Norm contoured distributions in R^2

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Abstract

Geometric measure and stochastic representations are derived for distributions of random vectors in $\mathbb{R}^2$ which result to be symmetric, when suitably shifted, according to an arbitrary norm. Integral representations of norm-disc circle numbers are also introduced by studying the normalizing constants of given density generating functions, allowing to model heavy and light distribution tails. Cavalieri's and Torricelli's method of indivisibles is sharpened and extended. Various examples are presented with emphasis on regularly varying vectors. Independent coordinate representations are proved as well as a thin layers property.

Keywords: Uniform basis, generalized uniform distribution, geometric measure representation, non-Euclidean arc-length, stochastic representation, log-concave density, regularly varying vector, spectral measure, tail index
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1 Introduction

In probability theory and in mathematical statistic, different research areas call for considering a great variety of geometric objects and for reflecting their essential properties in suitable probabilistic models. Just to mention two specific directions of such studies, we refer to data depth analysis and meta density analysis. Data depth analysis finds its roots already in [81]. A well illustrated insight into the nowadays used methods is given, e.g., in [44], where the diversity of geometric objects of interest is pinpointed. Meta density analysis started with asymptotic distributional studies of multivariate extremes and their geometry, e.g. in [4].

In the present paper, we focus our attention on geometric objects such as balls or discs, generated by norms on $\mathbb{R}^2$, and study probability distributions that reflect norm contours on their density level sets. Another question faced in the mentioned papers, and many others, is whether distribution tails are heavy or light. Several examples of both types are surveyed, e.g., in [19] and [35].

Norms corresponding to ellipsoids centered at the origin are basic in dealing with the popular class of elliptically contoured distributions which have been extensively studied since 1938 in [75], an then [36], [11], [33], [2],[20] and [19]. Note that modeling and estimating by using elements of the class of logarithmic-concave densities is dealt with, e.g., in [82] and [1], respectively. We refer to [29],[41], [30] and [16] for regularly varying distributions, a specific class of heavy tailed distributions.

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The aim of this contribution is to introduce and to describe the family of norm contoured distributions in R^2 (which are naturally often dealing even with the more general finite dimensional case) on the one hand and on the other to give a survey of papers which may be considered the most influential in this field. Special emphasis will be on geometric and stochastic representations which have been discussed in the past by the author for several subclasses and specific cases. The geometric measure representations are essentially based upon the norm-circle intersection proportion function (ipf) of a given set which is defined in terms of certain non-Euclidean arc-lengths. The choice of the suitable non-Euclidean geometry will be made in accordance with Theorem 2.11 and the rotated gradient condition (2.34) in [62]. Because this condition is actually available only in dimension two, the discussion is restricted to this case throughout the paper. Note that the differential geometric approach in [62] is closely connected with the perspective discussed in [10].

Let $\|.\|$ denote any norm on $\mathbb{R}^2$. Imagine that the level sets of the norm are best adopted in some sense to a suitably centered cloud of points coming from a two-dimensional data set of a large sample size. The set $K = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ will be called the unit disc and its boundary $C = \partial K = \bar{K} \setminus K^o$ the unit circle with respect to $\|.\|$. Because any norm is positively homogeneous of degree one, for $r > 0$ we have $rK = \{rx : x \in K\} = \{x \in \mathbb{R}^2 : \|x\| \leq r\}$ where rx denotes componentwise scalar multiplication. That's why we call r the norm-radius of the disc $K(r) = rK$ and $C(r) = \partial K(r)$ the circle of the same radius.

The aim of this paper is to study probability laws whose geometric form of mass concentration is suitably described by any norm and which allows us modeling heavy and light distribution tails at the same time. We will derive both geometric measure representations of those distributions and stochastic representations of the **corresponding random** vectors. For a general introduction into the basic theoretical material of geometric measures, we refer to [21], [15], [43], [74] and [38]. Specific norm contoured distributions have been studied in [77], [24], [47], [79], [46], [73], [53], [48], [22], [25], [76] and [78]. For a new type of geometric representation of the heteroscedastic normal and of the homoscedastic $l_{n,p}$-symmetric distributions in the spirit of the present paper we refer to [63], [65] and [61], respectively. The specific case $p = 1$ was dealt with in [27] and consequences were drawn in [3] for the much more general case of skewed $l_{n,p}$-symmetric distributions.

The paper is organized as follows. Section 2 deals with densities having level sets that are circles w.r.t. any norm. Such density is defined choosing first a density generating function (dgf) and then determining a suitable normalizing constant. In Section 3, we derive geometric representations for the corresponding probability distributions which essentially rely on non-Euclidean or Minkowski arc-length measures. In order to keep the probabilistic thread of the paper, the differential geometric approach to arc-length measures is discussed in Appendix A. Moreover, a wide variety of applications of both Euclidean and non-Euclidean geometric measure representations are surveyed in Section 3. Section 3 presents also stochastic representations of random vectors **following** a norm contoured density. The general distribution class will be considered in Section 4. Section 5 deals with examples of dgf's but also defined in a different way. In Appendix C we discuss the normalizing constants, which are closely connected with Minkowski space circle numbers, i.e. with circle numbers of discs w.r.t. arbitrary norms. In Sections 6 and 7, an independent coordinate representation is given based upon non-Euclidean polar coordinates and certain thin layers properties of the Lebesgue measure are proved as well as for some new distribution laws. Appendix B is not only devoted to the basics of the generalized geometric indivisibles method, used throughout this note, but goes beyond it.

2 Continuous distributions

In the present section, we define a density function having the same level sets of a fixed norm and such that **the levels** can be thought as well adopted, in some sense, to the relative frequencies of those sampled points lying in suitable neighborhoods of the norm levels. To this end we introduce a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$0 < I(g) < \infty \quad \text{where} \quad I(g) = \int_0^\infty r g(r) \mathrm{d}r.$$

We call g a density generating function (dgf) and

$$\varphi_{g,\, \|.\|}(x) = C(g, \|.\|) g(\|x\|),\ x \in \mathbb{R}^2$$

a norm (level set) contoured density, centered at the origin, if $C(g, \|.\|) \in (0, \infty)$ is a suitable normalizing constant. Later, we discuss in details the consequences of this assumption, i.e., how the constant $C(g, \|.\|)$ can be computed for any dgf g and any norm $\|.\|$.

The corresponding probability distribution on the Borel σ-field $\mathcal{B}^2$ in $\mathbb{R}^2$ will be denoted by $\Phi_{g,\, \|.\|}$ and the class of all such centered continuous norm contoured distributions by $CCNC$, i.e.

$$CCNC = \{\Phi_{g,\, \|.\|} : g \text{ is a dgf and } \|.\| \text{ is a norm on } \mathbb{R}^2\}.$$

Remark 1. In the next section, we introduce a geometric disintegration formula for the measure $\Phi_{g,\, \|.\|}$ which makes use of an arc-length measure on the Borel σ-field of subsets of the norm-circle C. From [62], it is well known that, due to the co-area formula [17], the usual Euclidean arc-length measure is not suitable in general for this purpose. That's why we introduce a suitable non-Euclidean arc-length measure, see Appendix A. Note that according to [10], see also [23], the given norm-circle solves the isoperimetric problem w.r.t. this arc-length measure.

