Extreme value distributions for dependent jointly $l_{n,p}$-symmetrically distributed random variables

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Running headline: **Extremes of dependent variables**
Received:

ABSTRACT: A measure-of-cone representation of skewed continuous $l_{n,p}$-symmetric distributions, $n \in \mathbb{N}$, $p > 0$, is proved following the geometric approach known for elliptically contoured distributions. On this basis, distributions of extreme values of n dependent random variables are derived if the latter follow a joint continuous $l_{n,p}$-symmetric distribution. Light, heavy, and extremely far tails as well as tail indices are discussed, and new parameters of multivariate tail behavior are introduced.

Key words: measure-of-cone representation, p-generalized Laplace and Gaussian distributions, skewed $l_{n,p}$-symmetric distribution, tail index, light/ heavy center of distribution

1 Introduction

Extreme value statistics are of interest not only in probability theory and mathematical statistics, but also in many fields of natural sciences and technique. While Fortin and Clusel (2015) present numerous applications of extreme value statistics to physics, Shibata (1994), Majumdar and Krapivsky (2002), and Castillo (2012) deal with application to corrosion, computer science, and engineering, respectively. Furthermore, the appearance of extreme value statistics in finance, insurance and actuarial science is dealt with for instance in Embrechts, Klüppelberg, and Mikosch (1997) and Reiss and Thomas (1997). For an application to reliability theory, see Müller and Richter (2015b). General introductions into and surveys over the theory and practice of extreme value distributions are presented, among other, in David and Nagaraja (2003), Leadbetter, Lindgren, and Rootzén (1983), Galambos (1987), Pfeifer (1989), Reiss (1989), Reiss, Haßmann, and Thomas (1994), and Galambos, Lechner, and Simiu (1994).

Distributions of extreme value statistics of independent and identically distributed random variables (rvs) are already determined in Gumbel (1958). The case of correlated rvs with a joint normal distribution is dealt with, i.a., in Gupta and Pillai (1965), Nagaraja (1982) and Kella (1986).

The probability density functions (pdfs) of the maximum statistic and of linear combinations of order statistics of arbitrary absolutely continuous dependent rvs, with an emphasis on elliptically contoured sample distributions, are considered in Arellano-Valle and Genton (2008) and Arellano-Valle and Genton (2007). These considerations are followed up in Jamalizadeh and Balakrishnan (2010) and some paper referred to there, and further developed by representing the results with the help of skewed distributions. Earlier results dealing with this relationship can be found in Loperfido (2002) where the two-dimensional Gaussian case is considered. A geometric approach to bivariate and multivariate skewed elliptically contoured distributions is presented in Günzel, Richter, Scheutzow, Schicker, and Venz (2012) and Richter and Venz (2014), respectively, where the measure-of-cone representation of these distributions is worked out.

In Müller and Richter (2015a), some steps of the development of the theory of $l_{n,p}$-symmetric distributions and their applications are reviewed. An emphasis of this overview is on geometric measure representations and on a methodological study of their applications to the derivation of exact statistical distributions if samples follow a joint continuous $l_{n,p}$-symmetric distribution. Such exact results extend those valid for multivariate spherically symmetric sample distributions. Notice that results on exact extreme value distributions holding if the sample distribution is an arbitrary element of the larger class of elliptically contoured distributions may be again further extended assuming a p-generalized elliptically contoured sample distribution. The latter distributions appeared already in Section 3.5 in Arellano-Valle and Richter (2012) and were studied from the point of view of star-shaped distributions in Richter (2014) and from that of convex and radially concave contoured distributions in Richter (2015a). All these distribution classes provide more flexibility in modeling data. For more details, we refer to Section 2.

Skewed $l_{n,p}$-symmetric distributions are introduced in Arellano-Valle and Richter (2012) and applied to the maximum distribution of continuous $l_{2,p}$-symmetrically distributed random vectors in Batún-Cutz, González-Farías, and Richter (2013). From a certain point of view, the aim of the present paper is to extend this result to the finite number n of dependent rvs following a joint continuous $l_{n,p}$-symmetric distribution. To this end, a geometric approach to skewed $l_{n,p}$-symmetric distributions, following and generalizing main ideas of the measure-of-cone representations from Richter and Venz (2014) in the present situation, will be developed and, afterwards, used for the derivation of extreme value distributions. From another point of view, our present results extend, although in a slightly different form, those derived in Müller and Richter (2015a) for three or four dependent random variables to the case of an arbitrary finite number of dependent rvs following a joint continuous $l_{n,p}$-symmetric distribution. As an additional result of the present paper it becomes obvious that the explicit representations of distributions of extreme value statistics derived in the two earlier papers of the authors may be considered as representations of skewed distributions being alternatives to the known ones.

In order to represent maximum distributions in terms of skewed distributions, there are two basic approaches. On the one hand, the authors of Batún-Cutz et al. (2013) present a full-length proof of transforming the result on the maximum pdf from Müller and Richter (2015b) directly into the language of skewed distributions. To this end, they start from the Laplace and Gaussian cases, respectively, and extend the results, passing the two-dimensional p-power exponential case, stepwise in quick succession to the $l_{2,p}$-symmetric case with an arbitrary density generator (dg). On the other hand, the authors of Günzel et al. (2012) and Richter and Venz (2014) deduce a certain measure-of-cone representation of skewed elliptically contoured distributions.

