thin layers properties for the Lebesgue measure of $l_{n,p}$-balls and ellipsoids are introduced to give an asymptotic of the Lebesgue measure of layers that are becoming asymptotically thinner. For the Lebesgue measure of the platonic bodies we can derive a thin layers property, too.

Theorem 6 *Let $L(r,\epsilon) = \{x \in \mathbb{R}^3 : r \le h_{Plat}(x) \le r+\epsilon\}$. The Lebesgue measure has the platonic thin layers property*

$$\frac{\lambda(L(r,\epsilon))}{3\pi_{Plat}r^2\epsilon} \longrightarrow 1 \ as\ \epsilon \to +0.$$

Proof We start from the volume of a platonic body with generalized radius $r > 0$ which is equivalent to the Lebesgue measure of $K_{Plat}(r) = \{x \in \mathbb{R}^3 : h_{Plat}(x) \le r\}$, i.e.

$$\lambda(K_{Plat}(r)) = V_{Plat}(r) = \int_0^r \rho^2 \cdot \mathfrak{O}_{Plat}(S_{Plat})d\rho.$$

Since the platonic ball number is $\pi_{Plat} = \frac{1}{3}\mathfrak{O}_{Plat}(S_{Plat})$, it follows

$$\lambda(K_{Plat}(r)) = 3\pi_{Plat}\int_0^r \rho^2 d\rho.$$

We can now calculate the Lebesgue measure by

$$\lambda(L(r,\epsilon)) = 3\pi_{Plat}\int_r^{r+\epsilon}\rho^2 d\rho = 3\pi_{Plat}(\frac{1}{3}(r+\epsilon)^3 - \frac{1}{3}r^3) = 3\pi_{Plat}(r^2\epsilon + r\epsilon^2 + \frac{1}{3}\epsilon^3) \qquad \blacksquare$$

Appendix

lateral faces	plane equations	lateral faces	plane equations
$a_1a_3a_2$	$(2-g)(x+y+z)=1$	$c_1c_3b_3$	$(2-\sqrt{5})x-(g-1)y=1$
$d_2a_3b_1$	$(2-g)(-x+y+z)=1$	$c_3d_1b_3$	$-(2-\sqrt{5})x-(g-1)y=1$
$a_2b_3d_1$	$(2-g)(x-y+z)=1$	$d_2b_3a_2$	$(2-\sqrt{5})y+(g-1)z=1$
$a_1b_2d_3$	$(2-g)(x+y-z)=1$	$a_2a_3d_2$	$-(2-\sqrt{5})y+(g-1)z=1$
$d_2c_1b_3$	$(2-g)(-x-y+z)=1$	$c_3b_2c_2$	$(2-\sqrt{5})y-(g-1)z=1$
$c_3b_2d_1$	$(2-g)(x-y-z)=1$	$d_3c_2b_2$	$-(2-\sqrt{5})y-(g-1)z=1$
$d_3c_2b_1$	$(2-g)(-x+y-z)=1$	$d_1b_2a_1$	$(g-1)x+(2-\sqrt{5})z=1$
$c_1c_2c_3$	$(2-g)(-x-y-z)=1$	$b_1c_2c_1$	$-(g-1)x+(2-\sqrt{5})z=1$
$a_3d_3b_1$	$(2-\sqrt{5})x+(g-1)y=1$	$a_2d_1a_1$	$(g-1)x-(2-\sqrt{5})z=1$
$a_1d_3a_3$	$-(2-\sqrt{5})x+(g-1)y=1$	$d_2b_1c_1$	$-(g-1)x-(2-\sqrt{5})z=1$

Table 1: Plane equations corresponding to the unit icosahedron

normal vectors	plane equations	normal vectors	plane equations
$\vec{n}_{a_1}$	$\frac{1}{\sqrt{5}}x+\frac{2}{5+\sqrt{5}}y=1$	$\vec{n}_{c_2}$	$-\frac{1}{\sqrt{5}}z-\frac{2}{5+\sqrt{5}}x=1$
$\vec{n}_{b_1}$	$-\frac{1}{\sqrt{5}}x+\frac{2}{5+\sqrt{5}}y=1$	$\vec{n}_{d_2}$	$-\frac{1}{\sqrt{5}}z+\frac{2}{5+\sqrt{5}}x=1$
$\vec{n}_{c_1}$	$-\frac{1}{\sqrt{5}}x-\frac{2}{5+\sqrt{5}}y=1$	$\vec{n}_{a_3}$	$\frac{1}{\sqrt{5}}y+\frac{2}{5+\sqrt{5}}z=1$
$\vec{n}_{d_1}$	$\frac{1}{\sqrt{5}}x-\frac{2}{5+\sqrt{5}}y=1$	$\vec{n}_{b_3}$	$-\frac{1}{\sqrt{5}}y+\frac{2}{5+\sqrt{5}}z=1$
$\vec{n}_{a_2}$	$\frac{1}{\sqrt{5}}z+\frac{2}{5+\sqrt{5}}x=1$	$\vec{n}_{c_3}$	$-\frac{1}{\sqrt{5}}y-\frac{2}{5+\sqrt{5}}z=1$
$\vec{n}_{b_2}$	$\frac{1}{\sqrt{5}}z-\frac{2}{5+\sqrt{5}}x=1$	$\vec{n}_{d_3}$	$\frac{1}{\sqrt{5}}y-\frac{2}{5+\sqrt{5}}z=1$

Table 2: Plane equations corresponding to the unit dodecahedron

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