As final step in constructing a general norm contoured density, we allow orthogonal transformations and shifting of a random vector, say X, considered so far. To this end, let O denote any orthogonal 2×2-matrix and μ any vector of $\mathbb{R}^2$. Set $Y = OX + \mu$. The density of Y is then

$$\varphi\big(x; g, \|.\|, O, \mu\big) = \varphi_{g, \|.\|}\big(O^{-1}(x - \mu)\big). \tag{1}$$

Let us denote by CNC the class of all such continuous norm contoured distributions, that is $CNC = \{\Phi_{g, \|.\|, O, \mu}\}$ for short.

Note that **any** density from this class may be generated starting from different norms because any orthogonally transformed norm-disc is a norm-disc itself. Moreover, for Gaussian distributions, in [13] different consequences are discussed depending if O is any regular matrix or just orthogonal. The more general class $\mathcal{CNC}$ of centered norm contoured distributions and the class $\mathcal{NC}$ of just norm contoured distributions will be considered in Section 4.

3 Geometric and stochastic representations

In this section, we consider geometric and stochastic representations of continuous norm contoured distributions and random vectors having such distributions, respectively.

The classical principle of Cavalieri states that two regions R_1 and R_2, located between two parallel lines l_1, l_2 in the plane $\mathbb{R}^2$, have equal area content if the intersections ("indivisibles") $l \cap R_1$ and $l \cap R_2$ have the same lengths for arbitrary lines l parallel to l_1. Using in a similar way arc-lengths of intersections of the regions with concentric circles, instead of parallel lines, may be referred to as a Torricelli modification of Cavalieri's method.

By sharpening these principles, our intention is to prove that, under certain conditions, integration of those arc-lengths of indivisibles is equivalent to the determination of the area contents of the identified regions. Extensions of these principles include the use of non-Euclidean arc-lengths measures and the introduction of additional functions, assigning to every indivisible an own weight.

Taking into account both of these two aspects of geometric integration, the ipf of a Borel set has been proved in [55] and a subsequent series of papers of the author and several co-authors result**s** to be a very useful tool for analyzing a variety of problems from probability theory to mathematical statistics. The introduction of this function is also closely connected with generalizations of the circle number π. The ipf of the Gaussian distribution law has been faced in [55] whereas spherical, asymmetric $l_{n,1}^{+}$-, $l_{n,p}$-symmetric and elliptically contoured distributions are studied in [57],[27],[59], [61], and in [65], respectively.

According to [62], the norm-circle ipf of a set is defined as

$$\mathcal{F}_{\|.\|}(A, r) = \frac{AL_B([\frac{1}{r}A] \cap C)}{AL_B(C)}, r > 0$$

where the non-Euclidean arc-length measure AL_B is defined in Appendix A. The function $\omega_C : \mathcal{B}^2 \cap C \to [0, 1]$ defined by

$$\omega_C(A) = \frac{AL_B(A)}{AL_B(C)}$$

will be called the Minkowski space $(\mathbb{R}^2, \|.\|_B)$ uniform distribution on C.

The reader, interested in the genesis of the notion of ipf, might additionally have a look into [54] and [6] where, when reading afterwards, one can feel already the naturalness of introducing this notion in subsequent research.

Next, we mention some alternative possibilities to represent the Minkowski space uniform distributions. To this end, let

$$CPC(M) = \left\{ x \in \mathbb{R}^2 : \frac{x}{\|x\|} \in M \right\}, \ M \in \mathcal{B}^2 \cap C,$$

be the central projection cone generated by the set M, and

$$sector(M, \varrho) = CPC(M) \cap K(\varrho) = [0, \varrho] \cdot M$$

its intersection with the norm-disc of norm-radius ϱ. Moreover, let us recall that the unit norm-circle C uniquely defines a function $R_C : [0, 2\pi) \to \mathbb{R}^+$ by assuming

$$C = \{R_C(\varphi) \cdot (\cos\varphi, \sin\varphi), \varphi \in [0, 2\pi)\}.$$

Finally, let Pol_K denote the generalized polar coordinate transformation defined in [62], Pol_K^* its restriction to generalized radius 1, λ the Lebesgue measure, or Euclidean area content measure on $\mathcal{B}^2$, and $\pi(K)$ the generalized circle number defined in (8), see Appendix B.

Theorem 1. *For all $A \in \mathcal{B}^2 \cap C$, the Minkowski space $(\mathbb{R}^2, \|.\|_B)$ uniform distribution on C satisfies the following representations:*

(a) the angular integral representation

$$\omega_C(A) = \frac{1}{\pi(K)} \int\limits_{Pol_K^{*-1}(A)} R_C^2(\varphi)\mathrm{d}\varphi,$$

(b) the sector measure representation

$$\omega_C(A) = \frac{\lambda(sector(A,1))}{\lambda(K)},$$

(c) the centered norm contoured cone measure representation

$$\omega_C(A) = \Phi_{g,\|.\|}\left(CPC(A)\right), \quad \textit{for any dgf } g.$$

Proof. The first assertion is a consequence of the proof of Lemma 1 in Appendix B, because AL_B coincides with $\mathfrak{U}_C$ defined in **(6), Appendix A**. The second representation follows from the identities $AL_B(A) = 2\lambda(sector(A,1))$ and $AL_B(C) = 2\lambda(K)$. For the last representation, observe that,

$$\mathcal{F}_{\|.\|}\left(CPC(A), r\right) = \frac{AL_B\left([\frac{1}{r} \cdot CPC(A)] \cap C\right)}{AL_B(C)} = \frac{AL_B(A)}{AL_B(C)} = \omega_C(A), \quad \text{for all } r > 0.$$

Hence, for arbitrary dgf g,

$$\begin{aligned}\Phi_{g,\,\|.\|}\left(CPC(A)\right) &= 2\pi(K)\,C\left(g, \|.\|\right) \int\limits_0^\infty r\,g(r)\mathcal{F}_{\|.\|}\left(CPC(A), r\right)\,\mathrm{d}r \\ &= \omega_C(A)\,2\pi(K)\,C(g, \|.\|)\,I(g) = \omega_C(A).\end{aligned}$$

□

Theorem 2. *For arbitrary norm $\|.\|$ and arbitrary dgf g, centered continuous norm contoured distributions allow the non-Euclidean indivisibles representation, or geometric disintegration formula, that is*

$$\Phi_{g,\|.\|}(A) = 2\pi(K)\,C\left(g, \|.\|\right) \int\limits_0^\infty r\,g(r)\,\mathcal{F}_{\|.\|}(A, r)\mathrm{d}r.$$

Proof. By definition and by using generalized polar coordinate transformation, with r_1 and r_2 suitably chosen in $[0, \infty)$ we have

$$\Phi_{g,\,\|.\|}(A) = \int\limits_A C\left(g, \|.\|\right) g\left(\|x\|\right)\,\mathrm{d}x = C\left(g, \|.\|\right) \int\limits_{r_1}^{r_2} g(r) \int\limits_{Pol_{\|.\|}^{*-1}(A)} r \cdot R_C^2(\varphi)\mathrm{d}\,\varphi\mathrm{d}r.$$

Because the inner part of this iterated integral equals $\mathfrak{U}_C(A \cap C(r))$, it follows

$$\Phi_{g,\,\|.\|}(A) = 2\pi(K)\,C\left(g, \|.\|\right) \int\limits_{r_1}^{r_2} r\,g(r)\,\frac{\mathfrak{U}_C\left([\frac{1}{r}A] \cap C\right)}{\mathfrak{U}_C\left(C\right)}\,\mathrm{d}r.$$

Finally, formula **(7)** in Appendix A applies. □

Theorem 2 was proved for the (multivariate) Gaussian, elliptically contoured and $l_{n,p}$-symmetric distributions in [55], [57], [65], and [61], respectively. The asymmetric $l_{n,1}^{+}$-case has been dealt with in [27].

