Exact distributions of order statistics of dependent random variables from $l_{n,p}$-symmetric sample distributions, $n \in \{3,4\}$

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ABSTRACT: Integral representations of the exact distributions of order statistics are derived in a geometric way when three or four random variables depend from each other as the components of continuous $l_{n,p}$-symmetrically distributed random vectors do, $n \in \{3,4\}$, $p > 0$. Once the representations are implemented in a computer program, it is easy to change the density generator of the $l_{n,p}$-symmetric distribution with another one for newly evaluating the distribution of interest.

Key words: density generator, extreme value statistics, geometric measure representation, p-generalized Gaussian and Laplace distributions, heavy tails

1 Introduction

It is well known that uncorrelatedness of a finite number of random variables (rvs) implies their independence if their joint multidimensional distribution is a Gaussian one. More specifically, if the density generating function (dgf) of a spherically distributed random vector is that of a Gaussian vector then the components of this vector are independent rvs. For any other choice of the dgf, these rvs depend from each other in a certain way. Similarly, if a random vector follows a continuous $l_{n,p}$-symmetric or $l_{n,p}$-spherical distribution, $p > 0$, its n components are independent if its dgf is that of a suitably defined n-dimensional p-power exponential distribution, and only in this case.

Therefore, studying distributions of functions of spherically or $l_{n,p}$-symmetrically distributed random vectors means, in general, studying distributions under specific dependence assumptions w.r.t. the joint sampling distribution. Note that the class of

$l_{n,2}$-symmetric distributions is just that of all spherical distributions. The type of dependence among the components of a continuous $l_{n,p}$-symmetrically distributed random vector depends on both the dgf and the parameter p. One might call, for short, this dependence the $l_{n,p}$-symmetry dependence.

Order statistics are useful tools in parametric and nonparametric statistics as well as, e.g., in reliability theory and other applied research areas. The distributions of order statistics of independent and identically distributed random variables are exhaustively dealt with in the last decades, see e.g. David and Nagaraja (2003). The probability density function (pdf) of the maximum statistic as well as that of a linear combination of order statistics of arbitrary absolutely continuous dependent random variables is studied in Arellano-Valle and Genton (2008) and Arellano-Valle and Genton (2007), respectively. In both papers, special emphasis is on the case that the joint multivariate sample distribution is an elliptically contoured distribution. In Jamalizadeh and Balakrishnan (2010) and some papers referred to there, the latter investigations are followed up and further developed by representing the results with the help of skewed distributions. For a related result for continuous $l_{2,p}$-symmetrically distributed sample vectors, see Batún-Cutz, González-Farías, and Richter (2013).

The class of elliptically contoured distributions extends that of spherical distributions, see Fang, Kotz, and Ng (1990). Another extension is the class of $l_{n,p}$-symmetric or $l_{n,p}$-spherical distributions. This class has been introduced in Osiewalski and Steel (1993), and dealt with later on, e.g., in Gupta and Song (1997). A geometric measure representation of these distributions was proved in Richter (2009), see equation (3) in Section 5.1. This representation found applications to simulation in Kalke and Richter (2013) and to the derivation of certain exact distributions in Kalke, Richter, and Thauer (2013). In Müller and Richter (2015), integral representations of exact distributions of extreme value statistics of $l_{2,p}$-symmetrically distributed samples are proved making possible to easily change a given density generator with another one. Here, we extend these results to dimensions three and four.

The aim of the present paper, however, is twice. On the one hand, as indicated, we contribute new results on the exact distribution of order statistics of three or four dependent rvs if the sample distribution is an $l_{n,p}$-symmetric one, $n \in \{3, 4\}$. On the other hand, we conduce to a systematic study of cases in which the geometric measure representation successfully applies. While the first aim of this paper needs no further explanation, the second one will to be discussed a bit closer in the following.

In Richter (2014a), a problem is dealt with which was considered beforehand by several authors in a series of papers and by using different methods. Reproving certain of the known results with a new method, applying the geometric measure representation, actually needs most of the space in Richter (2014a). For the subsequent step of substantially extending the class of random variables possessing the same property of interest, however, only little additional effort is needed. This way, a sometimes involved method suddenly mutates to a powerful ancillary tool of mathematical work. For a general discussion on the value of reproving, see Silverman (1994).

Another effect of systematically applying a geometric measure representation is discovered in Dietrich, Kalke, and Richter (2013). Among other things, the authors develop a new integral representation of the cumulative distribution function (cdf) of the largest eigenvalue of a certain Wishart distributed random matrix although another representation in terms of hypergeometric functions has been well established already in the literature for a long time. The non-anticipated wage for these methodological efforts was in numerical stability properties of the new result. Furthermore, the systematic geometric measure theoretical studies in Günzel, Richter, Scheutzow, Schicker, and Venz (2012) and Richter and Venz (2014) bring more structure into a variety of well-known proofs and results on skewed distributions, and noticeable generalize several well established results.

In all these cases, new results are proved for rvs depending from each other under the influence of a dgf and possibly additional parameters.

In order to summarize the two main aims of this paper, besides proving new results on distributions of order statistics, we extend the range where geometric measure representations successfully apply. This way, we contribute to establish such representations as standard ancillary tools of practical work in probability theory and statistics.

The rest of the present paper is organized as follows. In Section 2, general information on the model class of $l_{n,p}$-symmetric distributions are given. Assuming this $l_{n,p}$-symmetric model class, in Section 3, our main results on the cdf and pdf of maximum, median, and minimum statistics of three dependent rvs and on the cdf of extreme value statistics of four dependent rvs are presented. The pdf of the median is visualized, one the one hand, for trivariate p-generalized Gaussian distributed populations, $p = 3$, jointly with histogram plots for increasing sample sizes and, on the other hand, for $l_{3,p}$-symmetrically Kotz type and Pearson Type VII distributed populations for several choices of parameters. Section 4 is aimed to discuss some figures given beforehand and to interrelate the underlying distributions with heavy tailed ones. In Section 5, the results of Section 3 are proved. In particular, basics of the geometric method of proof are explained in Section 5.1. In the final Section 6, some conclusions are drawn from the results in the present paper.

2 The model class

We consider the model class of continuous $l_{n,p}$-symmetric distributions in this paper as a subclass of the class of star-shaped distributions. This point of view leads to a slight change of notation for continuous $l_{n,p}$-symmetric distributions, compared with previous papers dealing with these distributions.

Let $K \subset \mathbb{R}^n$ be a star body having the origin in its interior and let S denote its topological boundary. The functional $h_K\colon \mathbb{R}^n \to [0,\infty)$ defined by $h_K(x) = \inf\{\lambda > 0\colon x \in \lambda K\}$, $x \in \mathbb{R}^n$, is known as the Minkowski functional of K where $\lambda K = \{(\lambda x_1, \dots, \lambda x_n)^{\mathsf{T}}\colon (x_1, \dots, x_n)^{\mathsf{T}} \in K\}$. A function $g\colon (0,\infty) \to (0,\infty)$ satisfying the assumption $0 < I(g) < \infty$ is called density generating function (dgf) of an n-variate distribution where $I(g) = \int\limits_0^\infty r^{n-1} g(r)\, dr$. According to Richter (2014b), moreover assuming the homogeneity of degree one and a certain smoothness of h_K, a probability measure having the pdf

$$\varphi_{g,K}(x) = C(g,K) g(h_K(x)), \quad x \in \mathbb{R}^n,$$

is called a star-shaped distribution with density contour defining star body K, and denoted by $\Phi_{g,K}$. The normalizing constant allows the representation

$$C(g,K) = \frac{1}{\mathfrak{O}_S(S)\, I(g)}$$

where $\mathfrak{O}_S$ denotes the star generalized surface measure on S and is defined as well in Richter (2014b). If K is the unit ball of the finite-dimensional normed or antinormed space $(\mathbb{R}^n, \|\cdot\|)$ then $h_K(x) = \|x\|$, and $\Phi_{g,K}$ is called a norm or antinorm contoured distribution in $\mathbb{R}^n$, respectively, see Richter (2015b) for the 2-dimensional and Richter (2015a) for the general case. For the notion of an antinorm, we refer to Moszyńska and Richter (2012).

Throughout this paper, let $p > 0$. We denote the $l_{n,p}$-unit ball and the $l_{n,p}$-unit sphere by $K_{n,p} = \{x \in \mathbb{R}^n\colon |x|_p \leq 1\}$ and $S_{n,p} = \{x \in \mathbb{R}^n\colon |x|_p = 1\}$, respectively, where $|x|_p = \left(\sum\limits_{k=1}^n |x_k|^p\right)^{\frac{1}{p}}$, $x = (x_1, \dots, x_n)^{\mathsf{T}} \in \mathbb{R}^n$, stands for the p-functional. Then, $h_{K_{n,p}}(x) = |x|_p$, and $h_{K_{n,p}}$ is a norm if $p \geq 1$ and, according to Moszyńska and Richter (2012), an antinorm if $p \in (0,1)$. Further, the star generalized surface measure $\mathfrak{O}_{S_{n,p}}$ matches with the $l_{n,p}$-generalized surface measure $\mathfrak{O}_p$ defined in Richter (2009), and $\omega_{n,p}$ denotes the $l_{n,p}$-generalized surface content of $S_{n,p}$, $\omega_{n,p} = \frac{\left(2\Gamma\left(\frac{1}{p}\right)\right)^n}{p^{n-1}\Gamma\left(\frac{n}{p}\right)}$.

In particular, an n-dimensional random vector $X\colon \Omega \to \mathbb{R}^n$ defined on a probability space $(\Omega, \mathfrak{A}, P)$ and having a pdf $f_X(x) = \varphi_{g,K_{n,p}}(x) = \frac{g(|x|_p)}{\omega_{n,p}\, I(g)}$, $x \in \mathbb{R}^n$, is said to follow the continuous $l_{n,p}$-symmetric distribution $\Phi_{g,p}$ with dgf g. For short, the density $f_X = \varphi_{g,K_{n,p}}$ is written as $f_X = \varphi_{g,p}$. This pdf is norm contoured if $p \geq 1$ and radially concave star-shaped if $p \in (0,1)$. From now on, we assume that g is specifically chosen as a density generator (dg), i.e. the normalizing constant meets the condition $\omega_{n,p}\, I(g) = 1$. In other words, g is chosen in such a way that

$$\varphi_{g,p}(x) = g(|x|_p), \quad x \in \mathbb{R}^n.$$

This notation of an $l_{n,p}$-symmetric pdf slightly differs from the notation $f_X(x) = \tilde{g}(|x|_p^p)$, $x \in \mathbb{R}^n$, used in Gupta and Song (1997), Richter (2009), Arellano-Valle and Richter (2012), Batún-Cutz et al. (2013), Kalke et al. (2013), Müller and Richter (2015), as well as Fang et al. (1990) and Günzel et al. (2012) in the spherical case. Because of $g(c) = \tilde{g}(c^p)$, $c > 0$, we obtain $I(g) = I_{n,\tilde{g},p}$ where the notation $I_{n,\tilde{g},p} = \int\limits_0^\infty r^{n-1}\tilde{g}(r^p)\, dr$ is used in previous papers.

The remaining part of this section deals with examples of density generators of continuous $l_{n,p}$-symmetric distributions. In slightly other notation, these and other examples can be found already in Gupta and Song (1997), and for the case $p = 2$ in Fang et al. (1990). Note that only $g = g_{PE}$ in Example 3 generates independence of the components of the random vector.

Example 1. The dg of the $l_{n,p}$-symmetric (or n-dimensional p-generalized) Kotz type distribution with parameters $\beta, \gamma > 0$ and $M > 1 - \frac{n}{p}$ is

$$g_{Kt;M,\beta,\gamma}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\gamma\beta^{\frac{n+p(M-1)}{p\gamma}}\Gamma\left(\frac{n}{p}\right)}{\Gamma\left(\frac{n+p(M-1)}{p\gamma}\right)} r^{p(M-1)} \exp\{-\beta r^{p\gamma}\}, \quad r > 0.$$

If $p = 2$, paying attention to the change of notation, this is the dg of standardized Kotz type distribution, see Nadarajah (2003). In Gupta and Song (1997), $\Phi_{g_{Kt;M,\beta,\gamma},p}$ has parameter $N = M - 1 \geq 0$, and is called p-generalized Weibull distribution.

Example 2. The dg of the $l_{n,p}$-symmetric power exponential distribution with parameter $\gamma > 0$ is

$$g_{PE;\gamma}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\gamma\,\Gamma\left(\frac{n}{p}\right)}{p^{\frac{n}{p\gamma}}\,\Gamma\left(\frac{n}{p\gamma}\right)} \exp\left\{-\frac{r^{p\gamma}}{p}\right\}, \quad r > 0,$$

i.e. $g_{PE;\gamma} = g_{Kt;1,1/p,\gamma}$. If $p = 2$, this dg generates the standardized multivariate power exponential distribution, see Gómez, Gómez-Villegas, and Marín (1998), whose univariate form is introduced in Subbotin (1923). Under various parameterizations, and sometimes called exponential power distribution, the univariate distribution $\Phi_{g_{PE;\gamma},2}$ is studied in Box and Tiao (1973), Osiewalski and Steel (1993), and Nadarajah (2005, 2006).