If one compares the numerous applications of geometric measure representation done so far in the literature, one may distinguish between the direct and more advanced applications. To roughly define these terms, direct applications deal with immediate calculations of the so called intersection proportion function of a particular random event under consideration, and more advanced applications deal with types of intersection proportion functions being typical for whole classes of random events. While the small sample studies in the earlier papers of the authors belong to the direct type of applications of the geometric measure representation, the present paper deals with a more advanced one in Section 5.2.

The rest of the present paper is organized as follows. In Section 2, general information about the considered class of $l_{n,p}$-symmetric distributions and the corresponding class of skewed distributions are given. Based on the $l_{n,p}$-symmetric model assumption, in Section 3, the cumulative distribution function (cdf) and the pdf of extreme value statistics for a finite number of dependent rvs are considered. The density of the maximum is graphically illustrated for several choices of dgs of the $l_{3,p}$-symmetric sample distributions and for certain values of the shape/ tail parameter $p > 0$. Studying the asymptotic behavior of the pdf of the maximum statistic of the components of an n-dimensional p-generalized Gaussian distributed random vector is another aim of Section 3. In Section 4.1, the figures of the maximum pdf of three rvs following a joint $l_{3,p}$-symmetric Pearson Type VII or Kotz type distribution are discussed w.r.t. the heaviness of their tails. The tail index and two new parameters describing the tail behavior of some $l_{n,p}$-symmetric distributions, and the centers of $l_{3,5}$-symmetric Kotz type distributions are considered in Sections 4.2 and 4.3, respectively. In Section 5, first, a geometric measure-of-cone representation of skewed $l_{n,p}$-symmetric distributions is introduced leading afterwards to an advanced geometric method of proof. Second, based upon this, the results of Section 3 are proved and the advanced geometric method of proof is concisely compared to the direct one from Müller and Richter (2015a). In Section 6, some conclusions are drawn from the results of the present paper. Appendix A provides density generators of and some more basic facts on subclasses of $l_{n,p}$-symmetric distributions. Appendix B deals with the influence which parameters of density generators have onto the heaviness or lightness of multivariate distribution tails.

2 Preliminaries

In this section, the model class of $l_{n,p}$-symmetric distributions and the class of skewed $l_{n,p}$-symmetric distributions are introduced and some of their basic properties are recalled. Throughout this paper, let $p > 0$ be arbitrary but fixed and let $|x|_p = \left(\sum\limits_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$, $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, denote the p-functional which is a norm if $p \geq 1$ and, according to Moszyńska and Richter (2012), an antinorm if $p \in (0, 1)$. A function $g\colon (0, \infty) \to (0, \infty)$ satisfying the assumption $0 < I(g) < \infty$ is called a density generating function (dgf) of an n-variate distribution where $I(g) = \int\limits_0^\infty r^{n-1} g(r)\, dr$.

An random vector $X\colon \Omega \to \mathbb{R}^n$ defined on a probability space $(\Omega, \mathfrak{A}, P)$ and having the pdf

$$f_X(x) = \frac{g(|x|_p)}{\omega_{n,p}\, I(g)}, \quad x \in \mathbb{R}^n, \tag{1}$$

is said to follow the continuous $l_{n,p}$-symmetric distribution with dgf g where $\omega_{n,p} = \frac{\left(2\Gamma\left(\frac{1}{p}\right)\right)^n}{p^{n-1}\Gamma\left(\frac{n}{p}\right)}$ denotes the $l_{n,p}$-generalized surface content of the $l_{n,p}$-unit sphere $S_{n,p} = \{x \in \mathbb{R}^n\colon |x|_p = 1\}$, see Richter (2009).

Further, a dgf g of a continuous $l_{n,p}$-symmetric distribution meeting the condition $I(g) = \frac{1}{\omega_{n,p}}$ is called dg of this distribution, and denoted by $g^{(n)}$. We denote the cdf and the pdf of the corresponding distribution by $\Phi_{g^{(n)},p}$ and $\varphi_{g^{(n)},p}$, respectively, where

$$\varphi_{g^{(n)},p}(x) = g^{(n)}(|x|_p), \quad x \in \mathbb{R}^n. \tag{2}$$

For examples of dgs, for references to the literature on $l_{n,p}$-symmetric distributions, as well as for a discussion of a further aspect of notation, we refer to Müller and Richter (2015a) and the Appendix A, respectively.

Remark 1. According to Richter (2009), an $l_{n,p}$-symmetrically distributed random vector

X with dg $g^{(n)}$ satisfies the stochastic representation

$$X \stackrel{d}{=} R\,U_p^{(n)} \tag{3}$$

where $U_p^{(n)}$ is n-dimensional p-generalized uniformly distributed on the $l_{n,p}$-unit sphere $S_{n,p}$, R and $U_p^{(n)}$ are stochastically independent and R is a nonnegative random variable with pdf

$$f_R(r) = \omega_{n,p}\, r^{n-1} g^{(n)}(r), \quad r > 0. \tag{4}$$

Here and in what follows $X \stackrel{d}{=} Z$ and $X \sim \Psi$ means that the random vectors X and Z follow the same distribution law and that the random vector X follows the distribution law Φ, respectively. Moreover, let I_n be the $n \times n$ unit matrix and 0_n the zero vector in $\mathbb{R}^n$.