There is a broad variety of applications of geometric measure representations as introduced in this paper. Just to mention a few of them, we refer to

- the derivation of high precise large deviation asymptotic results in [55], [6], [49], [69] and in a series of subsequent papers;
- the geometric approach to finite sample and large deviation properties of `ANOVA` in [71];
- the evaluation of probabilities of the two-dimensional Gaussian law in [70];
- the construction of an exact modified Student test for two-parameter exponential distribution in [12];
- the generalizations of chi-square, Student and Fisher distributions and statistics in [58], [59], [61] and [65];
- the asymptotic expansions for large deviations in [8], [69] and [31];
- the approximation of percentiles of noncentral Chi-square, Student- and Fisher distributions in [31];
- the direct description of probabilities of correct classification in [39], and their description using the doubly non-central Fisher distribution in [40];
- the construction of an exact test in nonlinear regression in [58] and [32];
- the derivation of diverse exact distributions in [35] and [45];
- the geometric representations of skewed distributions in [26], [5] and [72];
- the construction of non-concentric elliptically contoured distributions in [67];
- the construction of generalized von Mises densities in [67] and [14];
- the construction of a class of Gaussian distributed random variables in [66].

The derivation of the ipf has been dealt with in several papers for (shifted) cones and balls, one- and two-sided cones, half spaces and their intersections, as well as other sets. General systems of sets with a known ipf are dealt with in [58] and subsequently used in various situations.

Let us recall that the class of centered continuous norm contoured distributions have been defined assuming that a suitably chosen normalizing constant $C(g, \|.\|)$ has been fixed for a given dgf g. We are now in a position to show how this constant generally looks like.

Remark 2. From Theorem 2 and $\Phi_{g,\|.\|}(\mathbb{R}^2) = 1$ we have

$$C(g, \|.\|) = \frac{1}{2\pi(K)I(g)}. \tag{2}$$

A possible statistical consequence of (2) might be that one could try to take into account it's validity when one estimates the parameters $||.||$ and g determining shape and tail behaviour of a density, respectively.

We shall discuss in Appendix C how the Minkowski space circle numbers $\pi(K)$ can be evaluated in the general case.

Let **us recall that** $\Phi_{g,\|.\|,O,\mu}(A)$, $A \in \mathcal{B}^2$ **denotes** the norm contoured probability law corresponding to the density (1).

Corollary 1. *Continuous norm contoured probability measures allow the representation*

$$\Phi_{g,\|.\|,O,\mu}(A) = 2\,\pi(K)\,C(g,\|.\|) \int_0^\infty r\,g(r)\,\mathcal{F}_{\|.\|}\left(O^{-1}(A-\mu),r\right)\,\mathrm{d}r.$$

The geometric measure representation in Theorem 2 has a counterpart in terms of random variables which will be stated and proved now. Note that $X \overset{d}{=} Y$ means that two random variables X and Y follow the same distribution law, while $X \sim \omega$ indicates that the random element X follows the probability distribution ω.

Theorem 3. *A random vector satisfying* $(\xi,\eta)^T \sim \Phi_{g,\|.\|}$*, for an arbitrary dgf* g *and any norm* $\|.\|$*, allows the stochastic representation*

$$(\xi,\eta)^T \overset{d}{=} R_g \cdot (X,Y)^T$$

where R_g *and* $(X,Y)^T = U_C$ *are independent,* $U_C \sim \omega_C$ *and* R_g *has the density*

$$f_{R_g}(t) = \frac{t\,g(t)}{I(g)} I_{(0,\infty)}(t).$$

Proof. With $R_g = \|(\xi,\eta)^T\| > 0, a.s.$, let $(X,Y)^T = \frac{1}{R_g}(\xi,\eta)^T$. For $A \in \mathcal{B}^2 \cap C$, we have

$$P\left((X,Y)^T \in A\right) = P\left((\xi,\eta)^T \in CPC(A)\right) = \Phi_{g,\|.\|}\left(CPC(A)\right).$$

Then, from the cone measure representation of ω_C, it follows $U_C \sim \omega_C$. The cumulative distribution function (cdf) of R_g at the point t, with $t > 0$, is

$$\begin{aligned} P(R_g < t) &= P\left(R_g(X,Y)^T \in t\,K\right) = \Phi_{g,\|.\|}(t\,K) \\ &= 2\,\pi(K)\,C(g,\|.\|) \int_0^\infty r\,g(r)\,\mathcal{F}_{\|.\|}(t\,K,r)\,\mathrm{d}r \\ &= 2\,\pi(K)\,C(g,\|.\|) \int_0^t r\,g(r)\,\mathrm{d}r, \end{aligned}$$

thus the density of R_g allows the representation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(R_g < t) = \frac{t\,g(t)}{I(g)}, \quad t > 0.$$

Finally, we consider

$$\begin{aligned} P(U_C \in A) \cdot P(R_g < t) &= \omega_C(A)\,\frac{1}{I(g)} \int_0^t r\,g(r)\,\mathrm{d}r = \frac{1}{I(g)} \int_0^t r\,g(r) \mathcal{F}_{\|.\|}\left(CPC(A),r\right)\,\mathrm{d}r \\ &= \frac{1}{I(g)} \int_0^\infty r\,g(r) \mathcal{F}_{\|.\|}\left(sector(A,t),r\right)\,\mathrm{d}r \\ &= 2\,\pi(K)\,C(g,\|.\|) \int_0^\infty r\,g(r) \mathcal{F}_{\|.\|}\left(sector(A,t),r\right)\,\mathrm{d}r \\ &= \Phi_{g,\|.\|}\left(sector(A,t)\right) = P(R_g < t, U_C \in A) \end{aligned}$$

proving that R_g and U_C are independent $\square$

For the special cases of spherical and elliptically contoured distributions, this result has been dealt with on the basis of Euclidean geometry, e.g., in [75],[36], [11], [33],[2],[20], [19] and on the basis of non-Euclidean geometry in [65]. For typical applications of stochastic representations of the given type see, e.g., [19], [52], [34] and [13].

Remark 3. The results of this and the previous section can be used for developing simulation algorithms. For example, if a sample is given of the basis vector $U_C \sim \omega_C$, then we can simulate any vector $(\xi, \eta)^T \sim \Phi_{g,\|.\|}$ just by simulating (or even exact evaluating) the cdf of R_g. Additionally, next Theorem 4 clarifies how to simulate the angular variable defined in the transformation Pol_K. For further information, we refer to [34] where the Box-Muller method and the Marsaglia-Bray rejecting polar method for the simulation of the Gaussian distribution are generalized to simulate the p-power exponential **(or** p-**generalized Gaussian)** distribution.

Let us emphasize finally the remarkable fact that the density function f_{R_g} does in no way depend on the underlying norm $\|.\|$. Moreover, the stochastic representation derived in the present section enables us to define a much more general class of norm contoured distributions which includes the class of continuous distributions as a subclass. This will be done in the next section.