Example 3. The particular function $g_{PE;1} = g_{PE}$ is called the dg of the n-dimensional

p-power exponential or p-generalized Gaussian or p-generalized Laplace distribution,

$$g_{PE}(r) = \left(\frac{p^{1-\frac{1}{p}}}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \exp\left\{-\frac{r^p}{p}\right\}, \quad r > 0.$$

If $p = 1$ or $p = 2$, g_{PE} is the dg of the n-dimensional Laplace or Gaussian distribution, respectively.

Example 4. The dg of the $l_{n,p}$-symmetric Pearson Type VII distribution with parameters $\nu > 0$ and $M > \frac{n}{p}$ is

$$g_{PT7;M,\nu}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma(M)}{\nu^{\frac{n}{p}}\Gamma\left(M - \frac{n}{p}\right)} \left(1 + \frac{r^p}{\nu}\right)^{-M}, \quad r > 0.$$

Example 5. The dg $g_{St;\nu}$ of the $l_{n,p}$-symmetric Student-t distribution with $\nu > 0$ degrees of freedom is defined as

$$g_{St;\nu}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma(\frac{n+\nu}{p})}{\nu^{\frac{n}{p}}\Gamma\left(\frac{\nu}{p}\right)} \left(1 + \frac{r^p}{\nu}\right)^{-\frac{n+\nu}{p}}, \quad r > 0.$$

In addition, $g_{St;\nu} = g_{PT7;\frac{n+\nu}{p},\nu}$.

Example 6. The dg g_C of the $l_{n,p}$-symmetric Cauchy distribution satisfies

$$g_C = g_{St;1}$$

as it is well known in the spherical case $p = 2$.

Let $\mathbb{1}_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{otherwise} \end{cases}$ denote the indicator function of the set A.

Example 7. The dg of the $l_{n,p}$-symmetric Pearson Type II distribution with parameter $\nu > 0$ is

$$g_{PT2;\nu}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma\left(\frac{n}{p} + \nu + 1\right)}{\Gamma(\nu + 1)} (1 - r^p)^\nu \mathbb{1}_{(0,1)}(r), \quad r > 0.$$

3 Exact distributions of order statistics

For the rest of the paper, we assume that the random variables $X_1, \ldots, X_n$ are the components of the random vector X, $X \sim \Phi_{g,p}$, for an arbitrary shape/ tail parameter

$p > 0$ as well as an arbitrary dg g. Furthermore, we denote the corresponding vector of order statistics by

$$X_{(n)}^{(ord)} = (X_{1:n}, \dots, X_{n:n})^{\mathtt{T}}$$

and the cdf and pdf of $X_{k:n}$, $k = 1, \dots, n$, by $F_{k:n}$ and $f_{k:n}$, respectively. The following result describes the basic structure of our representations of $F_{k:n}$ and $f_{k:n}$.

Lemma 1 **(Separating property).** *The cdf and the pdf of $X_{k:n}$ allow the representations*

$$F_{k:n}(t) = \int\limits_0^\infty f(t,r)\, r^{n-1} g(r)\, dr, \tag{1}$$

$$f_{k:n}(t) = \int\limits_0^\infty h(t,r)\, r^{n-1} g(r)\, dr, \tag{2}$$

$t \in \mathbb{R}$, *with functions $f, h \colon \mathbb{R} \times (0,\infty) \to \mathbb{R}$ not depending on the dg g.*

For simplicity of notation, we do not indicate that the functions f and h depend on the integers n and k, and on the parameter p. Note that the influence of g onto the distribution of $X_{k:n}$ is separated in Lemma 1 from that of all the other parameters. Once the functions f and h are implemented in a computer program, it is easy to change a certain dg g with another one for newly evaluating the functions $F_{k:n}$ and $f_{k:n}$. Figures 1-3 show the median density for different types of the dg (recognize different scaling in different pictures). The underlying results of Sections 3.1 and 3.2 specify the functions f and h, and will be derived on using the geometric measure representation (3), see Section 5.1.

3.1 Maximum, median, and minimum distributions of three dependent rvs

In order to define the function f in (1) for $n = 3$ in a dense form, we will make use of the following notations. For any reals $\rho_a < \rho_b$ from $[0,\infty)$ and any functions φ_a and φ_b mapping $[0,\infty)$ to $[0, 2\pi)$ and satisfying $\varphi_a(\rho) < \varphi_b(\rho)$ for all $\rho \in (0,\infty)$, let

$$H(\rho_a, \rho_b; \varphi_a, \varphi_b) = \int\limits_{\rho_a}^{\rho_b} \rho\, (1-\rho^p)^{\frac{1-p}{p}}\, \mathfrak{G}_p^{(2)}(\varphi_a(\rho), \varphi_b(\rho))\, d\rho$$

where $\mathfrak{G}_p^{(2)}(\varphi_a(\rho), \varphi_b(\rho)) = \int\limits_{\varphi_a(\rho)}^{\varphi_b(\rho)} \frac{d\varphi}{(N_p(\varphi))^2}$ with $N_p(\varphi) = (|\cos(\varphi)|^p + |\sin(\varphi)|^p)^{\frac{1}{p}}$ denotes the p-generalized uniform distribution on $S_{2,p}(\rho)$, see Richter (2008a, 2008b). Here, $S_{n,p}(\rho) = \rho \cdot S_{n,p} = \{x \in \mathbb{R}^n \colon |x|_p = \rho\}$ denotes the $l_{n,p}$-sphere with p-radius $\rho \in$

$(0,\infty)$. Furthermore, for all $r\in(0,\infty)$ and $t\in\mathbb{R}$, let $\alpha_{t,r}(\rho)=\arctan\left(\frac{|t|}{\sqrt[p]{\rho^p r^p-|t|^p}}\right)$ and $H_i(\rho_a,\rho_b)=H(\rho_a,\rho_b;\varphi_{a,i},\varphi_{b,i})$, $i=1,\dots,5$, where the functions $\varphi_{a,i}$ and $\varphi_{b,i}$ are given in Table 1.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$\varphi_{a,i}(\rho)$	$\pi+\alpha_{t,r}(\rho)$	$\pi-\alpha_{t,r}(\rho)$	$\frac{\pi}{2}-\alpha_{t,r}(\rho)$	$\pi-\alpha_{t,r}(\rho)$	$\frac{\pi}{2}+\alpha_{t,r}(\rho)$
$\varphi_{b,i}(\rho)$	$\frac{3\pi}{2}-\alpha_{t,r}(\rho)$	$\frac{\pi}{2}$	$\alpha_{t,r}(\rho)$	$\frac{3\pi}{2}+\alpha_{t,r}(\rho)$	$\frac{3\pi}{2}-\alpha_{t,r}(\rho)$

Table 1: Definitions of the functions $\varphi_{a,i}(\rho)$ and $\varphi_{b,i}(\rho)$, $i=1,\dots,5$, $\rho>0$.

Theorem 1. *The cdf of the maximum statistic satisfies representation* (1) *with* $n=k=3$ *and*

$$\begin{aligned}
f(t,r)=&\mathbb{1}_{(-\infty,0]}(t)\mathbb{1}_{(\sqrt[p]{3}|t|,\infty)}(r)H_1\Big(\frac{\sqrt[p]{2}|t|}{r},\frac{1}{r}\sqrt[p]{r^p-|t|^p}\Big)+\mathbb{1}_{(0,\infty)}(t)\Bigg[\mathbb{1}_{(0,t)}(r)\omega_{3,p}\\
&+\mathbb{1}_{[t,\sqrt[p]{2}t)}(r)\left(\omega_{3,p}\left(1-\frac{1}{2r^2}\left(r^p-t^p\right)^{\frac{2}{p}}\right)-8H_2\Big(\frac{t}{r},1\Big)\right)+\mathbb{1}_{[\sqrt[p]{2}t,\sqrt[p]{3}t)}(r)\Bigg(\omega_{3,p}\frac{t^2}{2r^2}\\
&+H_3\Big(\frac{1}{r}\sqrt[p]{r^p-t^p},\frac{\sqrt[p]{2}t}{r}\Big)+H_4\Big(\frac{1}{r}\sqrt[p]{r^p-t^p},1\Big)+H_3\Big(\frac{t}{r},\frac{\sqrt[p]{2}t}{r}\Big)+H_4\Big(\frac{t}{r},1\Big)\Bigg)\\
&+\mathbb{1}_{[\sqrt[p]{3}t,\infty)}(r)\left(\omega_{3,p}\frac{t^2}{2r^2}+H_4\Big(\frac{t}{r},1\Big)+H_3\Big(\frac{1}{r}\sqrt[p]{r^p-t^p},1\Big)+H_3\Big(\frac{t}{r},\frac{\sqrt[p]{2}t}{r}\Big)\right)\Bigg].
\end{aligned}$$

In order to derive from this result a tightly looking representation of the function h in (2), for any $\rho_a<\rho_b$ from $(0,\infty)$ and $\psi_a<\psi_b$ from $[0,2\pi)$, we use the notations

$$\begin{aligned}
k_\rho(\rho_a,\rho_b;\varphi_a,\varphi_b)&=\int_{\rho_a}^{\rho_b}\rho\left(1-\rho^p\right)^{\frac{1-p}{p}}\alpha'_{t,r}(\rho)\left[N_p^{-2}\left(\varphi_a(\rho)\right)+N_p^{-2}\left(\varphi_b(\rho)\right)\right]\,d\rho,\\
k_\psi(\psi_a,\psi_b)&=r^{p-3}\left(r^p-|t|^p\right)^{\frac{2-p}{p}}\mathfrak{G}_p^{(2)}(\psi_a,\psi_b)
\end{aligned}$$

and

$$\beta_{t,r}=\alpha_{t,r}\left(\frac{1}{r}\sqrt[p]{r^p-|t|^p}\right)=\arctan\left(\frac{|t|}{\sqrt[p]{r^p-2\,|t|^p}}\right)$$

where k_ψ and $\beta_{t,r}$ are defined for all $r\in(0,\infty)$ and $t\in\mathbb{R}$. Note that

$$\alpha'_{t,r}(\rho)=\frac{d}{dt}\alpha_{t,r}(\rho)=\frac{\left(\rho^p r^p-|t|^p\right)^{\frac{1}{p}}+|t|^p\left(\rho^p r^p-|t|^p\right)^{\frac{1-p}{p}}}{t^2+\left(\rho^p r^p-|t|^p\right)^{\frac{2}{p}}}.$$

Finally, we put $k_{\rho,i}(\rho_a,\rho_b) = k_\rho(\rho_a,\rho_b;\varphi_{a,i},\varphi_{b,i})$ for $i = 1,\ldots,5$ and $k_{\rho,6}(\rho_a,\rho_b) = k_\rho(\rho_a,\rho_b;\varphi_{a,2},\varphi_{a,2})$.

Corollary 1. *The pdf of the maximum statistic satisfies representation* (2) *with* $n = k = 3$ *and*

$$\begin{aligned} h(t,r) &= \mathbb{1}_{(-\infty,0]}(t)\mathbb{1}_{(\sqrt[p]{3}|t|,\infty)}(r)\left(k_{\rho,1}(\frac{\sqrt[p]{2}|t|}{r},\frac{1}{r}\sqrt[p]{r^p-|t|^p}) + k_\psi(\pi+\beta_{t,r},\frac{3\pi}{2}-\beta_{t,r})\right) \\ &\quad + \mathbb{1}_{(0,\infty)}(t)\left[\mathbb{1}_{[t,\sqrt[p]{2}t)}(r)\left(\frac{t^{p-1}}{r^2}\,(r^p-t^p)^{\frac{2-p}{p}}\,\omega_{3,p} + 4k_{\rho,6}(\frac{t}{r},1)\right) + \mathbb{1}_{[\sqrt[p]{2}t,\sqrt[p]{3}t)}(r)\right. \\ &\quad \left(k_\psi(\frac{\pi}{2}-\beta_{t,r},\beta_{t,r}) + k_{\rho,3}(\frac{1}{r}\sqrt[p]{r^p-t^p},\frac{\sqrt[p]{2}t}{r})\right) + \mathbb{1}_{[\sqrt[p]{3}t,\infty)}(r)\left(\omega_{3,p}\frac{t}{r^2} + k_{\rho,4}(\frac{t}{r},1)\right. \\ &\quad + k_\psi(\pi-\beta_{t,r},\frac{3\pi}{2}+\beta_{t,r}) - r^{p-3}t\,(r^p-t^p)^{\frac{1-p}{p}}\,2\pi(p) + k_{\rho,4}(\frac{1}{r}\sqrt[p]{r^p-t^p},1) \\ &\quad \left.\left. + k_{\rho,3}(\frac{t}{r},\frac{\sqrt[p]{2}t}{r})\right)\right]. \end{aligned}$$

It is worthwhile to mention that the cdf and the pdf of the minimum statistic satisfy the representations $F_{1:3}(t) = 1 - F_{3:3}(-t)$ and $f_{1:3}(t) = f_{3:3}(-t)$, $t \in \mathbb{R}$, respectively.