Remark 2. The density of the nonnegative random variable R^p is a $(g^{(n)}, p)$-generalization of the χ^2-density,

$$f^{\chi}_{g^{(n)},p}(y) = \frac{\omega_{n,p}}{p} y^{\frac{n}{p}-1} g^{(n)}\left(y^{\frac{1}{p}}\right) = \frac{y^{\frac{n}{p}-1} g^{(n)}\left(y^{\frac{1}{p}}\right)}{\int\limits_0^{\infty} \rho^{\frac{n}{p}-1} g^{(n)}\left(\rho^{\frac{1}{p}}\right) d\rho}, \quad y > 0.$$

Having in mind the slight change of notation from Richter (2014) and Müller and Richter (2015a), respectively, this distribution law was defined for arbitrary dgf g in Richter (2009) having pdf

$$f^{\chi}_{g,p}(y) = \frac{y^{\frac{n}{p}-1} g(y)}{p\int\limits_0^{\infty} r^{n-1} g(r^p)\, dr} = \frac{y^{\frac{n}{p}-1} g(y)}{\int\limits_0^{\infty} \rho^{\frac{n}{p}-1} g(\rho)\, d\rho}, \quad y > 0.$$

and was considered earlier in Richter (1991, 2007) for the particular cases $p = 2$ and $g^{(n)} = g^{(n)}_{PE}$, respectively.

Remark 3. Let $X \sim \Phi_{g^{(n)},p}$. According to Arellano-Valle and Richter (2012), $\mathbb{E}(X) = 0_n$ if $\mathbb{E}(R)$ is finite and $\mathrm{Cov}(X) = \mathbb{E}(XX^{\mathsf{T}}) = \sigma^2_{p,g^{(n)}} I_n$ if $\mathbb{E}(R^2)$ is finite where $\sigma^2_{p,g^{(n)}} = \tau_{n,p}\mathbb{E}(R^2)$ is called the univariate variance component and $\tau_{n,p} = \frac{\Gamma(\frac{3}{p})\Gamma(\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{n+2}{p})}$. Consequently, the components of X are uncorrelated. The components of X are only independent if $g^{(n)} = g^{(n)}_{PE}$, see Remark 5 and Appendix A.

As announced in Section 1, now, we further discuss the connection between $l_{n,p}$-symmetric and standardized multivariate elliptically contoured (or spherical) distributions. As it is well known according to Cambanis, Huang, and Simons (1981), the Euclidean stochastic representation of an n-dimensional with parameter vector $\mu \in \mathbb{R}^n$

and nonnegative definite matrix $\Sigma \in \mathbb{R}^{n\times n}$ having rank k elliptically contoured random vector Z is given by

$$Z \overset{d}{=} \mu + RU^{(k)}A \tag{5}$$

where $U^{(k)}$ is uniformly distributed on the unit sphere in $\mathbb{R}^k$,$R \geq 0$ is independent of $U^{(k)}$, $\Sigma = A^{\mathsf{T}}A$ is a rank factorization of Σ, and the cdf F of R is connected to the characteristic generator ϕ of Z by $\phi(u) = \int\limits_0^\infty \Omega_k(r^2u)\,dF(r)$, $u \geq 0$, where Ω_k denoted the characteristic function of $U^{(k)}$. In contrast to (5), a non-Euclidean stochastic representation of Z is given for regular Σ by

$$Z \overset{d}{=} \mu + RU \tag{6}$$

where R and U are independent, R^2 follows a g-generalized χ^2-distribution which is defined in Richter (1991), and U follows the ellipsoidal or $|\cdot|_{\frac{1}{a}}$-generalized uniform probability distribution introduced in Definition 3.2 in Richter (2013). A generalization of (6) is given in Section 4.7 in Richter (2014) where p-generalized elliptically contoured distributions are considered. Definition 8 in Section 4.4 of the same paper deals with a stochastic representation of even more general star-shaped distributions.

It is well known that some elliptical distributions can be obtained as normal variance mixtures. For the general case, see, e.g., McNeil, Frey, and Embrechts (2005), and for the particular case of the multivariate skew-t distribution, see Demarta and McNeil (2005). In Arslan and Genç (2003), a scale mixture expression of exponential power distributions is given for a generalized-t distribution defined in McDonald and Newey (1988). A general scale mixture of p-generalized normal distributions is dealt with in Section 3.3 in Arellano-Valle and Richter (2012) including the special case of p-generalized Student-t distribution.

For getting a first impression of the heaviness of the multivariate tails of $l_{n,p}$-symmetric distributions one may study typical values of the quantile function $\mathcal{Q}_{g^{(n)},p}\colon [0,1] \to \mathbb{R}$ w.r.t. the domain $B_{n,p} = \left\{x \in \mathbb{R}^n\colon\ |x|_p \leq 1\right\}$ defined by

$$\mathcal{Q}_{g^{(n)},p}(q) = \inf\left\{r > 0\colon \Phi_{g^{(n)},p}\left(r\,B_{n,p}\right) \geq q\right\}, \quad q \in [0,1]\,. \tag{7}$$

According to (3) and Remark 2,

$$\mathcal{Q}_{g^{(n)},p}(q) = F_R^{-1}(q), \quad q \in [0,1]\,.$$

where F_R has the density (4). In Appendix B, the quantiles $\mathcal{Q}_{g^{(n)},p}(q)$ are computed for several values of n, different dgs and $q \in \{0.9, 0.95, 0.99, 0.995, 0.999\}$.