4 The general distribution class

In the previous section, we have seen that all centered continuous norm contoured **distributed two-dimensional** random vectors may be considered as the product of an univariate non-negative random variable R_g and a Minkowski space $(\mathbb{R}^2, \|.\|_B)$ uniformly distributed random vector U_C which is independent of R_g. This observation can motivate the introduction of a more general class of norm contoured distributions. Doing this, we follow the line in [19], [57], [27] and [61]. We denote by $\mathcal{R}$ the set of all nonnegative random variables defined on the same probability space where R_g and U_C are defined. Let F be any cdf of a non-negative random variable and set

$$L_C(F) = \left\{ X : X \stackrel{d}{=} R \cdot U_C,\ R \in \mathcal{R} \,\text{with cdf}\, F,\ R \,\text{and}\, U_C \text{ stochastically independent} \right\}.$$

We call $L_C(F)$ the class of random vectors **having** a centered norm contoured distribution, taking their values in $\mathbb{R}^2$ and with generating variable R having the cdf F. The random vector U_C will be called the uniform basis of this class.

Let us recall that K is a convex body symmetric w.r.t. the origin. Let further $\mathcal{O}_2$ be the set of all orthogonal transformations $O : \mathbb{R}^2 \to \mathbb{R}^2$. Then, for any $O \in \mathcal{O}_2$, OK is a convex body symmetric w.r.t. the origin and its Minkowski functional [1] $\|.\|_O$ is a norm. The class $N(\|.\|)$ generated as

$$N(\|.\|) = \{\|.\|_O : O \in \mathcal{O}_2\},$$

is invariant w.r.t. all orthogonal transformations, and an analogous statement is true for the class of probability distributions

$$\Phi_g(\|.\|) = \left\{\Phi_{g,\|.\|_O} : O \in \mathcal{O}_2\right\}.$$

[1]For an introduction and a discussion on the employment of Minkowski functionals in stochastic field, we refer to [37] and [68], respectively.

Here, we define

$$O_1\Phi_{g,\|.\|_{O_2}} = \Phi_{g,\|.\|_{O_1 O_2}},\ \mathbf{O_i} \in \mathcal{O}_\mathbf{2}, \mathbf{i} \in \{\mathbf{1},\mathbf{2}\}.$$

For every dgf g, we call $\Phi_g(\|.\|)$ a class of centered norm contoured distributions and denote the collection of all such centered norm contoured distributions $\mathcal{CNC}$. Finally, we say that a random vector belongs to the class $\mathcal{NC}$ of norm contoured distributions if it is norm contoured distributed after suitably centering it. In case of existence, a density in this general class may be represented as in (1).

5 Examples

Let us recall that any nonnegative and integrable function on R^+ may serve as a dgf. Thus, in general it is possible to model both heavy and light distribution tails. In this section, we present some examples of dgf's which appeal attention in the literature, and refer additionally to lists of some more examples.

Example 1. Let the dgf be $g(r) = I_{(0,1)}(r)$ then, for an arbitrary norm $\|.\|$ on $\mathbb{R}^2$, the density of the norm-radius variable is $f_R(r) = 2\,r\,I_{(0,1)}(r)$. Let (ξ,η) be a random vector being centered norm contoured distributed w.r.t. this norm (and having dgf g). Then (ξ,η) follows the uniform distribution ω_K on the convex body $K = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. Let us emphasize that R is not uniformly distributed on the interval $(0,1)$ but has a triangular density. Finally, notice that $I(g) = 1/2$. According to the geometric definition of the uniform distribution on K, we have

$$\omega_K(A) = \frac{\lambda(A)}{\lambda(K)},\ A \in K \cap \mathcal{B}^2.$$

For the evaluation of the volume of any convex body, which is needed here, we refer on the one hand to [50], where the author makes extensively use of methods of the local theory of Banach spaces for proving inequalities on the volumes of convex bodies in $\mathbb{R}^n$. Thereby, in particular estimates of generalized Gaussian measures are used to evaluate certain volumes. On the other hand, we refer to Lemma 1 and Appendix C which present methods for exactly evaluating $\lambda(K)$ in the general case.

Example 2. Let the dgf be $g(r) = e^{-r^p/p}\,I_{(0,\infty)}(r)$, with $p > 0$. Then, for an arbitrary norm $\|.\|$ on $\mathbb{R}^2$ the density of the norm-radius variable is

$$f_R(r) = \frac{I_{(0,\infty)}(r)}{p^{2/p-1}\Gamma(p/2)}\,r\,e^{-r^p/p}.$$

This means that R follows the χ^p- or p-generalized χ^2-distribution, with two degrees of freedom (df) defined in [59]. The centered norm contoured random vector (ξ,η) follows the density function $\varphi_{g,\,\|.\|}(x) = C(g,\|.\|)\,e^{-\|x\|^p/p}, x \in \mathbb{R}^2, p > 0$. This density is the p-power exponential law if $\|.\| = |.|_p$ and $p \geq 1$ where $|(x,y)^T|_p = (|x|^p + |y|^p)^{1/p}$. The best known special cases of the power exponential distribution are the Laplace distribution, for p=1, and the Gaussian distribution, for $p = 2$. Finally, notice that $I(g) = p^{\frac{2}{p}-1}\Gamma(\frac{2}{p}-1)$.

A list of numerous authors who dealt with this distribution class can be found in the Introduction and in [61].

Example 3. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex and nondecreasing function such that $g(r) = I_{(0,\infty)}(r) \exp\{-h(r)\}$ is a dgf. For every norm $\|.\|$, the density $\varphi_{g,\|.\|}(x) = C(g, \|.\|)g(\|x\|)$ is log-concave (and, vice versa, every log-concave density allows the representation $C \exp\{-H(x)\}$ where $H : \mathbb{R}^2 \to \mathbb{R}$ is convex). We recall that according to [51], a random vector X, with values in $\mathbb{R}^2$, is called log-concave if for all compact nonempty subsets A, B of $\mathbb{R}^2$ and $\theta \in [0, 1]$,

$$P\left(X \in \theta A \oplus (1-\theta)B\right) \geq P(X \in A)^{\theta} P(X \in B)^{1-\theta}.$$

Due to a result in [7], a continuous vector X is log-concave iff for all $x, y \in \mathbb{R}^2$ and $\theta \in [0, 1]$, its density f satisfies

$$\ln f\left(\theta x + (1-\theta)y\right) \geq \theta \ln f(x) + (1-\theta) \ln f(y).$$

It is worthy to be mentioned that such distributions have found considerable attention both in probability and statistics. For only to mention one paper in each direction, we refer to [1] and [82]. Finally notice that here $I(g)$ strongly depends on the convex function h.

For a collection of several further classes of dgf's we refer to [19] and [35]. Let us **just mention** two types of univariate densities having considerable heavy tails: the p-generalized Cauchy density

$$f_p(t) = \frac{1}{\pi(p)(1+|t|^p)^{\frac{2}{p}}},\ t \in R,\ p > 0$$

and the p-generalized Student density with n df

$$f_{n,p}(x) = \frac{p\Gamma\left(\frac{n+1}{p}\right)}{2\, n^{\frac{1}{p}}\, \Gamma\left(\frac{n}{p}\right)\, \Gamma\left(\frac{1}{p}\right)} \left(1 + \frac{|x|^p}{n}\right)^{-\frac{n+1}{p}},\ x \in R,\ p > 0.$$

In [78], the density f_p has been proved to be the ratio distribution for the components of $l_{2,p}$-symmetrically distributed vectors. A geometric measure theoretical reproof of this result was given in [35] for the special cases $p \in \{1, 2\}$ and for the general case of any $p > 0$ in [45]. The p-generalized circle number $\pi(p)$ is defined in [60]. The function $f_{n,p}$ is log-convex on $(0, \infty)$. It was introduced in [59], as the density of a modified Student statistic, and generalized to the multidimensional case in [3].