Theorem 2. *The cdf of the median statistic satisfies representation* (1) *with* $n = 3$, $k = 2$ *and*

$$\begin{aligned} f(t,r) &= \mathbb{1}_{[\sqrt[p]{2}t,\sqrt[p]{3}t)}(r)\left(2H_5(\frac{|t|}{r},\frac{1}{r}\sqrt[p]{r^p-|t|^p}) + 2H_1(\frac{\sqrt[p]{2}|t|}{r},1)\right) \\ &\quad + \mathbb{1}_{[\sqrt[p]{3}t,\infty)}(r)\left(2H_5(\frac{|t|}{r},\frac{1}{r}\sqrt[p]{r^p-|t|^p}) + 2H_1(\frac{1}{r}\sqrt[p]{r^p-|t|^p},1)\right), \end{aligned}$$

if $t \le 0$, *and* $f(t,r) = \omega_{3,p} - f(-t,r)$, *if* $t > 0$.

Corollary 2. *The pdf of the median statistic satisfies representation* (2) *with* $n = 3$, $k = 2$ *and*

$$\begin{aligned} h(t,r) &= \mathbb{1}_{[\sqrt[p]{2}|t|,\sqrt[p]{3}|t|)}(r)\left(2k_{\rho,5}(\frac{|t|}{r},\frac{1}{r}\sqrt[p]{r^p-|t|^p}) + 2k_\psi(\frac{\pi}{2}+\beta_{t,r},\frac{3\pi}{2}-\beta_{t,r})\right. \\ &\quad \left. + 2k_{\rho,1}(\frac{\sqrt[p]{2}|t|}{r},1)\right) + \mathbb{1}_{[\sqrt[p]{3}|t|,\infty)}(r)\left(r^{p-3}\,(r^p-|t|^p)^{\frac{1-p}{p}}\,\pi(p)\right. \\ &\quad \left. + 2k_{\rho,5}(\frac{|t|}{r},\frac{1}{r}\sqrt[p]{r^p-|t|^p}) + 2k_{\rho,1}(\frac{1}{r}\sqrt[p]{r^p-|t|^p},1)\right). \end{aligned}$$

Figures 1-3 give an impression of how the density $f_{2:3}$ in representation (2) looks like for different types of dgs and several different parameter choices. In particular, Figure 1 deals with the particular independence case of dg $g = g_{PE}$ and $p = 3$ (c.f. Example 3, formula (2) and Corollary 2). Note that the numerical correctness of our evaluations is revealed by adding histogram plots of samples of increasing sizes from 10^3 up to 2.5×10^5.

Also illustrating the pdf of the median statistic but now of three dependent rvs following a joint continuous $l_{3,p}$-symmetric distribution, Figures 2 and 3 deal with generally dependence generating dgs of Kotz type and Pearson Type VII, respectively. Note the different scales of axes of ordinates as well as of abscissas, and that Figures 3(a) and 3(b) are further discussed under several aspects in Section 4.

3.2 Extreme value distributions of four dependent rvs

In the present section, we restrict our considerations to the cdfs of extreme values, i.e. to the functions $F_{k:4}$ in (1) with $k = 4$ and $k = 1$, respectively. For any $\rho_a < \rho_b$ from $(0, \infty)$, any functions φ_a and φ_b mapping $(0, \infty)$ to $[0, 2\pi)$ and satisfying $\varphi_a(\rho) < \varphi_b(\rho)$ for all $\rho \in (0, \infty)$, and any functions θ_a and θ_b mapping $(0, \infty) \times [0, 2\pi)$ to $[0, 2\pi)$ and satisfying $\theta_a(\rho, \varphi) < \theta_b(\rho, \varphi)$ for all $(\rho, \varphi) \in (0, \infty) \times [0, 2\pi)$, let

$$L(\rho_a, \rho_b; \varphi_a, \varphi_b; \theta_a, \theta_b) = \int\limits_{\rho_a}^{\rho_b} \rho^2 \left(1 - \rho^p\right)^{\frac{1-p}{p}} \mathfrak{G}_p^{(3)}(\varphi_a(\rho), \varphi_b(\rho); \theta_a(\rho, \varphi_a(\rho)), \theta_b(\rho, \varphi_b(\rho)))\, d\rho$$

where

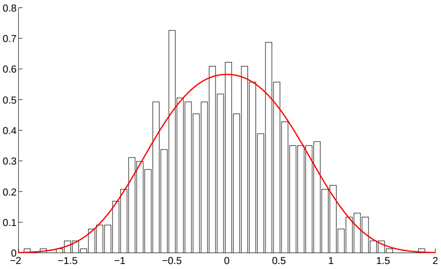
$$\mathfrak{G}_p^{(3)}(\varphi_a(\rho), \varphi_b(\rho); \theta_a(\rho, \varphi_a(\rho)), \theta_b(\rho, \varphi_b(\rho))) = \int\limits_{\varphi_a(\rho)}^{\varphi_b(\rho)} \int\limits_{\theta_a(\rho,\varphi_a(\rho))}^{\theta_b(\rho,\varphi_b(\rho))} \frac{\sin_p(\theta)}{N_p^2(\theta) N_p^2(\varphi)}\, d\theta\, d\varphi$$

denotes the p-generalized uniform distribution on $S_{3,p}(\rho)$ with $\rho \in (0, \infty)$, and $\sin_p$ is the p-generalized sine function defined in Richter (2007). Moreover, let

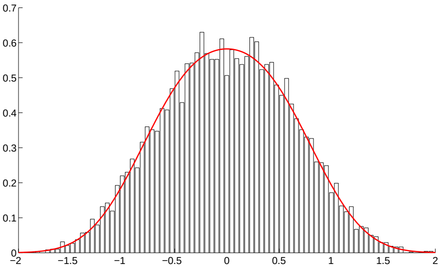
$$\gamma_{t,r}(\rho) = \arctan\left(\frac{|t|}{\sqrt[p]{\rho^p r^p - 2|t|^p}}\right)$$

define a parametric function mapping $[0, \infty)$ to $[0, \pi/2)$, for all $r \in [0, \infty)$ and $t \in \mathbb{R}$. Recognize that $\gamma_{t,r}(1) = \beta_{t,r}$ for all t and r. Additionally, let

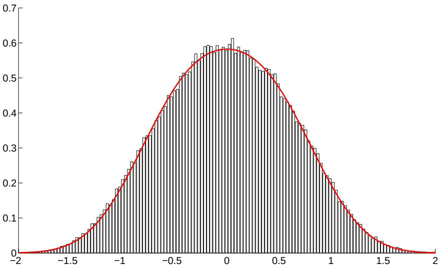
$$\delta_{t,r}^{-}(\rho, \varphi) = \frac{\pi}{2} + \arctan\left(\frac{\sqrt[p]{r^p \rho^p\, |\sin_p(\varphi)|^p - |t|^p}}{|t|}\right) \quad \text{and} \quad \delta_{t,r}^{+}(\rho, \varphi) = \arctan\left(\sqrt[p]{|\cot(\varphi)|^p + 1}\right)$$



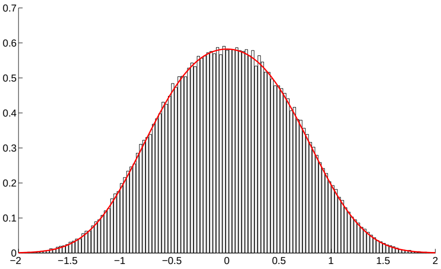
(a) sample size 10^3



(b) sample size 10^4

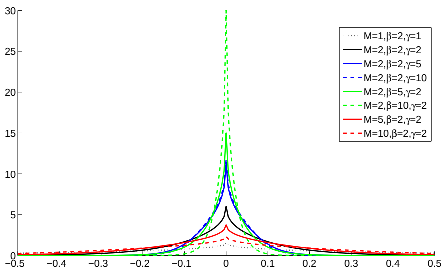


(c) sample size 10^5

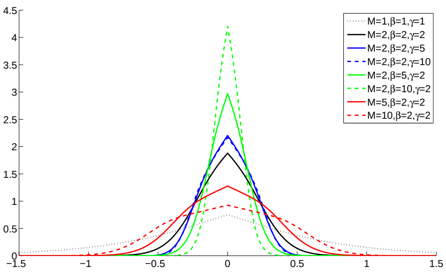


(d) sample size 2.5×10^5

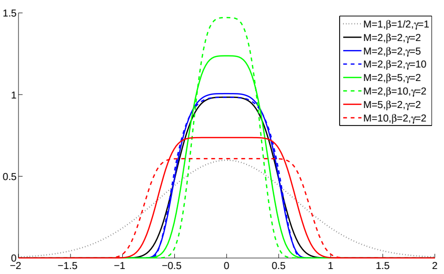
Figure 1: Median pdf $f_{2:3}$ and histogram for $p = 3$, increasing sample sizes and dg $g = g_{PE}$.



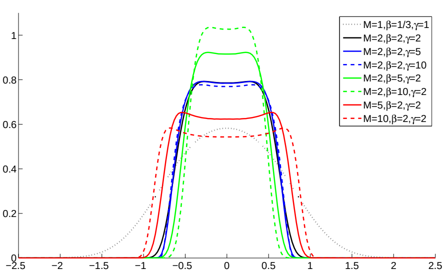
(a) $p = \frac{1}{2}$



(b) $p = 1$

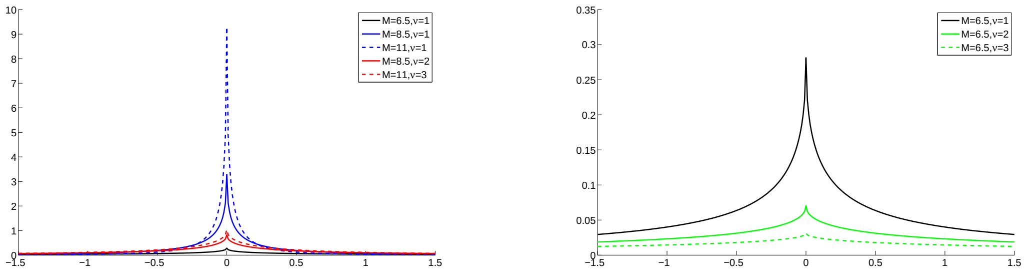


(c) $p = 2$

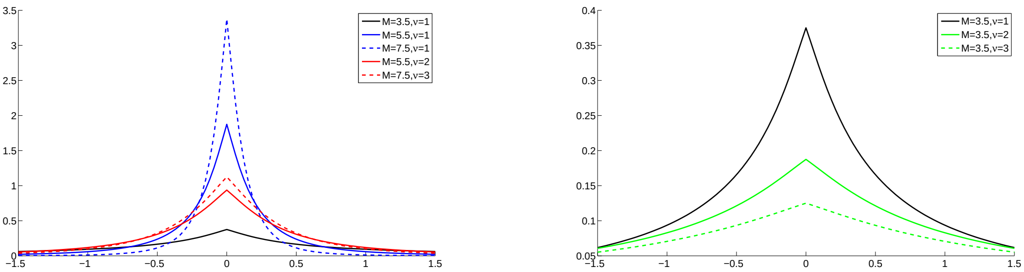


(d) $p = 3$

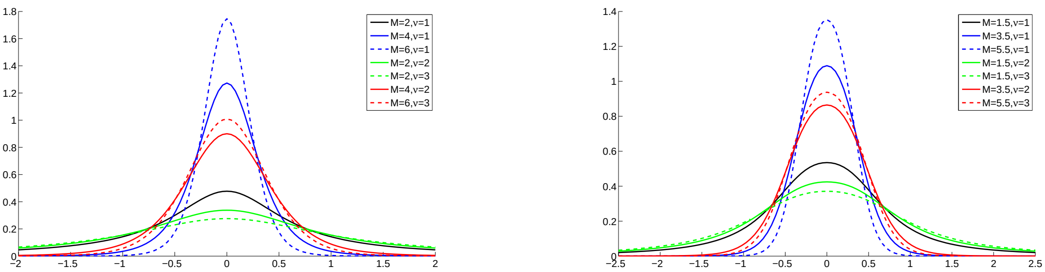
Figure 2: Median pdf $f_{2:3}$ for $p \in \left\{\frac{1}{2}, 1, 2, 3\right\}$, dg $g = g_{Kt;M,\beta,\gamma}$, and several choices of the parameters $M > 1 - \frac{3}{p}$, $\beta > 0$, and $\gamma > 0$.