According to the work of Loperfido (2002), Jamalizadeh and Balakrishnan (2010), Batún-Cutz et al. (2013), and other authors, distributions of extreme value statistics are intrinsically connected with certain skewed versions derived from the considered sample distributions. Skewed versions of $l_{n,p}$-symmetric distributions are studied in Arellano-Valle and Richter (2012). To follow these authors, let $X = \left(X^{(1)\mathsf{T}}, X^{(2)\mathsf{T}}\right)^{\mathsf{T}}$ be a random vector having a continuous $l_{k+m,p}$-symmetric distribution with dg $g^{(k+m)}$ where $X^{(1)}\colon \Omega \to \mathbb{R}^k$ and $X^{(2)}\colon \Omega \to \mathbb{R}^m$. We recall that, differing from (2), the density of X was represented in Arellano-Valle and Richter (2012) as $g^{(k+m)}(|x|_p^p)$. Taking, here and later, this minor change of notation into account, the dg $g^{(k)}_{(k+m)}$ the dg of the marginal distribution of $X^{(1)}$ in $\mathbb{R}^k$ satisfies

$$g^{(k)}_{(k+m)}(z) = \frac{\omega_{m,p}}{p} \int_{z^p}^{\infty} (y - z^p)^{\frac{m}{p}-1}\, g^{(k+m)}(\sqrt[p]{y})\, dy, \quad z \in (0, \infty).$$

Furthermore, for $\Lambda \in \mathbb{R}^{m\times k}$, $\Gamma = (\Lambda, -I_m)$, and $\Sigma = \Gamma\Gamma^{\mathsf{T}} = I_m + \Lambda\Lambda^{\mathsf{T}}$, the cdf of ΓX will be denoted by $F^{(2)}_{m,p}\left(x; \Sigma, g^{(m)}_{(k+m)}\right)$, $x \in \mathbb{R}^m$. Moreover, for every $x^{(1)} \in \mathbb{R}^k$, the conditional density of $X^{(2)}$ given $X^{(1)} = x^{(1)}$ is

$$\frac{g^{(k+m)}(\sqrt[p]{|x^{(1)}|_p^p + |x^{(2)}|_p^p})}{g^{(k)}_{(k+m)}(|x^{(1)}|_p)} = g^{(m)}_{[|x^{(1)}|_p]}(|x^{(2)}|_p), \quad x^{(2)} \in \mathbb{R}^m,$$

and the corresponding distribution law is $\Phi_{g^{(m)}_{[|x^{(1)}|_p]},p}$. Let Y be a random vector following this distribution, $Y \sim \Phi_{g^{(m)}_{[|x^{(1)}|_p]},p}$, then its cdf is

$$F^{(1)}_{m,p}\left(x; g^{(m)}_{[|x^{(1)}|_p]}\right) = \int_{\mathbb{R}^m_+} g^{(m)}_{[|x^{(1)}|_p]}\left(|x-u|_p\right)\, du, \quad x \in \mathbb{R}^m.$$

A k-dimensional random vector Z having a pdf of the form

$$f_Z(z) = \frac{1}{F^{(2)}_{m,p}(0_m; \Sigma, g^{(m)}_{(k+m)})} g^{(k)}_{(k+m)}\left(|z|_p\right) F^{(1)}_{m,p}\left(\Lambda z; g^{(m)}_{[|z|_p]}\right), \quad z \in \mathbb{R}^k, \tag{8}$$

is said to follow the skewed $l_{k,p}$-symmetric distribution $SS_{k,m,p}\left(\Lambda, g^{(k+m)}\right)$ with dimensionality parameter m, dg $g^{(k+m)}$ and skewness/ shape matrix-parameter Λ. Further, the parameter k is called the co-dimensionality parameter and the cdf of Z is denoted by $F_{k,m,p}\left(\cdot; \Lambda, g^{(k+m)}\right)$.

Notice that $F^{(1)}_{m,p}\left(x; g^{(m)}_{[|x^{(1)}|_p]}\right) = \int_{v<x} g^{(m)}_{[|x^{(1)}|_p]}(|v|_p)\, dv$, and that $F^{(2)}_{m,p}\left(0_m; \Sigma, g^{(m)}_{(k+m)}\right) = 2^{-m}$ if Σ is diagonal. The following remark deals with the effects of interchanging columns or rows in the matrix-parameter Λ and is proven in Section 5.1.