The following example deals with a specific class of norm contoured distributions having heavy tails. The notion of regular variation of a random vector (or its distribution) is widely used in extreme value theory and has various applications in insurance, finance and many other fields, see the papers [29],[41],[30],[16] upon which the introductory part of this example is based. The tail index of X reflects to a certain extent the radial decay of probability mass, and the spectral measure of X w.r.t. a given norm indicates in which directions proportions of extreme realizations of X are. At hand of a specific example, we demonstrate how the spectral measure may change if the norm in *a)* is replaced by another one in *b)*.

Example 4. Let the bivariate Cauchy density be given by

$$f_{C,2}(x) = \frac{1}{2\,\pi(1+|x|_2^2)^{3/2}},\ x \in \mathbb{R}^2$$

where $|x|_2 = (x_1^2 + x_2^2)^{1/2}$ denotes the Euclidean norm. According to Theorem 2 (see also [57] and [35]), the corresponding probability law satisfies the geometric measure representation

$$\Phi_C(A) = \int_0^\infty \mathcal{F}_{|.|_2}(A, r)\, \frac{r\,\mathrm{d}r}{(1+r^2)^{3/2}},\ A \in \mathcal{B}^2. \tag{3}$$

A random vector X following Φ_C allows the stochastic representation $X \stackrel{d}{=} R \cdot U$, where the nonnegative random variable R and the random vector U, uniformly distributed on the Euclidean unit circle C_2, are independent. For any Borel subset M of C_2, set

$$\mu_u(M; \|.\|) = \frac{P\left(\|X\| > ux, X/\|X\| \in M\right)}{P(\|X\| > u)},\ u > 0,\ x > 0.$$

A random vector X is often called regularly varying w.r.t. the norm $\|.\|$ if there exist a positive constant α and a probability law σ, on the Borel σ-field of subsets of the $\|.\|$-unit sphere, such that $x^\alpha \mu_u(.; \|.\|) \Rightarrow \sigma$ as $u \to \infty$, where $\Rightarrow$ means weak convergence. Moreover, if X is regularly varying with tail index $\alpha > 0$ w.r.t. a norm $\|.\|$ then it is also regularly varying with the same tail index w.r.t. any other norm, and vice versa.

a) From (3) and the stochastic representation of X, it can immediately be derived that X is regularly varying w.r.t. the Euclidean norm. To this end, let $A(u; \|.\|) = \{y \in R^2 : \|y\| \geq u\}$. Then,

$$\mathcal{F}_{|.|_2}\left(A(u; |.|_2)\right) = I_{[u,\infty)}(r) = \begin{cases} 0 & \text{if } 0 \leq r < u, \\ 1 & \text{if } r \geq u, \end{cases}$$

and $P(|X|_2 > u) = \Phi_C\left(A(u; |.|_2)\right) = \int\limits_u^\infty \frac{r\,\mathrm{d}r}{(1+r^2)^{3/2}} = \frac{1}{\sqrt{1+u^2}}$. Moreover,

$$P\left(X \in A(ux, |.|_2), \frac{X}{|X|_2} \in M\right) = P\left(X \in A(ux, |.|_2)\right) P(U \in M),$$

thus

$$\mu_u(M; |.|_2) = \frac{P(U \in M)\sqrt{1 + (1/u)^2}}{x\sqrt{1 + (1/(ux))^2}} \text{ and } x\mu_u(.; |.|_2) \Rightarrow P(U \in .), \text{ for } u \to \infty,$$

where $M \to P(U \in M)$ is called the spectral measure of X w.r.t. the norm $|.|_2$, and the tail index of X equals 1.

b) For studying $\mu_u(M; |.|_\infty)$ where $|(x_1, x_2)^T|_\infty = \max\{|x_1|, |x_2|\}$ and M is any Borel subset of $C_\infty = \{y \in R^2 : |y|_\infty = 1\}$, we note that

$$\mathcal{F}_{|.|_2}\left(A(u; |.|_\infty), r\right) = \frac{4}{\pi} I_{[u,\sqrt{2}u)}(r) \arccos\left(\frac{u}{r}\right) + I_{[\sqrt{2}u,\infty)}(r). \tag{4}$$

Integrating by parts the integral (3) obtained by replacing $\mathcal{F}_{|.|_2}(A, r)$ with (4), we obtain

$$P(|X|_\infty > u) = \frac{4u}{\pi} \int_u^{\sqrt{2}u} \frac{\mathrm{d}r}{r^2\sqrt{(1 - (u/r)^2)(1 + r^2)}} \sim \frac{4}{\pi u} \int_1^{\sqrt{2}} \frac{\mathrm{d}t}{t^2\sqrt{t^2 - 1}} \tag{5}$$

where $\sim$ means asymptotic equivalence.

b1) Let us consider a first particular class of subsets of C_∞,

$$M_1(\epsilon) = \left\{ \begin{pmatrix} \cos_\infty(\phi) \\ \sin_\infty(\phi) \end{pmatrix} \in C_\infty : \frac{\pi}{4} - \epsilon \le \phi < \frac{\pi}{4} + \epsilon \right\}$$

where the generalized trigonometric functions $\cos_\infty$ and $\sin_\infty$ are defined in [59]. The set $M_1(\epsilon)$ may be written

$$M_1(\epsilon) = \left\{ r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \in C_\infty : \frac{\pi}{4} - \epsilon \le \phi < \frac{\pi}{4} + \epsilon,\ r \text{ suitably chosen from } [r_0, r_1] \right\}$$

where $\epsilon \in (0, \frac{\pi}{8})$, $r_0 = (1 + \tan^2(\frac{\pi}{4} - \epsilon))^{1/2}$ and $r_1 = \sqrt{2}$. Then,

$$P\left(X \in A(ux; |.|_\infty), \frac{X}{|X|_\infty} \in M_1(\epsilon) \right) = \Phi_C(A)$$

where

$$\frac{\pi}{4\epsilon} \cdot \mathcal{F}_{|.|_2}(A, r) = \frac{r - r_0}{r_1 - r_0} I_{[uxr_0, uxr_1]}(r) + I_{(uxr_1, \infty)}(r).$$

Consequently,

$$\Phi_C(A) = \frac{4\epsilon}{\pi} \frac{\ln \frac{\sqrt{2} + \sqrt{2 + \frac{1}{2u^2x^2}}}{\sqrt{1 + \tan^2(\frac{\pi}{4} - \epsilon)} + \sqrt{1 + \tan^2(\frac{\pi}{4} - \epsilon) + \frac{1}{u^2x^2}}}}{9ux[\sqrt{2} - \sqrt{1 + \tan^2(\frac{\pi}{4} - \epsilon)}]} \sim \frac{const_1(\epsilon)}{ux}$$

as $u \to \infty$, where

$$const_1(\epsilon) = \frac{4\epsilon \ln \sqrt{2/\left(1 + \tan^2(\frac{\pi}{4} - \epsilon)\right)}}{\pi \left[\sqrt{2} - \sqrt{1 + \tan^2(\frac{\pi}{4} - \epsilon)} \right]}.$$