(a) $p = \frac{1}{2}$



(b) $p = 1$



(c) $p = 2$ (d) $p = 3$

Figure 3: Median pdf $f_{2:3}$ for $p \in \left\{\frac{1}{2}, 1, 2, 3\right\}$, dg $g = g_{PT7;M,\nu}$, and several choices of the parameters $M > \frac{3}{p}$ and $\nu > 0$.

be parametric functions defined on $(0, \infty) \times [0, 2\pi)$, and, for all parameters r and t,

$$
\begin{aligned}
L_1(\varphi_a(\rho), \varphi_b(\rho)) &= L(\frac{\sqrt[p]{3}|t|}{r}, \frac{1}{r}\sqrt[p]{r^p - |t|^p}; \varphi_a(\rho), \varphi_b(\rho); \frac{\pi}{2} + \alpha_{t,r}, \delta^-_{t,r}), \\
L_{2,1}(\varphi_a(\rho), \varphi_b(\rho)) &= L(\frac{t}{r}, 1; \varphi_a(\rho), \varphi_b(\rho); 0, \frac{\pi}{2} - \alpha_{t,r}), \\
L_{2,2}(\varphi_a(\rho), \varphi_b(\rho)) &= L(\frac{t}{r}, 1; \varphi_a(\rho), \varphi_b(\rho); 0, \delta^+_{t,r}), \\
L_{3,1}(\varphi_a(\rho), \varphi_b(\rho)) &= L(\frac{t}{r}, \frac{1}{r}\sqrt[p]{r^p - |t|^p}; \varphi_a(\rho), \varphi_b(\rho); 0, \frac{\pi}{2} - \alpha_{t,r}), \\
L_{3,2}(\varphi_a(\rho), \varphi_b(\rho)) &= L(\frac{t}{r}, \frac{1}{r}\sqrt[p]{r^p - |t|^p}; \varphi_a(\rho), \varphi_b(\rho); 0, \delta^+_{t,r}).
\end{aligned}
$$

Theorem 3. *The cdf of the maximum statistic satisfies representation* (1) *with* $n = k = 4$

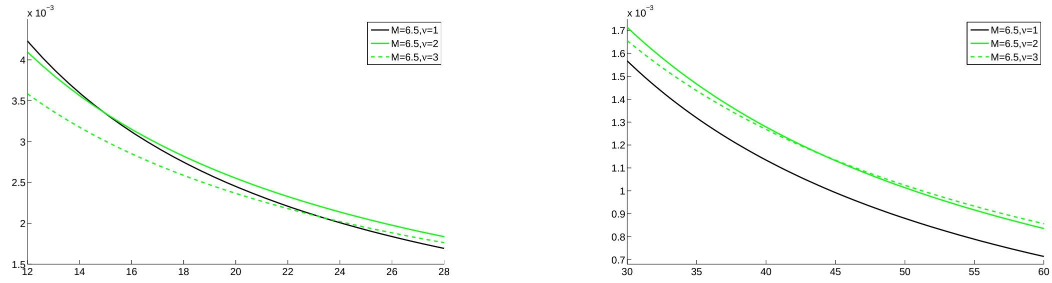
and

$$f(t,r) = \mathbb{1}_{(-\infty,0]}(t)\mathbb{1}_{(\sqrt[p]{4}|t|,\infty)}(r)2L_1(\pi+\gamma_{t,r},\frac{5\pi}{4}) + \mathbb{1}_{(0,\infty)}(t)\Bigg[\mathbb{1}_{(0,t)}(r)\omega_{4,p} + \mathbb{1}_{[t,\sqrt[p]{2}t)}(r)$$
$$\left(\omega_{4,p} - \frac{\omega_{4,p}}{2}\left(1-\left|\frac{t}{r}\right|^p\right)^{\frac{3}{p}} - 12L_{2,1}(0,\pi)\right) + \mathbb{1}_{[\sqrt[p]{2}t,\sqrt[p]{3}t)}(r)\Bigg(\omega_{4,p} + 6L_{3,1}(0,\pi)$$
$$-\frac{\omega_{4,p}}{2}\left(1-\left|\frac{t}{r}\right|^p\right)^{\frac{3}{p}} - 6L_{2,1}(\frac{\pi}{2}-\gamma_{t,r},\gamma_{t,r}) - 6L_{2,1}(\pi-\gamma_{t,r},\frac{3\pi}{2}+\gamma_{t,r})$$
$$-12L_{2,2}(\gamma_{t,r},\pi-\gamma_{t,r})\Bigg) + \mathbb{1}_{[\sqrt[p]{3}t,\sqrt[p]{4}t)}(r)\Bigg(\omega_{4,p} - 6L_{2,1}(\pi-\gamma_{t,r},\frac{3\pi}{2}+\gamma_{t,r})$$
$$-\frac{\omega_{4,p}}{2}\left(1-\left|\frac{t}{r}\right|^p\right)^{\frac{3}{p}} - 12L_{2,2}(\frac{\pi}{4},\pi-\gamma_{t,r}) + 3L_{3,1}(\pi-\gamma_{t,r},\frac{3\pi}{2}+\gamma_{t,r})$$
$$+3L_{3,1}(\frac{\pi}{2}-\gamma_{t,r},\gamma_{t,r}) + 6L_{3,2}(\gamma_{t,r},\pi-\gamma_{t,r})\Bigg) + \mathbb{1}_{[\sqrt[p]{4}t,\infty)}(r)\Bigg(\omega_{4,p}$$
$$-\frac{\omega_{4,p}}{2}\left(1-\left|\frac{t}{r}\right|^p\right)^{\frac{3}{p}} - 12H_{2,2}(\frac{\pi}{4},\pi-\gamma_{t,r}) - 6H_{2,1}(\pi-\gamma_{t,r},\frac{3\pi}{2}+\gamma_{t,r})$$
$$+3H_{3,1}(\pi-\gamma_{t,r},\frac{3\pi}{2}+\gamma_{t,r}) + 6H_{3,2}(\frac{\pi}{4},\pi-\gamma_{t,r})\Bigg)\Bigg].$$

4 Heavy tails

Distributions having heavy tails play an important role in statistical practice and find especially many applications to insurance and financial mathematics. The median pdf $f_{2:3}$ plotted in Figure 3 deals with heavy tails where the dg of X is of $l_{3,p}$-symmetric Pearson Type VII which includes both Student and Cauchy type sample distributions. It appears to be typical in such cases that only very few probability mass is concentrated around the distribution center leading on the right hand sides of Figures 3(a) and 3(b) to the misleading impression that the drawn densities could build a monotonically decreasing sequence of functions. By zooming into the right hand side of Figures 3(a), however, and taking the symmetry w.r.t. axis of ordinates into account, one detects the points of intersection of the black and the green solid, the black and the green dashed, and the green solid and the green dashed graphs at $t_{1/2} \approx \pm 15$, $t_{3/4} \approx \pm 23$, and $t_{5/6} \approx \pm 44$, respectively, see Figure 4. A similar explanation avoids a potential misunderstanding in the case of Figure 3(b).

Furthermore, the Figures 3(a) and 4 suggest the visual impression that the tail heaviness of the distribution of the median statistic of three dependent rvs following a joint $l_{3,\frac{1}{2}}$-symmetric Pearson Type VII distribution increases if the parameter M is



(a) (b)

Figure 4: Zoom into the right hand side of Figure 3(a).

$A_\nu(z)$	$z = 1.5$	$z = 100$	$z = 10^3$	$z = 10^4$	$z = 10^6$
$\nu = 1$	0.1903	0.5906	0.7368	0.8249	0.9077
$\nu = 2$	0.0889	0.4689	0.6571	0.7777	0.8943
$\nu = 3$	0.0512	0.3884	0.5988	0.7416	0.8832

Table 2: Interval probabilities of the sample median in jointly $l_{3,\frac{1}{2}}$-symmetrically Pearson Type VII distributed samples with $M = \frac{13}{2}$ and $\nu \in \{1, 2, 3\}$.

constant and the parameter $\nu > 0$ increases. For specific values of interval probabilities see Table 2 where the values $A_\nu(z) = \int\limits_{-z}^{z} f_{2:3}(t)\, dt$ are numerically computed for $z \in \{1.5, 100, 10^3, 10^4, 10^6\}$, shape/ tail parameter $p = \frac{1}{2}$, and dg $g_{PT7;M,\nu}$ with parameters $M = \frac{13}{2}$ and $\nu \in \{1, 2, 3\}$. If M is constant and $\nu > 0$ increases, such a manner of heaviness of tails can be observed in all cases of Figure 3.

5 Proofs

In order to proof the assertions from Section 3, the general method of proof and some basics on applying this method to order statistics are presented in Sections 5.1 and 5.2, respectively. Afterwards, the claimed results on the distributions of order statistics for three dependent rvs and extreme value statistics for four dependent rvs are established whereas the details of proofs decrease in quantity in later parts of these sections.

5.1 Basics of the geometric method of proof

Let $T\colon \mathbb{R}^n \to \mathbb{R}$ be any statistic and $A(t) = \{x \in \mathbb{R}^n \colon T(x) < t\}$ a sublevel set (sls) generated by it. If $X \sim \Phi_{g,p}$ with an arbitrary dg g, the cdf of $T(X)$ is

$$F_T(t) = \Phi_{g,p}\left(A(t)\right), \quad t \in \mathbb{R}.$$

The geometric measure representation in Richter (2009), with notations as described in Section 2 suitably adapted to the ones used in Richter (2014b) and here, applies

$$\Phi_{g,p}\left(A(t)\right) = \omega_{n,p} \int_0^\infty \mathfrak{F}_p\left(A(t), r\right) r^{n-1} g(r)\, dr \tag{3}$$

where the $l_{n,p}$-sphere intersection-proportion function (ipf) $\mathfrak{F}_p\colon (0,\infty) \to [0,\infty)$ is defined on $\mathfrak{B}^n$ by

$$r \mapsto \mathfrak{F}_p\left(A, r\right) = \frac{\mathfrak{O}_p\left(\left[\frac{1}{r}A\right] \cap S_{n,p}\right)}{\mathfrak{O}_p\left(S_{n,p}\right)}.$$

According to Richter (2009), the $l_{n,p}$-generalized surface content is defined on $\mathfrak{B}^n$ by

$$\mathfrak{O}_p(A) = \int\limits_{G\left(A\cap S_{n,p}^{-}\right)} \left(1 - \sum_{j=1}^{n-1} |x_j|^p\right)^{\frac{1-p}{p}} dx \;+ \int\limits_{G\left(A\cap S_{n,p}^{+}\right)} \left(1 - \sum_{j=1}^{n-1} |x_j|^p\right)^{\frac{1-p}{p}} dx$$

where

$$G(A) := \{(x_1, \ldots, x_{n-1}) \in K_{n-1,p} : \exists\, x_n \in \mathbb{R} : (x_1, \ldots, x_n) \in A\}$$

and $S_{n,p}^{+(-)} = \{x \in S_{n,p}\colon x_n \geq (\leq)\, 0\}$. Hence,

$$F_T(t) = \int_0^\infty \mathfrak{O}_p\left(\left[\frac{1}{r}A(t)\right] \cap S_{n,p}\right) r^{n-1} g(r)\, dr.$$

Since $\mathfrak{O}_p\left(\left[\frac{1}{r}A(t)\right] \cap S_{n,p}\right)$ does not depend on the dg g, this representation proves the first part of Lemma 1, i.e. (1) and the independence of the function f from the dg g, whereas T is chosen as an arbitrary order statistic. The second part of Lemma 1 follows by the Leibniz integral rule.

Now, we prepare for the proofs of the results from Sections 3.1 and 3.2. With the help of the $l_{n-1,p}$-spherical coordinate transformation $SPH_p^{(n-1)}\colon [0,\infty) \times [0,\pi)^{\times(n-3)} \times [0, 2\pi) \to \mathbb{R}^{n-1}$ and its corresponding inverse mapping, see Richter (2007), the cdf of $T(X)$ allows the general representation

$$F_T(t) = \int_0^\infty \left(\int\limits_{M_{(t,r)}^{-}} \tilde{h}(\rho,\varphi)\, d(\rho,\varphi) + \int\limits_{M_{(t,r)}^{+}} \tilde{h}(\rho,\varphi)\, d(\rho,\varphi) \right) r^{n-1} g(r)\, dr \tag{4}$$

where $\tilde{h}(\rho,\varphi) = (1-\rho^p)^{\frac{1-p}{p}}\, J\left(SPH_p^{(n-1)}\right)(\rho,\varphi)$,

$$M_{(t,r)}^{+(-)} = \left(SPH_p^{(n-1)}\right)^{-1} \left(G\left(\left[\frac{1}{r}A(t)\right] \cap S_{n,p}^{+(-)} \right) \right),$$

and

$$J\left(SPH_p^{(n-1)}\right)(\rho,\varphi) = \rho^{n-2}\prod_{i=1}^{n-2}\frac{(\sin\varphi_i)^{n-2-i}}{(N_p(\varphi_i))^{n-i}}$$

is the Jacobian of $SPH_p^{(n-1)}$. Thus, it remains to determine the domains of integration $M^+_{(t,r)}$ and $M^-_{(t,r)}$ for all cases considered in Sections 3.1 and 3.2.