Remark 4. a) Let M_1 be a $k \times k$ permutation matrix and $Z \sim SS_{k,m,p}\left(\Lambda, g^{(k+m)}\right)$. Then $M_1 Z \sim SS_{k,m,p}\left(\Lambda M_1^{\mathsf{T}}, g^{(k+m)}\right)$ where ΛM_1^{T} arises from Λ by interchanging columns.

b) Let M_2 be a $m \times m$ permutation matrix. Then for $z \in \mathbb{R}^k$, $F_{k,m,p}\left(z; M_2\Lambda, g^{(k+m)}\right) = F_{k,m,p}\left(z; \Lambda, g^{(k+m)}\right)$, i.e. $SS_{k,m,p}\left(\Lambda, g^{(k+m)}\right) = SS_{k,m,p}\left(M_2\Lambda, g^{(k+m)}\right)$ where $M_2\Lambda$ arises from Λ by interchanging rows.

According to Arellano-Valle and Richter (2012), skewed $l_{k,p}$-symmetric distributions are constructed via selection mechanisms from $l_{n,p}$-symmetric distributions. Particularly, if $X^{(1)}: \Omega \to \mathbb{R}^k$ and $X^{(2)}: \Omega \to \mathbb{R}^m$ are again two subvectors of a random vector X, $X \sim \Phi_{g^{(k+m)},p}$, then

$$\mathfrak{L}\left(X^{(1)} \,\middle|\, X^{(2)} < \Lambda X^{(1)}\right) = SS_{k,m,p}\left(\Lambda, g^{(k+m)}\right) \tag{9}$$

where $\mathfrak{L}(Y)$ denotes the distribution law of the random vector Y. Therefore, part b) of Remark 4 reflects the exchangeability of the components of an $l_{n,p}$-symmetrically distributed random vector within the skewed $l_{k,p}$-symmetrical distributions.

3 Extreme value distributions for arbitrary finite sample sizes

We recall that specific results for exact distributions of order statistics of up to three and extreme value statistics up to four dependent rvs following a joint continuous $l_{n,p}$-symmetric distribution, $n \in \{2, 3, 4\}$, are proved in earlier papers of the authors by directly applying the geometric measure representation of $l_{n,p}$-symmetric distributions. In Section 3.1, exact extreme value distributions are derived if an arbitrary finite number of dependent rvs follows a continuous $l_{n,p}$-symmetric distribution. To this end, an advanced geometric method of proof will be developed in Section 5 following main ideas for geometrically representing skewed elliptically contoured distributions in Richter and Venz (2014). In Section 3.2, our results are graphically visualized. On the one hand, figures of densities are drawn for the special case of jointly trivariate 3-generalized Gaussian distributed rvs and, on the other hand, for the case of three dependent rvs following a joint $l_{3,p}$-symmetric Kotz type and a joint $l_{3,p}$-symmetric Pearson Type VII distribution, respectively. Another aim of this section is to provide an idea of the asymptotic behavior of the maximum pdf for n-variate p-generalized Gaussian sample distribution as n tends to infinity.

3.1 Dimension and co-dimension representations

The results of this section reflect strong connections between skewed distributions and distributions of extremes. Such type of connection can already be seen in Loperfido (2002), Jamalizadeh and Balakrishnan (2010), and in papers of several other authors. The present results are derived based upon the geometric measure representation in Richter (2009). This representation applies directly if only small numbers of dependent rvs are considered. In Batún-Cutz et al. (2013), the particular result on the maximum pdf of two dependent rvs being jointly $l_{2,p}$-symmetrically distributed is transformed directly into the typical representation of skewed distributions. Here, we use the geometric representation of $l_{n,p}$-symmetric measures in a more advanced way. To be more concrete, we derive from it a measure-of-cone representation of skewed $l_{k,p}$-symmetric distributions in Corollary 2, Section 5.

Let $E^{(\nu)}$ denote the $\nu \times (n-\nu)$ matrix whose 1st column is $1_\nu = (1, \dots, 1)^{\mathsf{T}} \in \mathbb{R}^\nu$ and whose remaining $n - \nu - 1$ columns are ν-dimensional zero vectors.

Theorem 1. *If $X \sim \Phi_{g^{(n)},p}$, for every $\nu \in \{1, \dots, n-1\}$, the cdf $F_{n:n}$ of the maximum statistic of the components of X satisfies the representation*

$$F_{n:n}(t) = (\nu+1) F^{(2)}_{\nu,p}\left(0_\nu; \Sigma, g^{(\nu)}_{(n)}\right) \cdot F_{n-\nu,\nu,p}\left(t 1_{n-\nu}; E^{(\nu)}, g^{(n)}\right), \quad t \in \mathbb{R}, \tag{10}$$

where $\Sigma = I_\nu + E^{(\nu)} E^{(\nu)\mathsf{T}} = I_\nu + 1_\nu 1_\nu^{\mathsf{T}}$.

For arbitrary $n \in \mathbb{N}$ and $p > 0$, Theorem 1 provides numerous representations of the cdf of the maximum statistic from $l_{n,p}$-symmetrically distributed populations in terms of skewed distributions. In particular, these are alternatives to that given in Theorem 1 in Müller and Richter (2015b) for $n = 2$ and Theorems 1 and 3 in Müller and Richter (2015a) for $n = 3$ and $n = 4$, respectively. In the case of $n = 2$, the equivalence of these two alternative representations is shown in Batún-Cutz et al. (2013) by direct integral transformation. Furthermore, in the specific case $p = 2$ and $\nu = n - 1$, the result of Theorem 1 is covered by Theorem 7 in Jamalizadeh and Balakrishnan (2010).