Finally,

$$\mu_u\left(M_1(\epsilon); |.|_\infty\right) = \frac{\Phi_C(A)}{P(|X|_\infty > u)} \sim \frac{\frac{\pi}{4x} const_1(\epsilon)}{\int\limits_1^{\sqrt{2}} \frac{dt}{t^2\sqrt{t^2 - 1}}},\ \text{as}\ u \to \infty$$

and the value of the spectral measure at the set $M_1(\epsilon)$ equals

$$\pi \cdot const_1(\epsilon) / \left(4 \int\limits_1^{\sqrt{2}} \frac{dt}{t^2\sqrt{t^2 - 1}} \right)$$

b2) Let us consider now another particular type of subsets of C_∞, that is

$$M_2(\epsilon) = \left\{ \begin{pmatrix} \cos_\infty(\phi) \\ \sin_\infty(\phi) \end{pmatrix} \in C_\infty : -\epsilon \le \phi < \epsilon \right\}$$

where $\epsilon \in (0, \frac{\pi}{8})$. Then,

$$P\left(X \in A(ux; |.|_\infty), \frac{X}{|X|_\infty} \in M_2(\epsilon) \right) = \Phi_C(A)$$

where

$$\frac{\pi}{4} \cdot \mathcal{F}_{|.|_2}(A, r) = \arccos\left(\frac{r_0}{r}\right) I_{[r_0, r_1)}(r) + I_{(r_1, \infty)}(r) \cdot \epsilon.$$

Consequently, $\Phi_C(A) \sim const_2(\epsilon)/(ux)$ as $u \to \infty$ where

$$const_2(\epsilon) = \frac{4}{\pi}\left[\frac{\epsilon - \arccos\frac{1}{\sqrt{1+\tan^2\epsilon}}}{\sqrt{1+\tan^2\epsilon}} + \int\limits_1^{\sqrt{1+\tan^2\epsilon}} \frac{\mathrm{d}t}{t^2\sqrt{(t^2-1)^{1/2}}}\right].$$

Finally,

$$\mu_u\left(M_2(\epsilon); |.|_\infty\right) = \frac{\Phi_C(A)}{P(|X|_\infty > u)} \sim \frac{\pi}{4x}\frac{const_2(\epsilon)}{\int\limits_1^{\sqrt{2}} \frac{\mathrm{d}t}{t^2\sqrt{t^2-1}}}, \text{ as } u \to \infty.$$

Thus, the value of the spectral measure at the set $M_2(\epsilon)$ equals

$$\pi \cdot const_2(\epsilon) / \left(4 \int\limits_1^{\sqrt{2}} \frac{\mathrm{d}t}{t^2\sqrt{t^2-1}}\right).$$

Extending the previous example, in the following one we indicate how the tail index may change if the dgf is replaced by another one.

Example 5. Let the bivariate spherical Pearson type VII density be given by

$$f_{PT7;M,m}(x) = \frac{M-1}{m\pi}\left(1 + \frac{|x|_2^2}{m}\right)^{-M}, \; x \in \mathbb{R}^2, M > 1, m > 0.$$

If $M = 1 + m/2$, we recover the Student density and the Cauchy density if additionally $m = 1$. While the spectral measure w.r.t $|.|_2$ is here the same as in Example 4, the tail index is now $2M - 2$.

Example 6. Let a random vector X follow the $l_{2,p}$-symmetric Pearson type VII density

$$f^{(2,p)}_{PT7;M,m}(x) = D(M, m, p) g_{M,m,p}(|x|_p), \; x \in \mathbb{R}^2, p > 0, M > \frac{2}{p}, m > 0$$

where

$$D(M, m, p) = \frac{(p/\Gamma(\frac{1}{p}))^2 \Gamma(M)}{4m^{2/p}\Gamma(M - \frac{2}{p})} \text{ and } g_{M,m,p}(r) = (1 + r^p/m)^{-M}, \; r \geq 0.$$

Define

$$\mu_{u,p}(M; \|.\|) = \frac{P\left(||X|| > ux, \frac{X}{\|X\|} \in M\right)}{P(\|X\| > u)}.$$

Then

$$x^{Mp-2}\mu_{u,p}(.; |.|_p) \Rightarrow P(U_p \in M), \text{ as } u \to \infty$$

where the vector U_p follows the $l_{2,p}$-generalized uniform probability distribution ω_p on the $l_{2,p}$-unit circle as defined in [60]. Thus, here the tail index is $Mp - 2$, and the spectral measure is $\omega_p(.)$.

6 Independent coordinate representations

From the theory of $l_{n,p}$-symmetrically distributed random vectors, it is well known that their Euclidean coordinates are not independent unless their joint dgf is chosen in a very specific way. The latter is the case for vectors with a p-power exponential distribution.

Independence of different types of common and generalized polar coordinates of two-dimensional Gaussian vectors have been discussed, e.g., in [13]. Here, we shall make use of the Minkowski space polar coordinate transformation $Pol_K : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$ as defined in [62]. In detail,

$$x = \frac{\cos\varphi}{M_K(\varphi)},\ y = \frac{\sin\varphi}{M_K(\varphi)}$$

where $M_K(\varphi) = h_K(\cos\varphi, \sin\varphi)$, and $J(r, \varphi) = rR_C^2(\varphi)$ is the Jacobian of this transformation.

In the following theorem we make use of the (p, g)-generalized Chi-distribution $\chi_g^p(n)$ with n df, having the density

$$f_{n,g,p}^{\chi}(y) = \frac{y^{\frac{n}{p}} g(y^p)}{pI_{n,g,p}},\ y > 0.$$

For more details concerning the derivation of $f_{n,g,p}^{\chi}$ in the case $p = 2$ and in the general case, we refer to [57] and [61], respectively.

Theorem 4. *If $(\xi, \eta)^T \sim \Phi_{g,\|\cdot\|}$ and $(R, \Psi)^T = Pol_K^{-1}(\xi, \eta)$ then*

i) the random variables R and Ψ are independent;

ii) Ψ has the density $f(\varphi) = \dfrac{R_C^2(\varphi)}{2\pi(K)}, \varphi \in [0, 2\pi)$, independently of the dgf g,

iii) R follows the $(1, g)$-generalized Chi-distribution with 2 df, that is $R \sim \chi_g^1(2)$.

Proof. The density of the random vector $(\xi, \eta)^T$ is $f_{(\xi,\eta)}(x, y) = C(g, \|.\|)g(\|(x, y)^T\|)$, $(x, y)^T \in \mathbb{R}^2$. By the density transformation formula, the density of (R, Ψ) is $f_{(R,\Psi)}(r, \varphi) = C(g, \|.\|)rg(r)R_C^2(\varphi)$, $(r, \varphi)^T \in [0, \infty) \times [0, 2\pi)$ where $C(g, \|.\|)$ is known, from Remark 2. $\square$

7 Thin layers property

Some of the most immediate applications of the geometric measure representations, given in Section 3 and Appendix B, as well as the circle numbers defined in Appendix C, deal with measuring thin layers like

$$L(r, \varepsilon) = \left\{x \in \mathbb{R}^2 : r \leq \|x\| \leq r + \varepsilon\right\},\ \varepsilon > 0.$$

Theorem 5. *a) The Lebesgue measure of a thin layer behaves asymptotically as the thickness of the layer and tends to zero as follows $\lambda(L(r, \varepsilon)) \sim 2\pi(K)r\varepsilon$, $\varepsilon \to 0^+$.*

b) For continuous dgf g and in the case of thin layers, the norm contoured distributions behave asymptotically as $\Phi_{g,\|\cdot\|}(L(r, \varepsilon)) \sim \varepsilon\,(rg(r))\,/I_g$, $\varepsilon \to 0^+$.