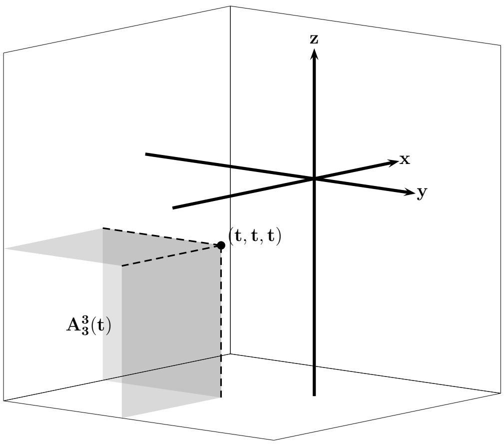
5.2 General representations of the domains of integration $M^{+(-)}_{(t,r)}$

As it can be seen from Section 5.1, studying the sets $G(\left[\frac{1}{r}A(t)\right]\cap S^-_{n,p})$ and $G(\left[\frac{1}{r}A(t)\right]\cap S^+_{n,p})$ plays a fundamental role for the application of the geometric measure representation formula. The present section is aimed to prove general representations of these sets if the generating statistic is any order statistic. More specific representations will be derived from it in the next section and will be used there to prove results for all cases considered in Sections 3.1 and 3.2.

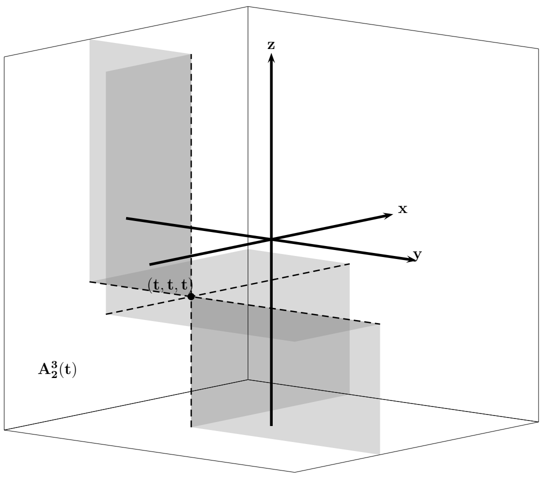
For $k \in \{1,\dots,n\}$, let

$$A^n_k(t) = \{x \in \mathbb{R}^n\colon \text{ at least } k \text{ components of } x \text{ are less than } t\}, \quad t \in \mathbb{R},$$

be a sls generated by the kth order statistic of an n-dimensional random vector. An illustration of the set $A^n_k(t)$ can be seen in Figure 5 for $(n,k) \in \{(3,3),(3,2)\}$ and $t<0$.



(a) Sls of the maximum



(b) Sls of the median

Figure 5: Sls of order statistics of three variables and $t<0$.

Remark 1. Let X be a continuous and symmetrically with respect to the origin distributed random vector, $X \sim -X$, and let $F_{k:n}(t) = P(X_{k:n} < t)$ be the cdf of the kth order statistic $X_{k:n}$ of X. Then, for $k = 1,\dots,n$ and every $t \in \mathbb{R}$, $F_{n-k+1,n}(t) = 1 - F_{k,n}(-t)$, and

$$F_{k:n}(t) = P\left(X \in A^n_k(t)\right).$$

Lemma 2. *If $t \leq 0$, then*

$$G\left(\left[\frac{1}{r}A_k^n(t)\right] \cap S_{n,p}^-\right) = \left[\mathring{K}_{n-1,p}\left(\sqrt[p]{1-\left|\frac{t}{r}\right|^p}\right) \cap \frac{1}{r}A_{k-1}^{n-1}(t)\right] \cup \left[K_{n-1,p} \cap \frac{1}{r}A_k^{n-1}(t)\right],$$

$$G\left(\left[\frac{1}{r}A_k^n(t)\right] \cap S_{n,p}^+\right) = K_{n-1,p} \cap \frac{1}{r}A_k^{n-1}(t)$$

and, if $t > 0$,

$$G\left(\left[\frac{1}{r}A_k^n(t)\right] \cap S_{n,p}^-\right) = K_{n-1,p} \cap \frac{1}{r}A_{k-1}^{n-1}(t),$$

$$G\left(\left[\frac{1}{r}A_k^n(t)\right] \cap S_{n,p}^+\right) = \left[R_{n-1,p}\left(\sqrt[p]{1-\left|\frac{t}{r}\right|^p}\right) \cap \frac{1}{r}A_{k-1}^{n-1}(t)\right] \cup \left[K_{n-1,p} \cap \frac{1}{r}A_k^{n-1}(t)\right]$$

where $A_l^m(t) = \emptyset$ if $l = 0$ or $l > m$, $\mathring{A}$ denotes the topological interior of the set $A \subseteq \mathbb{R}^n$, $K_{n,p}(\rho) = \{x \in \mathbb{R}^n \colon |x|_p \leq \rho\}$ the $l_{n,p}$-ball with p-radius $\rho \in (0, \infty)$, and

$$R_{n,p}(\rho_a, \rho_b) = K_{n,p}(\rho_b) \backslash K_{n,p}(\rho_a) = \{x \in \mathbb{R}^n \colon \rho_a < |x|_p \leq \rho_b\}$$

the $l_{n,p}$-layer with p-radii $\rho_a < \rho_b$.

Proof. Let $t \leq 0$, $B_1 = \left[\frac{1}{r}A_{k-1}^{n-1}(t)\right] \times \left\{x_n \in \mathbb{R} \colon x_n < \frac{t}{r}\right\}$, and $B_2 = \left[\frac{1}{r}A_k^{n-1}(t)\right] \times \mathbb{R}$. Then $\frac{1}{r}A_k^n(t) = B_1 \cup B_2$ where $B_1 = \emptyset$, if $k = 1$, and $B_2 = \emptyset$, if $k = n$. Note that, for every $x = (x_1, \ldots, x_n)^\mathsf{T} \in B_1 \cap S_{n,p}^-$,

$$(x_1, \ldots, x_{n-1})^\mathsf{T} \in \frac{1}{r}A_{k-1}^{n-1}(t) \quad , \quad x_n < \frac{t}{r} \leq 0 \quad \text{, and} \quad \sum_{i=1}^n |x_i|^p = 1.$$

Since $|x_n|^p > \left|\frac{t}{r}\right|^p \geq 0$, it follows

$$(x_1, \ldots, x_{n-1})^\mathsf{T} \in \frac{1}{r}A_{k-1}^{n-1}(t) \quad \text{and} \quad \sum_{i=1}^{n-1} |x_i|^p < 1 - \left|\frac{t}{r}\right|^p.$$

That is why every element of

$$G(B_1 \cap S_{n,p}^-) = \left\{(x_1, \ldots, x_{n-1})^\mathsf{T} \in \mathbb{R}^{n-1} \colon \exists!\, x_n \leq 0 : \sum_{i=1}^n |x_i|^p = 1,\ x_n < \frac{t}{r},\ (x_1, \ldots, x_{n-1})^\mathsf{T} \in \frac{1}{r}A_{k-1}^{n-1}(t)\right\}$$

is an element of

$$C_1 := \mathring{K}_{n-1,p}\left(\sqrt[p]{1-\left|\frac{t}{r}\right|^p}\right) \cap \frac{1}{r}A_{k-1}^{n-1}(t).$$

In other words, $G(B_1 \cap S^-_{n,p}) \subseteq C_1$. Let $(x_1, \ldots, x_{n-1})^\mathtt{T} \in C_1$. Choosing $\rho^p = \sum\limits_{i=1}^{n-1} |x_i|^p$, it follows $0 \leq \rho < \sqrt[p]{1 - \left|\frac{t}{r}\right|^p}$. Further, one can uniquely choose $x_n = -\sqrt[p]{1-\rho^p} < 0$ such that $\rho^p + |x_n|^p = 1$, i.e. $(x_1, \ldots, x_{n-1}, x_n) \in S^-_{n,p}$. Then, $x_n < \frac{t}{r}$ and $C_1 \subseteq G(B_1 \cap S^-_{n,p})$. Hence,

$$G(B_1 \cap S^-_{n,p}) = \mathring{K}_{n-1,p}\left(\sqrt[p]{1 - \left|\frac{t}{r}\right|^p}\right) \cap \frac{1}{r} A^{n-1}_{k-1}(t).$$

For any $x = (x_1, \ldots, x_n)^\mathtt{T} \in B_2 \cap S^-_{n,p}$,

$$(x_1, \ldots, x_{n-1})^\mathtt{T} \in \frac{1}{r} A^{n-1}_k(t) \quad \text{and} \quad x_n \leq 0.$$

As $G(B_2 \cap S^-_{n,p}) = \{(x_1, \ldots, x_{n-1})^\mathtt{T} \in \mathbb{R}^{n-1} \colon \exists!\, x_n \leq 0 : \sum\limits_{i=1}^{n} |x_i|^p = 1,\ (x_1, \ldots, x_{n-1})^\mathtt{T} \in \frac{1}{r} A^{n-1}_k(t)\}$, for any $(x_1, \ldots, x_{n-1})^\mathtt{T} \in G(B_2 \cap S^-_{n,p})$

$$(x_1, \ldots, x_{n-1})^\mathtt{T} \in C_2 := K_{n-1,p} \cap \frac{1}{r} A^{n-1}_k(t),$$

i.e., $G(B_2 \cap S^-_{n,p}) \subseteq C_2$. Let $(x_1, \ldots, x_{n-1})^\mathtt{T} \in C_2$. Choosing $\rho^p = \sum\limits_{i=1}^{n-1} |x_i|^p$, it follows $0 \leq \rho \leq 1$. Further, let $x_n = -\sqrt[p]{1-\rho^p} \leq 0$ such that $\rho^p + |x_n|^p = 1$. Because of $(x_1, \ldots, x_{n-1})^\mathtt{T} \in \left[\frac{1}{r} A^{n-1}_k(t)\right] \subseteq C_2$, we have $(x_1, \ldots, x_{n-1}, x_n)^\mathtt{T} \in \left[\frac{1}{r} A^{n-1}_k(t)\right] \times \{z \in \mathbb{R} \colon z \leq 0\} \subseteq B_2$. Thus, $C_2 \subseteq G(B_2 \cap S^-_{n,p})$, and consequently,

$$G(B_2 \cap S^-_{n,p}) = K_{n-1,p} \cap \frac{1}{r} A^{n-1}_k(t).$$

Summarizing the above results, the first assertion of the lemma follows. The other cases can be dealt with in an analogous way. $\square$

The next step of analyzing the sets $G(\left[\frac{1}{r} A(t)\right] \cap S^-_{n,p})$ and $G(\left[\frac{1}{r} A(t)\right] \cap S^+_{n,p})$ consists of numerous case studies. Because the number of cases increases if the number of rvs does, we restrict the outline of this way mainly to the case of three rvs.

5.3 Specific representations of the domains of integration for considering the maximum of three dependent rvs

This section demonstrates that, in the case of three dependent rvs, the geometric method of proof applies as successful as in Müller and Richter (2015) where the case of sample size two was dealt with. The present calculations may also serve as an orientation for the derivation of analogous results in more general star-shaped model classes.

Proof of Theorem 1. To get the exact cdf of the considered statistic, according to equation (4), it remains to represent the sets $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$, generally satisfying the representations of Lemma 2 for $k = n = 3$ and $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right)$, in $l_{2,p}$-spherical coordinates with arbitrary $r \in (0,\infty)$. For this purpose, let

$$R_i(t) = \left\{ x \in \mathbb{R}^3 \colon x = \frac{1}{r}\begin{pmatrix} t \\ t \\ t \end{pmatrix} - \lambda e_i^{(3)} \ , \ \lambda \geq 0 \right\},$$

$i \in \{1,2,3\}$, denote the rays, which represent the dashed edges of $\frac{1}{r}A_3^3(t)$, see Figure 5(a). Note that, without loss of generality, all figures are drawn throughout this proof for $p = 3$.

Case 1: Let $t \leq 0$. Because of $\frac{1}{r}(t,t,t)^{\mathrm{T}} \notin \frac{1}{r}A_3^3(t)$, the intersection $\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}$ is empty if the point $\frac{1}{r}(t,t,t)^{\mathrm{T}}$ is off or on the $l_{3,p}$-unit sphere, i.e. $1 \leq \frac{1}{r}\sqrt[p]{3}|t|$. Hence, $\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p} \neq \emptyset$ iff $r \in \left(\sqrt[p]{3}|t|, \infty\right)$, and $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right) = \emptyset$ for every $r \in \mathbb{R}_+$, since $\frac{1}{r}A_3^3(t) \subset \{x \in \mathbb{R}^3 : x_3 \leq 0\}$. The set $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$ is shown in Figure 6, where $P_i = R_i(t) \cap S_{3,p}$ and $P_i' = G(P_i)$ for $i \in \{1,2,3\}$. Note that $P_1 = \left(-\frac{1}{r}\sqrt[p]{r^p - 2|t|^p}, \frac{1}{r}t, \frac{1}{r}t\right)$, $P_2 = \left(\frac{1}{r}t, -\frac{1}{r}\sqrt[p]{r^p - 2|t|^p}, \frac{1}{r}t\right)$, and $P_3 = \left(\frac{1}{r}t, \frac{1}{r}t, -\frac{1}{r}\sqrt[p]{r^p - 2|t|^p}\right)$.