Now, we briefly discuss the impact of the parameter $\nu \in \{1, \dots, n-1\}$ in Theorem 1. Recalling that $F_{n-\nu,\nu,p}\left(t 1_{n-\nu}; E^{(\nu)}, g^{(n)}\right)$ is the cdf of $SS_{n-\nu,\nu,p}\left(E^{(\nu)}, g^{(n)}\right)$ and considering the construction of skewed $l_{n-\nu,p}$-symmetric distributions via select mechanisms again, $X^{(1)}$ and $X^{(2)}$ in equation (9) are real-valued $(n-\nu)$- and ν-dimensional subvectors of an $l_{n,p}$-symmetrically with dg $g^{(n)}$ distributed random vector X, respectively. Hence, the parameter ν defines the dimensionality of the conditioning subvector in equation (9) and, implicitly, the dimension of the skewed $l_{k,p}$-symmetric distribution which is used to represent the maximum cdf in Theorem 1. Especially for $\nu = 1$, $E^{(1)} = e_1^{(n-1)\mathsf{T}}$ where $e_j^{(n-1)}$ denotes the jth unit vector of $\mathbb{R}^{n-1}$, and the matrix

$I_1 + 1_1 1_1^{\mathsf{T}} = (2)$ is diagonal. Thus, $F_{1,p}^{(2)}\left(0; I_1 + 1_1 1_1^{\mathsf{T}}, g^{(1)}\right) = \frac{1}{2} = \frac{1}{\nu+1}$ and equation (10) reads as $F_{n:n}(t) = F_{n-1,1,p}\left(t1_{n-1}; e_1^{(n-1)^{\mathsf{T}}}, g^{(n)}\right)$, $t \in \mathbb{R}$. Therefore, maximum distributions for continuous $l_{n,p}$-symmetric vectors may particularly be represented as skewed $l_{n-1,p}$-symmetric distributions. For all the other parameters $\nu \in \{2, \ldots, n-1\}$, the matrix $I_\nu + 1_\nu 1_\nu^{\mathsf{T}}$ is not diagonal. Due to this, the normalizing constant $F_{\nu,p}^{(2)}\left(0_\nu; I_\nu + 1_\nu 1_\nu^{\mathsf{T}}, g_{(n)}^{(\nu)}\right)$ is not as easy to handle with as in the case $\nu = 1$ and, in general, its exact value is unknown. Nevertheless, all representations of the maximum cdf given in (10) are well treatable since the corresponding maximum pdfs, see Corollary 1, do not depend on the mentioned normalizing constant.

Vice versa, it may be sometimes of interest to read equation (10) in a reverse order, meaning that for every $\nu \in \{1, \ldots, n-1\}$, the cdf of Z, $Z \sim SS_{n-\nu,\nu,p}(E^{(\nu)}, g^{(n)})$, at the particular argument $t1_{n-\nu}$, $t \in \mathbb{R}$, satisfies

$$F_{n-\nu,\nu,p}\left(t1_{n-\nu}; E^{(\nu)}, g^{(n)}\right) = \frac{1}{(\nu+1) F_{\nu,p}^{(2)}\left(0_\nu; \Sigma, g_{(n)}^{(\nu)}\right)} F_{n:n}(t), \quad t \in \mathbb{R}, \tag{11}$$

where $\Sigma = I_\nu + 1_\nu 1_\nu^{\mathsf{T}}$ and $F_{n:n}$ denotes the cdf of the maximum statistic of the components of X, $X \sim \Phi_{g^{(n)},p}$.

Using the direct application of the geometric measure representation shortly discussed in Section 1, the cdf $F_{n:n}$ is already determined for $n \in \{2, 3, 4\}$ in earlier papers of the authors. Thus, by substituting $F_{n:n}$ on the right hand side of equation (11) for $n \in \{2, 3, 4\}$ by these previous representations, one gets alternative representations of the cdf $F_{n-\nu,\nu,p}\left(t1_{n-\nu}; E_i^{(\nu)}, g^{(n)}\right)$, $t \in \mathbb{R}$, to that following from Section 2 as the componentwise defined integral of the pdf of $SS_{n-\nu,\nu,p}(E_i^{(\nu)}, g^{(n)})$ over the region $\{z \in \mathbb{R}^{n-\nu} \colon z < t1_{n-\nu}\}$.

Summarizing this section up to here, there are $n-1$ equivalent possibilities of representing the maximum cdf $F_{n:n}$ using skewed distributions. This effect also occurs in the representation of the maximum pdf $f_{n:n}$ as it can be seen in the following corollary being proved in Section 5.2.