Proof. We start from the representation of the Lebesgue measure given in Corollary 2:

$$\lambda(L(r,\varepsilon)) = 2\pi(K)\int\limits_{r_1}^{r_2} \varrho\mathcal{F}_{\|.\|}\left(L(r,\varepsilon),\varrho\right)\mathrm{d}\varrho = 2\pi(K)\int\limits_{r}^{r+\varepsilon}\varrho\,\mathrm{d}\varrho = \pi(K)\left[(r+\varepsilon)^2 - r^2\right].$$

This proves part *a)*. For proving part *b)*, Theorems **2 and 1** apply:

$$\Phi_{g,\|.\|}\left(L(r,\varepsilon)\right) = \frac{\int\limits_{r_1}^{r_2}\varrho\, g(\varrho)\,\mathcal{F}_{\|.\|}\left(L(r,\varepsilon),\varrho\right)\mathrm{d}\varrho}{I_g}.$$

Due to the properties of the set $L(r,\varepsilon)$, we get $\Phi_{g,\|.\|}\left(L(r,\varepsilon)\right) = I_g^{-1}\int\limits_{r}^{r+\varepsilon}\varrho\, g(\varrho)\,\mathrm{d}\varrho$ □

Appendix A: Non-Euclidean arc-length measures

Let $\|.\|^*$ denote the norm being dual to the norm $\|.\|$ and $K^* = \{x \in \mathbb{R}^2 : \|x\|^* \leq 1\}$ the corresponding unit disc. Note that, for simplicity, we do not distinguish here between the vector space $E = \mathbb{R}^2$, where the norm $\|.\|$ is defined, and the space $F = \mathbb{R}^2$ of linear functionals on E, where $\|.\|^*$ is defined. The set $B = O(90^o)K^*$, with

$$O(90^o) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is the image of K^* under a map which forces anticlockwise rotation through 90^o. The Minkowski functional of this set is the norm $\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$, $x \in \mathbb{R}^2$. It was proven in [10], see also [23], that C solves the isoperimetric problem in $(\mathbb{R}^2, \|.\|_B)$, what means that K maximizes the area content among all sets circumscribed by simple closed curves of the same $\|.\|_B$-arc-length as C. In what follows, we consider curves being subsets of the norm-circles $C(r), r > 0$ as being subsets of the Minkowski space $(\mathbb{R}^2, \|.\|_B)$ and will measure their arc-lengths with respect to the metric geometry of this space. General metric geometries are studied, e.g., in [9]. Encouraging early remarks on the usage of suitable non-Euclidean geometries can be found in [28]. To start with, let M be an element of $\mathcal{B}^2 \cap C$, $\varrho > 0$, and let $Z_n = \{z_0, z_1, ..., z_n\}$ denote a successive positive oriented partition of ϱM. The Minkowski space arc-length of ϱM is then well defined by

$$AL_B(\varrho M) = \sup\left\{\sum_{j=1}^{n}\|z_j - z_{j-1}\|_B : Z_n \text{ is any partition of } \varrho M\right\}$$

and may be denoted in similarity to the standard differential-geometric approach as

$$AL_B(\varrho M) = \int\limits_{\Phi}\|(x'(\varphi), y'(\varphi))\|_B\,\mathrm{d}\varphi,$$

where Φ is the suitably defined angular range of integration. Alternatively, we may write

$$AL_B(\varrho M) = \left(\int\limits_{X^+} + \int\limits_{X^-}\right)\|(1, y'(x))\|_B\,\mathrm{d}x$$

with ranges X^+ and X^- reasonably chosen from the x-axis in order to describe those parts of ϱM corresponding to positive and negative values of y, respectively. The reader may check the correctness of such definitions by standard techniques. One may call this approach to defining an arc-length measure the differential geometric or a global one.

Going back to [59], the following approach to defining an arc-length measure may be alternatively called a local one. The function $f(M, \varrho) = \lambda(sector(M, \varrho))$ may be called a sector-measure. In this setting,

$$\mathfrak{U}_C(\varrho M) = \frac{\partial}{\partial \varrho} f(M, \varrho) \tag{6}$$

could be considered as another arc-length measure compared with the one previously considered. However, as a consequence of the considerations given in [62], this locally defined arc-length measure and the globally one are the same, i.e.

$$\mathfrak{U}_C(\varrho M) = AL_B(\varrho M),\ M \in \mathcal{B}^2 \cap C,\ \varrho > 0. \tag{7}$$

This measure is fundamental for deriving the geometric disintegration formulas in Section 3. Note that different strategies of measuring arc-lengths and generalizing the circle number π in Minkowski spaces are discussed in [80]. An alternative to the one chosen here could be to measure the circumference of a norm-circle w.r.t. the geometry generated by the corresponding norm-disc itself instead of the rotated dual one. The strategy followed here takes into account basic consequences of the co-area formula of measure theory and may be considered as a non-Euclidean method of indivisibles generalizing that of Cavalieri which was modified by Torricelli. For more details on this and the co-area formula, we refer to [61],[62],[64] and [17], respectively.

Still another possibility to assign an arc-length measure to the boundary of a convex body $M \subset \mathbb{R}^2$ is to define its K-based Minkowski length in the spirit of [42] by

$$ML_K(M) = \lim_{\epsilon \to +0} \frac{\lambda(M \oplus \epsilon K) - \lambda(M)}{\epsilon}.$$

Here, $A \oplus B$ denotes the Minkowski addition of two subsets $A, B \subset \mathbb{R}^2$, and K is a centrally symmetric convex body. Then, from $K \oplus \epsilon K = (1+\epsilon)K$ we have

$$ML_K(K) = \frac{\mathrm{d}}{\mathrm{d}\varrho} \lambda\left(K(\varrho)\right)\Big|_{\varrho=1} = 2\lambda(K) = \mathfrak{U}_C(C).$$

Appendix B: A non-Euclidean sharpening and extension of Cavalieri's and Torricelli's method of indivisibles

Cavalieri's and Torricelli's method deals with conditions that are assumed to be satisfied by the arc-lengths of what they call the indivisibles of two regions, to ensure equality of the two regions area contents. Their method, however, does not describe the area contents of the regions of interest in terms of integrals of suitably defined arc-lengths. Thus, our approach extends the classical principle just w.r.t. this aspect of sharpening, i.e. by proving a corresponding integral representation. Proofs in cases when the indivisibles are parts of Euclidean norm circles can be found in [55] and [57]. Proving the mentioned equality of integrals in rather general cases, we make use of non-Euclidean arc-lengths measures thus also generalizing the classical principle w.r.t. this circumstance. As already mentioned, a geometric disintegration formula may be alternatively considered as a generalization of the method of indivisibles of Cavalieri combined with the Fubini's method of interchanging order of integration. Both these methods, however, do originally and intrinsically not have any connection with measuring arc-lengths in a non-Euclidean way.

Although in this paper, we are mainly interested in probability distributions, we restrict our considerations **here** to an analysis of the area content measure, which is the dominating measure of all elements from the distribution class CNC. After showing a disintegration property of the Lebesgue measure, one may turn to the continuous norm contoured distributions, see Section 3. To start with, we consider a bounded Borel set, $A \in [K(r_2) \setminus K(r_1)] \cap \mathcal{B}^2, 0 \le r_1 < r_2 < \infty$.