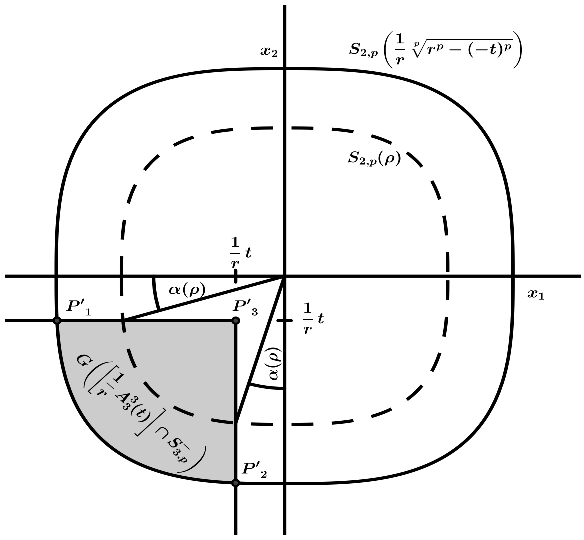


Figure 6: The set $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$ for $t \leq 0$ and $r \in \left(\sqrt[p]{3}|t|, \infty\right)$.

The rays starting in the origin and passing through the points $(z, 0)$ and $\left(z, \frac{1}{r}t\right)$, and $\left(\frac{1}{r}t, z\right)$ and $(0, z)$, respectively, enclose angles of the same magnitude $\alpha(\rho)$, where one has to determine $z < 0$ such that the point $\left(z, \frac{1}{r}t\right)$ belongs to the $l_{2,p}$-sphere with p-radius ρ, i.e. $\rho^p = |z|^p + \left|\frac{1}{r}t\right|^p$. Thus, $z = -\frac{1}{r}\sqrt[p]{\rho^p r^p - |t|^p}$. By the definition of the tangent function, and making use of the notation at the beginning of Section 3.1,

$$\alpha(\rho) = \alpha_{t,r}(\rho).$$

If $r \in \left(\sqrt[p]{3}|t|, \infty\right)$, the set $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$ satisfies the representation

$$\begin{aligned}
& G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right) \\
= \; & SPH_p^{(2)}\left(\left\{(\rho,\varphi)\colon \rho \in \left[\|P_3'\|_p, \|P_1'\|_p\right], \varphi \in \left(\pi + \alpha_{t,r}(\rho), \frac{3\pi}{2} - \alpha_{t,r}(\rho)\right)\right\}\right) \\
= \; & SPH_p^{(2)}\left(\left\{(\rho,\varphi)\colon \rho \in \left[\frac{\sqrt[p]{2}|t|}{r}, \frac{1}{r}\sqrt[p]{r^p - |t|^p}\right], \varphi \in (\varphi_{a,1}(\rho), \varphi_{b,1}(\rho))\right\}\right).
\end{aligned}$$

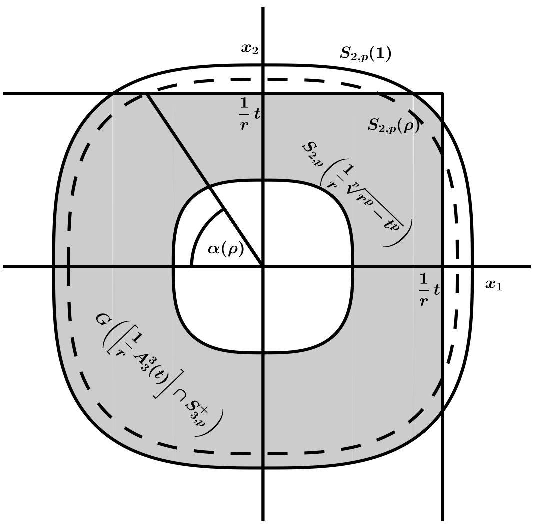
Case 2: Let $t > 0$. We consider $\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}$.

Case 2.1: Let $r \in (0, t)$. Then the $l_{3,p}$-unit sphere is completely contained in $\frac{1}{r}A_3^3(t)$. Therefore, $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right) = G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right) = K_{2,p}$.

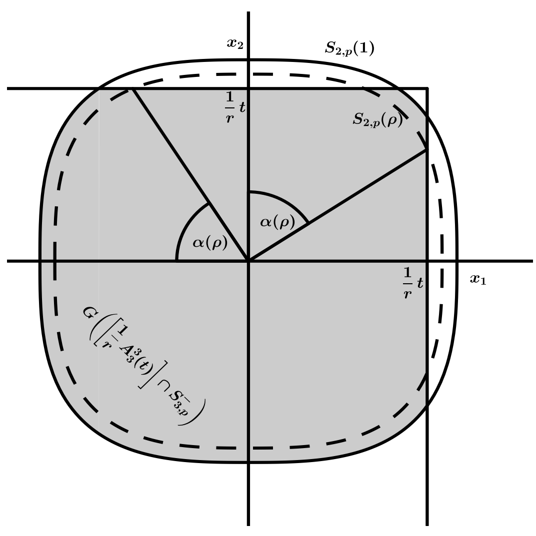
Case 2.2: Let $r \in [t, \sqrt[p]{2}t)$. This case occurs iff the rays $R_i(t)$, $i \in \{1,2,3\}$, do not intersect $S_{3,p}$, but the three planes, which are defined such that each of them contains exactly two of these rays, intersect the $l_{3,p}$-unit sphere, i.e. $R_i(t) \cap S_{3,p} = \emptyset$, and

$$\left\{(1-\lambda)z_1 + \lambda z_2 \in \mathbb{R}^3\colon z_1 \in R_i(t), z_2 \in R_j(t), \lambda \in [0,1]\right\} \cap S_{3,p} \neq \emptyset$$

for $i, j \in \{1,2,3\}$ with $i \neq j$. In other words, the range of r for this case ends if the rays are tangents to $S_{3,p}$ and, without any loss of generality, if $R_1(t)$ is a tangent to $S_{3,p}$, $\frac{1}{r}(0,t,t)$ is the boundary point.



(a) $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right)$



(b) $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$

Figure 7: The sets in case 2.2.

To achieve representations of the sets $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right)$ and $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$ for $r \in [t, \sqrt[p]{2}t)$, see Figure 7. With the help of $l_{2,p}$-spherical coordinates, we determine $z < 0$ such that the point $\left(z, \frac{1}{r}t\right)$ belongs to the $l_{2,p}$-sphere with p-radius ρ, i.e. $\rho^p = |z|^p + \left(\frac{1}{r}t\right)^p$. Thus, $z = -\frac{1}{r}\sqrt[p]{\rho^p r^p - t^p}$. Analogously to the case $t \leq 0$, the angle of the magnitude

$\alpha(\rho)$, enclosed by the rays starting in the origin and passing through the points $(z, 0)$ and $\left(z, \frac{1}{r}t\right)$, satisfies

$$\alpha(\rho) = \alpha_{t,r}(\rho).$$

Since $l_{2,p}$-spheres are invariant with respect to rotations of angles $\left\{k\frac{\pi}{2} : k \in \mathbb{N}\right\}$ around the origin, the set $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right)$ satisfies for $t > 0$ and $r \in [t, \sqrt[p]{2}t)$ the representation

$$G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right) = K_{2,p} \backslash \left(K_{2,p}\left(\frac{1}{r}\sqrt[p]{r^p - t^p}\right) \cup SPH_p^{(2)}(M)\right)$$

where

$$M = \left\{(\rho, \varphi) \colon \rho \in \left[\frac{t}{r}, 1\right], \varphi \in \left[\alpha(\rho) - \frac{\pi}{2}, \frac{\pi}{2} - \alpha(\rho)\right] \cup [\alpha(\rho), \pi - \alpha(\rho)]\right\}.$$

As the set $\frac{1}{r}A_3^3(t)$ is unbounded with respect to the x_3-direction, $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$ satisfies

$$G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right) = K_{2,p} \backslash SPH_p^{(2)}(M)$$

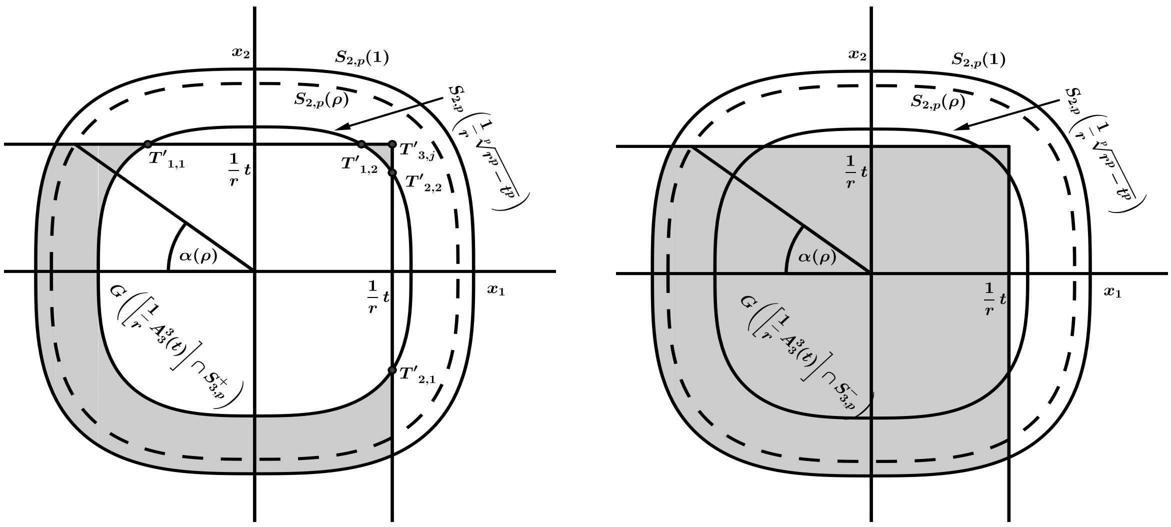
for $r \in [t, \sqrt[p]{2}t)$. Using the symmetry, one gets the claimed result.

Case 2.3: Let $r \in [\sqrt[p]{2}t, \sqrt[p]{3}t)$. The range of r for this case starts when the rays $R_i(t)$, $i \in \{1, 2, 3\}$, are tangents to $S_{3,p}$ and ends before the origin $\frac{1}{r}(t, t, t)$ of the rays is on $S_{3,p}$. Let $\{T_{i,1}, T_{i,2}\} = R_i(t) \cap S_{3,p}$ denote the set of points of intersection of the ray $R_i(t)$ and the $l_{3,p}$-unit sphere and $T'_{i,j} = G(T_{i,j})$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. Analogously to the case $t \leq 0$, $T_{1,j} = \left((-1)^j \frac{1}{r}\sqrt[p]{r^p - 2t^p}, \frac{1}{r}t, \frac{1}{r}t\right)$, $T_{2,j} = \left(\frac{1}{r}t, (-1)^j \frac{1}{r}\sqrt[p]{r^p - 2t^p}, \frac{1}{r}t\right)$, and $T_{3,j} = \left(\frac{1}{r}t, \frac{1}{r}t, (-1)^j \frac{1}{r}\sqrt[p]{r^p - 2t^p}\right)$ for $j \in \{1, 2\}$, and $T_{i,1} = T_{i,2}$, $i \in \{1, 2, 3\}$, if $r = \sqrt[p]{2}t$. Figure 8 illustrates that, as in the case before, the sets $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right)$ and $G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right)$ allow for $r \in [\sqrt[p]{2}t, \sqrt[p]{3}t)$ the representations

$$\begin{aligned} & G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^+\right) \\ = \; & SPH_p^{(2)}\left(\left\{(\rho, \varphi) \colon \rho \in \left(\frac{1}{r}\sqrt[p]{r^p - t^p}, 1\right], \varphi \in (\varphi_{a,4}(\rho), \varphi_{b,4}(\rho))\right\}\right) \\ & \cup SPH_p^{(2)}\left(\left\{(\rho, \varphi) \colon \rho \in \left[\frac{1}{r}\sqrt[p]{r^p - t^p}, \frac{\sqrt[p]{2}t}{r}\right], \varphi \in (\varphi_{a,3}(\rho), \varphi_{b,3}(\rho))\right\}\right) \end{aligned}$$