Corollary 1. *Let $X \sim \Phi_{g^{(n)},p}$. For every $t \in \mathbb{R}$ and $\nu \in \{1, \ldots, n-2\}$,*

$$\begin{aligned} f_{n:n}(t) = (\nu+1)(n-\nu-1) &\int_{z \in D(t)} g_{(n)}^{(n-\nu)}\left(\sqrt[p]{|t|^p + |z|_p^p}\right) F_{\nu,p}^{(1)}\left(z_1 1_\nu; g^{(\nu)}_{[\sqrt[p]{|t|^p + |z|_p^p}]}\right) dz \\ &+ (\nu+1) \int_{z \in D(t)} g_{(n)}^{(n-\nu)}\left(\sqrt[p]{|t|^p + |z|_p^p}\right) F_{\nu,p}^{(1)}\left(t1_\nu; g^{(\nu)}_{[\sqrt[p]{|t|^p + |z|_p^p}]}\right) dz \end{aligned}$$

where $D(t) = \{z \in \mathbb{R}^{n-\nu-1} \colon z < t1_{n-\nu-1}\}$. Moreover,

$$f_{n:n}(t) = n \cdot g_{(n)}^{(1)}(|t|) \cdot F_{n-1,p}^{(1)}\left(t1_{n-1}; g_{[|t|]}^{(n-1)}\right), \quad t \in \mathbb{R}. \tag{12}$$

Using the general relation between maximum and minimum statistics if the sample distribution is symmetric and continuous, and the functions provided by Theorem 1 and Corollary 1, the minimum cdf and pdf of the components of X, $X \sim \Phi_{g^{(n)},p}$, are given by $F_{1:n}(t) = 1 - F_{n:n}(-t)$ and $f_{1:n}(t) = f_{n:n}(-t)$, respectively. In particular, for $\nu = n-1$, this yields

$$f_{1:n}(t) = n \cdot g^{(1)}(|t|) \cdot F^{(1)}_{n-1,p}\left(-t1_{n-1}; g^{(n-1)}_{[|t|]}\right), \quad t \in \mathbb{R}. \tag{13}$$

In due consideration of the mentioned slight variation of notation for $l_{n,p}$-symmetric densities, formula (12) equals for $n = 2$ the result in Batún-Cutz et al. (2013). It is worthwhile to note that the structure of our results on extremes of several dependent variables in (12) and (13) is similar to that of the corresponding results of the pdf of the maximum and the minimum statistic, respectively, of n independent and identically distributed rvs, see David and Nagaraja (2003).

Remark 5. Let X be n-dimensional p-generalized Gaussian distributed, $X \sim \Phi_{g^{(n)}_{PE},p}$ with

$$g^{(n)}_{PE}(r) = \left(\frac{p^{1-\frac{1}{p}}}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \exp\left\{-\frac{r^p}{p}\right\}, \quad r > 0.$$

Further, let $\varphi_p(t) = g^{(1)}_{PE}(|t|)$, $t \in \mathbb{R}$, and $\Phi_p(t) = \int\limits_{-\infty}^{t} \varphi_p(s)\, ds$, $t \in \mathbb{R}$, denote the pdf and the cdf of the one-dimensional marginal distribution, respectively. The pdfs (12) and (13) of the extreme value statistics of the components of X simplify to

$$f_{n:n}(t) = n \cdot \varphi_p(t) \cdot (\Phi_p(t))^{n-1}, \quad t \in \mathbb{R},$$

and

$$f_{1:n}(t) = n \cdot \varphi_p(t) \cdot (1 - \Phi_p(t))^{n-1}, \quad t \in \mathbb{R},$$

respectively. Note that, in these specific cases, our representations (12) and (13) also follow from David and Nagaraja (2003) since the components of p-generalized Gaussian distributed random vectors are independent.

3.2 Visualization of the maximum density

In the present section, the pdf of the maximum statistic of dependent, jointly $l_{n,p}$-symmetrically distributed rvs is illustrated for some choices of the dg $g^{(n)}$ and the shape/ tail parameter $p > 0$. Figures 1-3 show the maximum pdf of the components of a three-dimensional p-generalized Gaussian distributed random vector, i.e. choosing the dg $g_{PE}^{(3)}$, accompanied by histogram plots of samples of sizes from 10^3 up to 2.5×10^5 for $p \in \left\{\frac{1}{2}, 1, 3\right\}$. These Figures do not only demonstrate the numerical correctness of our evaluations but it also indicate in a certain rough sense how large sample sizes should be when simulating extreme value densities under non-standard model assumptions.

Figures 4 and 5 visualize the pdf of the maximum statistic of the components of $l_{3,p}$-generalized Kotz type and Pearson Type VII, respectively, distributed random vectors for different values of the parameter p, $p > 0$. The definitions of the dgs $g_{Kt;M,\beta,\gamma}^{(n)}$ and $g_{PT7;M,\nu}^{(n)}$ of these subclasses of the $l_{n,p}$-symmetric distributions are given in the Appendix A. Note that the present choice of the shape/ tail parameter p and of the parameters appearing in the definitions of the dgs coincides with that in Figures 2 and 3 in Müller and Richter (2015a). Thus, one can compare the graphs of the pdf of the maximum statistics presented here with that of the median statistic drawn there.

Another aim of this section is to give a visual impression of the asymptotic behavior of the pdf of the maximum statistic for increasing sample sizes. This will be done in n-dimensional p-generalized normally distributed populations, i.e. in the case of independent components. Using the dg $g_{PE}^{(n)}$ and the four choices of the parameter p as in Figures 4 and 5, the impact of an increasing sample size onto the shape of the maximum pdf is reflected in Figure 6. Furthermore, in this figure, one can perceive the impact of parameter p which is, on the one hand, a shape parameter and, on the other hand, a tail parameter since the shape of the multivariate density level sets depends on it and the tail pdf of the underlying random vector becomes lighter if p increases, respectively.