Lemma 1. *The Lebesgue measure satisfies the disintegration formulas*

$$\lambda(A) = \int_{r_1}^{r_2} AL_B\left(A \cap C(r)\right) \mathrm{d}r \text{ and } \lambda(A) = \int_{r_1}^{r_2} r \left(\int_{Pol_K^{*-1}\left([\frac{1}{r}A] \cap C\right)} R_C^2(\varphi) \mathrm{d}\varphi \right) \mathrm{d}r.$$

Proof. Let $M \in \mathcal{B}^2 \cap C$ and $r_1 \le \varrho_1 < \varrho_2 \le r_2$. The collection σ^* of all sets

$$A = A(M, \varrho_1, \varrho_2) = sector(M, \varrho_2) \setminus sector(M, \varrho_1)$$

is a semi algebra in $\mathbb{R}^2$. The smallest algebra generated by σ^* will be denoted $\mathcal{A}$. The finitely additive set function on $\mathcal{A}$, $\lambda^*(A) = \int\limits_{\varrho_1}^{\varrho_2} \mathfrak{U}_C\left(A \cap C(r)\right) \mathrm{d}r$, allows because of $A(M, \varrho_1, \varrho_2) \cap C(r) = r \cdot M$ the representation $\lambda^*(A) = \int\limits_{\varrho_1}^{\varrho_2} \mathfrak{U}_C(rM)\, \mathrm{d}r$. Changing the Cartesian with the generalized polar coordinates, the Lebesgue measure of A can be written as

$$\lambda(A) = \int_{\varrho_1}^{\varrho_2} \left(\int_{Pol_K^{*-1}(M)} r\, R_C^2(\varphi)\, \mathrm{d}\varphi \right) \mathrm{d}r.$$

From

$$\frac{\mathrm{d}}{\mathrm{d}r}\lambda\left(sector(M, r)\right) = r \int_{Pol_K^{*-1}(M)} R_C^2(\varphi)\, \mathrm{d}\varphi = \mathfrak{U}_C(rM)$$

we have $\lambda(A) = \int\limits_{\varrho_1}^{\varrho_2} \mathfrak{U}_C(rM) \mathrm{d}r$. Hence, $\lambda(A) = \lambda^*(A)$, $\forall A \in \mathcal{A}$. Thus, by measure extension theorem, $\lambda(A) = \int\limits_{r_1}^{r_2} \mathfrak{U}_C(A \cap C(r))\, \mathrm{d}r$. Theorem 2.11 in [62] enables us to switch from $\mathfrak{U}_C$ to AL_B what ends the proof □

Let $\pi(K)$ denote the norm-circle number which can be assigned to any norm-disc $K(r)$ of norm-radius $r > 0$ according to [62]. The well known Euclidean area content and the Minkowski space $(\mathbb{R}^2, \|.\|_B)$ circumference properties of the norm-discs $K(r)$ are reflected by the equations

$$\pi(K) = \lambda(K) = \frac{1}{2} AL_B(C). \tag{8}$$

For determining the volume of a convex body, see [50]. The following corollary is now an immediate consequence of Lemma 1.

Corollary 2. *For arbitrary norm, the Lebesgue measure allows the geometric disintegration formula* $\lambda(A) = 2\,\pi(K) \int\limits_{r_1}^{r_2} r\,\mathcal{F}_{\|.\|}(A,r)\,\mathrm{d}r.$

The basic disintegration idea behind this corollary **has been** extended in Theorem 2 to norm contoured distributions, which are absolutely continuous with respect to the Lebesgue measure.

Appendix C: Circle numbers of norm discs

In this paper, we have explained the basic analytical and geometric structure of norm contoured distributions in $\mathbb{R}^2$. According to Remark 2, the normalizing constant $C(g, \|.\|)$ strongly depends upon which value takes the circle number function $K \to \pi(K)$ "at the point" K. Let us recall that the function $\pi(.)$ was studied in [60] and [62] and other papers, someone here mentioned. For a number called circle number of a norm-disc, the initial properties are expressed in the equations (8). Note, however, that several more properties of circle numbers have been discussed in [62] and that additional aspects of defining circle numbers are discussed in [68].

Remark 4. It is an immediate consequence of the angular integral representation of the uniform distribution on the unit circle C of the Minkowski space $(\mathbb{R}^2, \|.\|_B)$ that $\pi(K) = \int\limits_0^{2\pi} R_C^2(\varphi)\,\mathrm{d}\varphi$.

Remark 5. Note that, due to (8), at the same time $\lambda(K) = \int\limits_0^{2\pi} R_C^2(\varphi)\,\mathrm{d}\varphi$ is a formula for the area content of a convex body in $\mathbb{R}^2$ symmetric w.r.t. the origin.

Example 7. We consider the norm $\|(x,y)\|$ which admits the values $|x|+|y|$ and $\sqrt{x^2+y^2}$ if (x,y) belongs to $Q_1 \cup Q_3$ or $Q_2 \cup Q_4$, respectively, and the norm dual to it, $\|(x,y)\|^*$, which admits the values $\max\{|x|,|y|\}$ and $\sqrt{x^2+y^2}$ if (x,y) belongs to $Q_1 \cup Q_3$ or $Q_2 \cup Q_4$, respectively, where $Q_1, ..., Q_4$ are the anticlockwise ordered quadrants of $\mathbb{R}^2$.

Let K and K^* denote the unit discs generated by the norms $\|.\|$ and $\|.\|^*$, respectively. The norm $\|(x,y)\|_B$ corresponding to $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K^*$ admits the values $(x^2+y^2)^{1/2}$ and $\max\{|x|,|y|\}$ if (x,y) belongs to $Q_1 \cup Q_3$ or $Q_2 \cup Q_4$, respectively. The area content of $K(r) = r \cdot K = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| \le r\}$ is $f(K,r) = \lambda(K(r)) = \frac{1}{2}\pi r^2 + r^2$. Thus, $\pi(K) = 1 + \pi/2$. Note that this result can be reasonably considered as a linear combination $\pi(K) = (\pi(1) + \pi(2))/2$ of $l_{2,p}$-circle numbers $\pi(p), p \in \{1,2\}$.

Remark 6. A straightforward generalization of the norm considered in the latter example is the norm $\|(x,y)\|$ which admits the values $|(x,y)|_\alpha$ and $|(x,y)|_\beta$ if (x,y) belongs to $Q_1 \cup Q_3$ or $Q_2 \cup Q_4$, respectively, with $(\alpha,\beta) \in [1,\infty)^{\times 2}$.

Let us recall that a subset C of $\mathbb{R}^2$ is called a cone with vertex 0 whenever $tx \in C$ for every $x \in C$ and $t \ge 0$. Every closed convex cone C in $\mathbb{R}^2$ containing no half-space, with vertex in 0 and non-empty interior, will be called a sector of $\mathbb{R}^2$. According to the definition 1.7 of a "complete fan" in [18], a finite collection C of sectors in $\mathbb{R}^2$ will be called a fan if its members have pairwise disjoint interiors and their union is $\mathbb{R}^2$.

Example 8. We consider a fan $C = \bigcup_{i=1}^{k} C_i$ and assume that a convex body K symmetric w.r.t. the origin in $\mathbb{R}^2$ is given, with a certain function R, by

$$K = \{r \cdot (\cos\varphi, \sin\varphi), 0 \leq r \leq R(\varphi),\, \varphi \in Pol^{*-1}(C_i),\ i = 1, ..., k\}.$$

We assume further that K_i are norm-discs having generalized circle numbers $\pi(K_i)$ and satisfying

$$K_i \cap C_i = K \cap C_i,\, i = 1, ..., k.$$

Then $\pi(K)$ is a linear combination of the generalized circle numbers $\pi(K_i)$, $i = 1, ..., k$. Note that here Pol^* denotes the restriction of the usual polar coordinate transformation to the radius 1 and Pol^{*-1} the map being inverse to it. Special cases of this example are polygonal circle discs considered in [68].

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