and

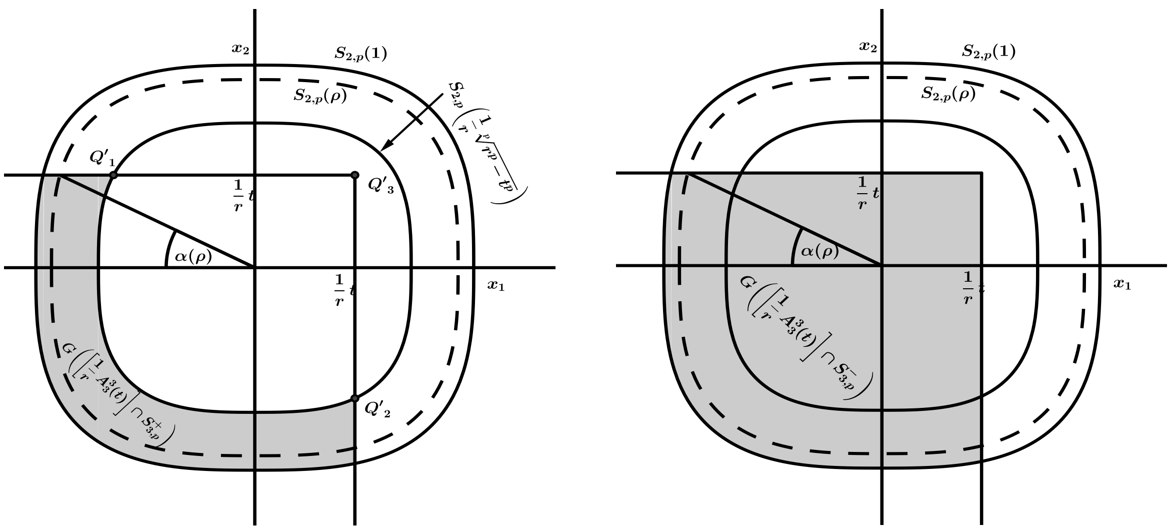
$$\begin{aligned} & G\left(\left[\frac{1}{r}A_3^3(t)\right] \cap S_{3,p}^-\right) \\ = \; & SPH_p^{(2)}\left(\left\{(\rho, \varphi) \colon \rho \in \left[\frac{t}{r}, 1\right], \varphi \in (\varphi_{a,4}(\rho), \varphi_{b,4}(\rho))\right\}\right) \\ & \cup SPH_p^{(2)}\left(\left\{(\rho, \varphi) \colon \rho \in \left[\frac{t}{r}, \frac{\sqrt[p]{2}t}{r}\right], \varphi \in (\varphi_{a,3}(\rho), \varphi_{b,3}(\rho))\right\}\right) \cup K_{2,p}\left(\frac{t}{r}\right). \end{aligned}$$



(a) $G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^+\right)$ (b) $G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^-\right)$

Figure 8: The sets in case 2.3.

Case 2.4: Let $r \in [\sqrt[p]{3}t, \infty)$, i.e. the range of r for this case begins when the point $\frac{1}{r}(t,t,t)$ is on the $l_{3,p}$-unit sphere. Unless for $r = \sqrt[p]{3}t$, every ray $R_i(t)$ has precisely one point of intersection with $S_{3,p}$ which is denoted by Q_i, $i \in \{1,2,3\}$. Then $Q_1 = \left(-\frac{1}{r}\sqrt[p]{r^p-2t^p}, \frac{1}{r}t, \frac{1}{r}t\right)$, $Q_2 = \left(\frac{1}{r}t, -\frac{1}{r}\sqrt[p]{r^p-2t^p}, \frac{1}{r}t\right)$, and $Q_3 = \left(\frac{1}{r}t, \frac{1}{r}t, -\frac{1}{r}\sqrt[p]{r^p-2t^p}\right)$. Let $Q'_i = G(Q_i)$, $i \in \{1,2,3\}$, and let the angle $\alpha(\rho)$ be defined as before. Figure 9 illustrates that the sets $G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^+\right)$ and $G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^-\right)$ can be represented as



(a) $G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^+\right)$ (b) $G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^-\right)$

Figure 9: The sets in case 2.4.

$$
\begin{aligned}
& G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^+\right)\\
= \; & SPH_p\left(\left\{(\rho,\varphi)\colon \rho\in\left(\frac{1}{r}\sqrt[p]{r^p-t^p},1\right],\varphi\in(\varphi_{a,4}(\rho),\varphi_{b,4}(\rho))\right\}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& G\left(\left[\frac{1}{r}A_3^3(t)\right]\cap S_{3,p}^-\right)\\
= \; & SPH_p\left(\left\{(\rho,\varphi)\colon \rho\in\left[\frac{1}{r}t,1\right],\varphi\in(\varphi_{a,4}(\rho),\varphi_{b,4}(\rho))\right\}\right)\\
& \cup SPH_p\left(\left\{(\rho,\varphi)\colon \rho\in\left[\frac{1}{r}t,\frac{1}{r}\sqrt[p]{2}t\right],\varphi\in(\varphi_{a,3}(\rho),\varphi_{b,3}(\rho))\right\}\right)\cup K_{2,p}\left(\frac{1}{r}t\right).
\end{aligned}
$$

□

5.4 Maximum pdfs as derivatives of parameter integrals

In this section, we establish the pdf of the maximum statistic in the case of three dependent rvs, see Corollary 1, taking the derivative of the parameter integral representation of the corresponding cdf given in Theorem 1.

Proof of Corollary 1. Case 1: Let $t<0$. Using the notation $P(t,r)=r^2g(r)f(t,r)$, function f from Theorem 1, and the Leibniz integral rule,

$$
f_{3:3}(t)=\int\limits_{\sqrt[p]{3}|t|}^{\infty} r^2g(r)\,\frac{\partial f}{\partial t}(t,r)\,dr+3^{\frac{3}{p}}t^2g(\sqrt[p]{3}|t|)f(t,\sqrt[p]{3}|t|).
$$

Note that $f(t,\sqrt[p]{3}|t|)=0$ and, because of $\alpha_{t,r}\left(\frac{\sqrt[p]{2}|t|}{r}\right)=\frac{\pi}{4}$,

$$
\begin{aligned}
\frac{\partial f}{\partial t}(t,r)= & \int\limits_{\frac{\sqrt[p]{2}|t|}{r}}^{\frac{1}{r}\sqrt[p]{r^p-(-t)^p}} \rho\,(1-\rho^p)^{\frac{1-p}{p}}\left(\frac{\partial}{\partial t}\int\limits_{\pi+\alpha_{t,r}(\rho)}^{\frac{3\pi}{2}-\alpha_{t,r}(\rho)}\frac{d\varphi}{N_p^2(\varphi)}\right)d\rho\\
& +r^{p-3}\left(r^p-|t|^p\right)^{\frac{2-p}{p}}\int\limits_{\pi+\beta_{t,r}}^{\frac{3\pi}{2}-\beta_{t,r}}\frac{d\varphi}{N_p^2(\varphi)}.
\end{aligned}
$$

Therefore, in the case $(t,r)\in(-\infty,0)\times\left(\sqrt[p]{3}|t|,\infty\right)$, it follows that $\frac{\partial f}{\partial t}(t,r)=k(t,r)$.

Case 2: Let $t>0$. Let f_1, f_2, f_3, and f_4 denote the restrictions of f (w.r.t. the variable r) to the intervals $(0,t)$, $[t,\sqrt[p]{2}t)$, $[\sqrt[p]{2}t,\sqrt[p]{3}t)$, and $[\sqrt[p]{3}t,\infty)$, respectively, and let

$$P_i(t,r) = r^2 g(r) f_i(t,r) \qquad \text{and} \qquad S_i(t) = \int\limits_0^\infty P_i(t,r)\, dr$$

for $i = 1, \ldots, 4$. In this part of the proof, the four summands $S_1(t), \ldots, S_4(t)$ are considered separately. Note that $\alpha_{t,r}\left(\frac{1}{r}\sqrt[p]{2}t\right) = \frac{\pi}{4}$ and $\lim\limits_{\rho \searrow \frac{1}{r}t} \alpha_{t,r}(\rho) = \frac{\pi}{2}$. We consider $\gamma_1 = \alpha_{t,\sqrt[p]{2}t}(\rho) = \arctan\left(\frac{1}{\sqrt[p]{2\rho^p - 1}}\right)$ and $\gamma_2 = \alpha_{t,\sqrt[p]{3}t}(\rho) = \arctan\left(\frac{1}{\sqrt[p]{3\rho^p - 1}}\right)$. The first summand satisfies

$$\frac{dS_1}{dt}(t) = \omega_{3,p} t^2 g(t),$$

and the second summand fulfills

$$\frac{dS_2}{dt}(t) = \int\limits_t^{\sqrt[p]{2}t} r^2 g(r) \frac{\partial f_2}{\partial t}(t,r)\, dr + \sqrt[p]{2}\, P_2(t, \sqrt[p]{2}t) - P_2(t,t).$$

Further, $P_2(t,t) = \omega_{3,p} t^2 g(t)$,

$$P_2(t, \sqrt[p]{2}t) = 2^{\frac{2}{p}} t^2 g(\sqrt[p]{2}t) \left(\omega_{3,p} \left(1 - 2^{-\frac{2+p}{p}}\right) - 8 \int\limits_{\sqrt[p]{\frac{1}{2}}}^{1} \int\limits_{\frac{\pi}{2}}^{\pi - \gamma_1} \frac{\rho \left(1 - \rho^p\right)^{\frac{1-p}{p}}}{\left(N_p(\varphi)\right)^2}\, d\varphi\, d\rho \right),$$

and

$$\begin{aligned} \frac{\partial f_2}{\partial t}(t,r) &= \omega_{3,p} \frac{t^{p-1}}{r^2} \left(r^p - t^p\right)^{\frac{2-p}{p}} + 8 \int\limits_{\frac{1}{r}t}^{1} \rho \left(1 - \rho^p\right)^{\frac{1-p}{p}} \alpha'_{t,r}(\rho) N_p^{-2} \left(\pi - \alpha_{t,r}(\rho)\right)\, d\rho \\ &= \omega_{3,p} \frac{t^{p-1}}{r^2} \left(r^p - t^p\right)^{\frac{2-p}{p}} + 4 k_{\rho,6}(\frac{t}{r}, 1). \end{aligned}$$

Taking the derivative of the fourth summand yields

$$\frac{dS_4}{dt}(t) = \int\limits_{\sqrt[p]{3}t}^{\infty} r^2 g(r) \frac{\partial f_4}{\partial t}(t,r)\, dr - \sqrt[p]{3}\, P_4(t, \sqrt[p]{3}t)$$

where

$$\begin{aligned} P_4(t, \sqrt[p]{3}t) = 3^{\frac{2}{p}} t^2 g(\sqrt[p]{3}t) \Bigg(&\omega_{3,p} \frac{1}{2} \left(\frac{1}{3}\right)^{\frac{2}{p}} + \int\limits_{\sqrt[p]{\frac{2}{3}}}^{1} \int\limits_{\pi - \gamma_2}^{\frac{3\pi}{2} + \gamma_2} \frac{\rho \left(1 - \rho^p\right)^{\frac{1-p}{p}}}{\left(N_p(\varphi)\right)^2}\, d\varphi\, d\rho \\ &+ \int\limits_{\sqrt[p]{\frac{1}{3}}}^{1} \int\limits_{\pi - \gamma_2}^{\frac{3\pi}{2} + \gamma_2} \frac{\rho \left(1 - \rho^p\right)^{\frac{1-p}{p}}}{\left(N_p(\varphi)\right)^2}\, d\varphi\, d\rho + \int\limits_{\sqrt[p]{\frac{1}{3}}}^{\sqrt[p]{\frac{2}{3}}} \int\limits_{\frac{\pi}{2} - \gamma_2}^{\gamma_2} \frac{\rho \left(1 - \rho^p\right)^{\frac{1-p}{p}}}{\left(N_p(\varphi)\right)^2}\, d\varphi\, d\rho \Bigg). \end{aligned}$$

The partial derivative $\frac{\partial f_4}{\partial t}$ satisfies the representation

$$\begin{aligned}\frac{\partial f_4}{\partial t}(t,r) = \omega_{3,p}\frac{1}{r^2}t - r^{p-3}t\,(r^p - t^p)^{\frac{1-p}{p}}\, 2\pi(p) + k_{\rho,4}(\frac{1}{r}\sqrt[p]{r^p - t^p}, 1)\\ + k_{\rho,4}(\frac{t}{r}, 1) + k_{\rho,3}(\frac{t}{r}, \frac{\sqrt[p]{2}t}{r})\\ + k_{\psi}(\pi - \beta_{t,r}, \frac{3\pi}{2} + \beta_{t,r})\end{aligned}$$

where k_ρ and k_ψ are introduced in Corollary 1.
The third summand S_3 satisfies

$$\frac{dS_3}{dt}(t) = \int_{\sqrt[p]{2}t}^{\sqrt[p]{3}t} r^2 g(r)\frac{\partial f_3}{\partial t}(t,r)\,dr + \sqrt[p]{3}\,P_3(t,\sqrt[p]{3}t) - \sqrt[p]{2}\,P_3(t,\sqrt[p]{2}t)$$

where $P_3(t,\sqrt[p]{2}t) = P_2(t,\sqrt[p]{2}t)$ and $P_3(t,\sqrt[p]{3}t) = P_4(t,\sqrt[p]{3}t)$. With

$$\tilde{f}_3(t,r) = \int_{\frac{1}{r}\sqrt[p]{r^p-t^p}}^{\frac{1}{r}\sqrt[p]{2}t} \int_{\frac{\pi}{2}-\alpha_{t,r}(\rho)}^{\alpha_{t,r}(\rho)} \frac{\rho\,(1-\rho^p)^{\frac{1-p}{p}}}{(N_p(\varphi))^2}\,d\varphi\,d\rho,$$

it follows $f_3 = f_4 + \tilde{f}_3$ and $\frac{\partial f_3}{\partial t} = \frac{\partial f_4}{\partial t} + \frac{\partial \tilde{f}_3}{\partial t}$, where

$$\frac{\partial \tilde{f}_3}{\partial t}(t,r) = k_{\rho,3}(\frac{1}{r}\sqrt[p]{r^p - t^p}, \frac{\sqrt[p]{2}t}{r}) + k_\psi(\frac{\pi}{2} - \beta_{t,r}, \beta_{t,r}).$$

Summarizing all intermediate results, the corollary follows from

$$f_{3:3}(t) = \frac{dS_1}{dt}(t) + \frac{dS_2}{dt}(t) + \frac{dS_3}{dt}(t) + \frac{dS_4}{dt}(t).$$

□

5.5 Median for $n = 3$ and Maximum for $n = 4$

Following the same line as in the last two sections, we prove here the representations of the cdf and the pdf of the median in the case of three dependent rvs, and the cdf of the maximum in the case of four dependent rvs. This proves Theorem 2, Corollary 2 and Theorem 3. Here, calculations will not be given as detailed as in the preceding sections.

Proof of Theorem 2. In an analogous manner as in the proof of Theorem 1, we use equation (4) for the median statistic in the case of $n = 3$ and represent the sets $G\left(\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^-\right)$ and $G\left(\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^-\right)$, given in Lemma 2 for $k = 2$ and $n = 3$, for an arbitrary $r \in (0,\infty)$ using $l_{2,p}$-spherical coordinates. In order to do this, if $t \leq 0$, the cases to be distinguished are $r \in (0, \sqrt[p]{2}|t|]$, $r \in (\sqrt[p]{2}|t|, \sqrt[p]{3}|t|]$, and $r \in (\sqrt[p]{3}|t|, \infty)$. In the first case, $\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^{-(+)} = \emptyset$. In the other two cases, the gray-colored sets shown in Figures 10 have to be considered. Here, in contrast to the proof of Theorem 1, but again without loss of generality, figures are drawn for $p = \frac{3}{2}$.

If $t > 0$, the different cases are $r \in (0, \sqrt[p]{2}t]$, $r \in (\sqrt[p]{2}t, \sqrt[p]{3}t]$, and $r \in (\sqrt[p]{3}t, \infty)$. In the first case, $\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^{-(+)} = S_{3,p}^{-(+)}$, and the sets $G\left(\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^{-(+)}\right)$ in the other cases are shown in Figure 11. Note that there is a helpful symmetry relation between the cases $t \leq 0$ and $t > 0$.

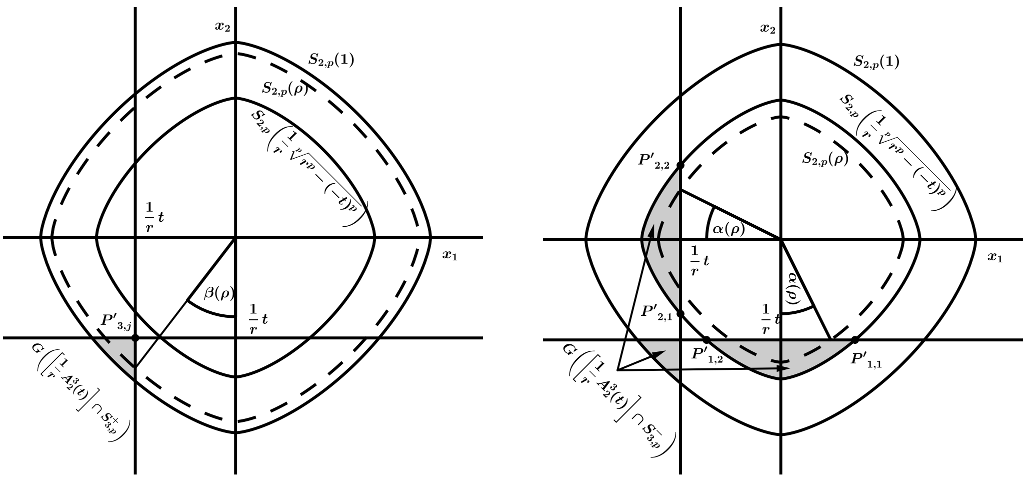
$\square$

Proof of Corollary 2. Taking the derivative of $F_{2:3}(t)$ directly yields the pdf $f_{2:3}(t)$. $\square$

Proof of Theorem 3. With the help of equation (4), for $n = 4$ and the maximum statistic, this proof follows analogously to that of Theorem 1 or 2. The sets $G\left(\left[\frac{1}{r}A_4^4(t)\right] \cap S_{4,p}^{-}\right)$ and $G\left(\left[\frac{1}{r}A_4^4(t)\right] \cap S_{4,p}^{-}\right)$ given in Cartesian coordinates by Lemma 2 need to be expressed using $l_{3,p}$-spherical coordinates. To this end, we consider the separate cases $r \in (0, \sqrt[p]{4}|t|]$ and $r \in (\sqrt[p]{4}|t|, \infty)$, if $t \leq 0$, and $r \in (0, t)$, $r \in [t, \sqrt[p]{2}t)$, $r \in [\sqrt[p]{2}t, \sqrt[p]{3}t)$, $r \in [\sqrt[p]{3}t, \sqrt[p]{4}t)$, and $r \in [\sqrt[p]{4}t, \infty)$, if $t > 0$. $\square$

6 Discussion

In Müller and Richter (2015), the exact extreme value distributions of the components of $l_{2,p}$-symmetrically distributed random vectors are derived explicitly. A reformulation in terms of skewed distributions was proved in Batún-Cutz et al. (2013). In the present paper, assuming again the model class of continuous $l_{n,p}$-symmetric distributions, the exact distributions of order statistics for three dependent and of extreme value statistics for four rvs are derived applying the geometric measure representation from Richter (2009)



(a) $r \in (\sqrt[p]{2}|t|, \sqrt[p]{3}|t|]$ (b) $r \in (\sqrt[p]{2}|t|, \sqrt[p]{3}|t|]$

directly. In contrast to other applications of this geometric method, results and proofs in the case of order statistics become increasingly involved if the dimension increases. This explains the need of finding a more advanced method to make use of the geometric measure representation in higher dimensions. In the spherical case $p = 2$, such a method was developed for $n = 2$ in Günzel et al. (2012) and, generalizing this, for arbitrary n in Richter and Venz (2014). We hope to report a p-generalization of this method in another paper.

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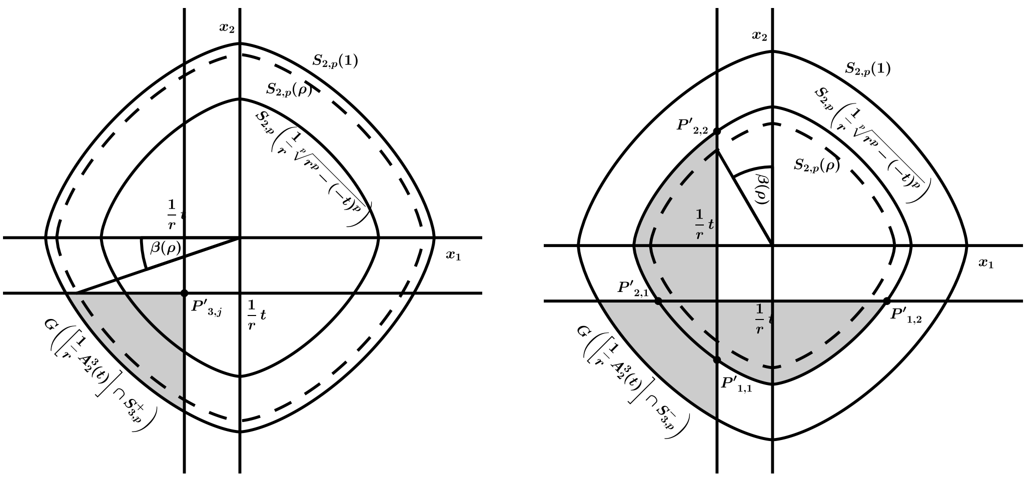
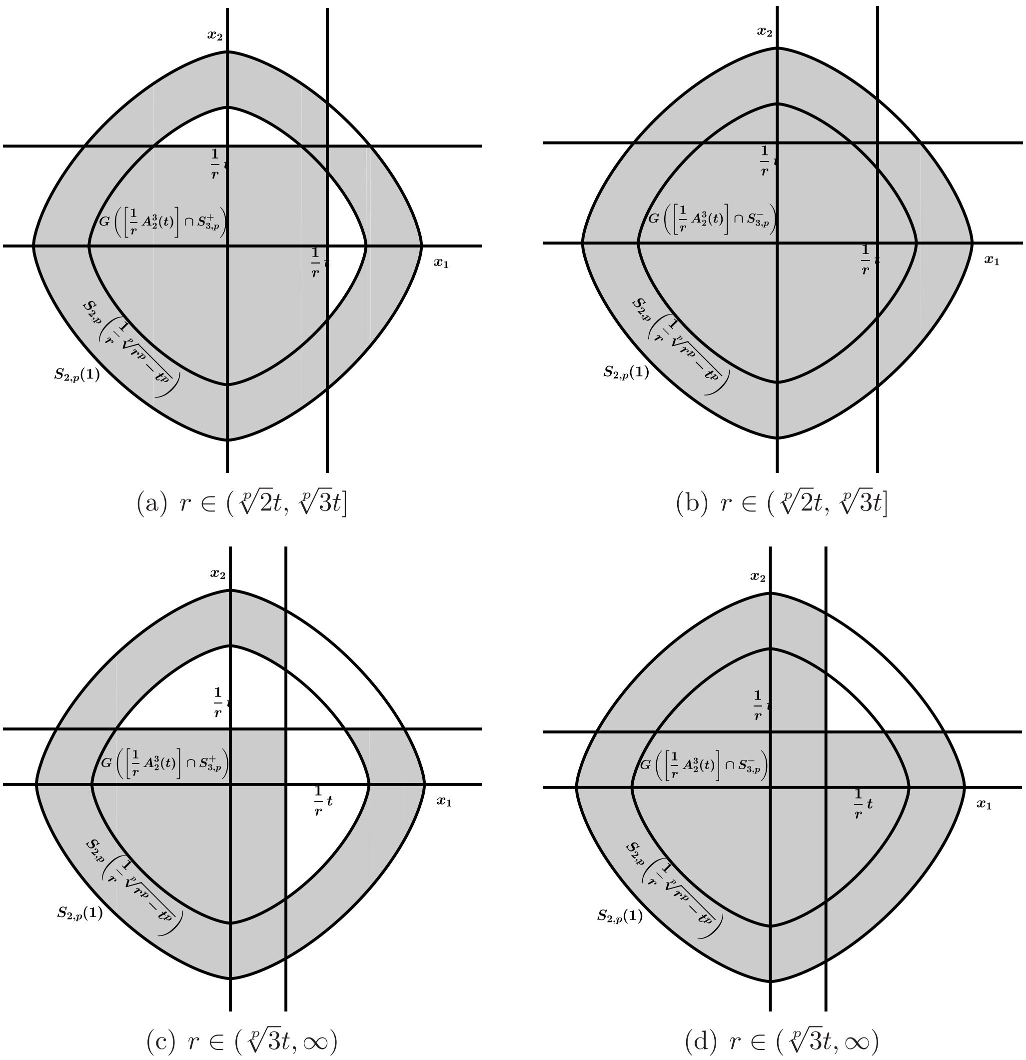


(c) $r \in (\sqrt[p]{3}|t|, \infty)$

(d) $r \in (\sqrt[p]{3}|t|, \infty)$

Figure 10: The sets $G\left(\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^+\right)$, on the left hand side, and $G\left(\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^-\right)$, on the right hand side, if $t \leq 0$.



(a) $r \in (\sqrt[p]{2}t,\, \sqrt[p]{3}t]$ (b) $r \in (\sqrt[p]{2}t,\, \sqrt[p]{3}t]$

(c) $r \in (\sqrt[p]{3}t, \infty)$ (d) $r \in (\sqrt[p]{3}t, \infty)$

Figure 11: The sets $G\left(\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^{+}\right)$, on the left hand side, and $G\left(\left[\frac{1}{r}A_2^3(t)\right] \cap S_{3,p}^{-}\right)$, on the right hand side, if $t > 0$.