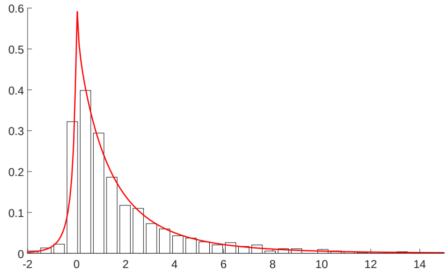
Note that the axes in Figures 4, 5 and 6 are scaled differently, and that both the left and the right hand sides of Figures 4(a), 5(a), 5(b) and 6(a) show a black graph as a respective benchmark. Moreover, note that illustrations of the minimum pdf can be received if the graphs of the corresponding maximum pdfs are mirrored at the ordinate axis.

4 Tails and centers of maximum distributions

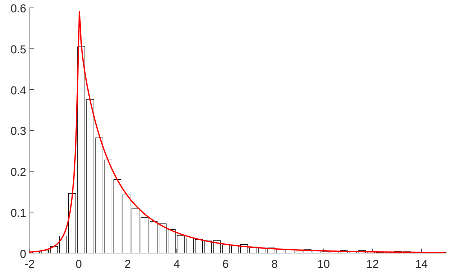
In this section, we review some of the figures of Section 3.2 in detail. Moreover, we give additional information on the tail index and on heaviness and on lightness of tails of $l_{n,p}$-symmetric distributions.

4.1 Light, heavy and extremely far tails

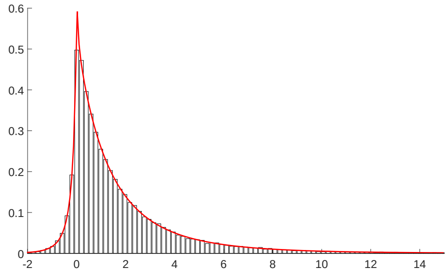
The influence that the parameter $\nu > 0$ of an $l_{3,\frac{1}{2}}$-symmetric Pearson Type VII distribution with parameter $M = \frac{13}{2}$ has onto the heaviness of the tails of the median distribution



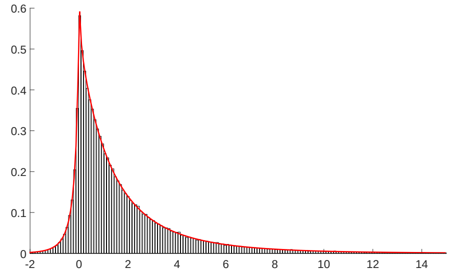
(a) sample size 10^3



(b) sample size 10^4

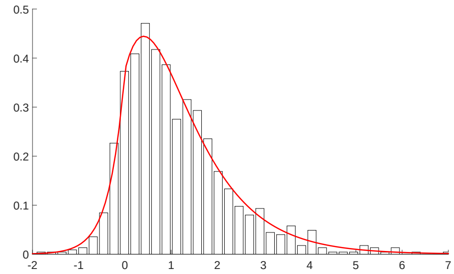


(c) sample size 10^5



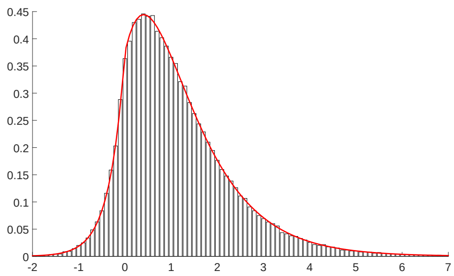
(d) sample size 2.5×10^5

Figure 1: Maximum pdf $f_{3:3}$ and histogram for $p = \frac{1}{2}$, increasing sample sizes and dg $g^{(3)}_{PE}$.

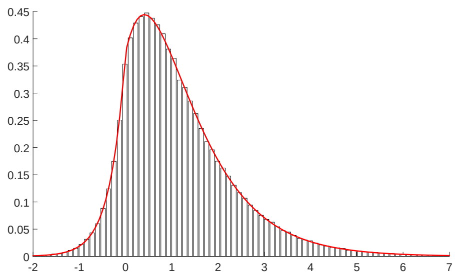


(a) sample size 10^3

(b) sample size 10^4

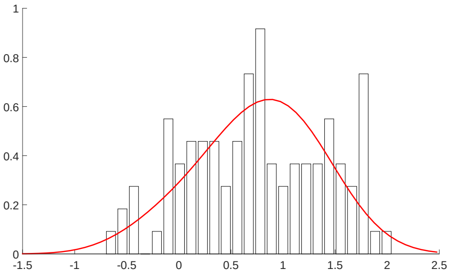


(c) sample size 10^5

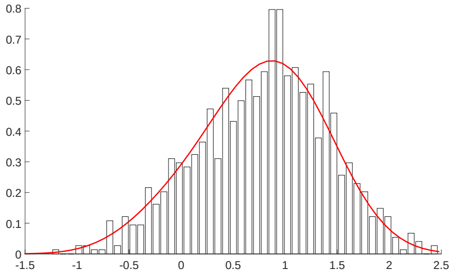


(d) sample size 2.5×10^5

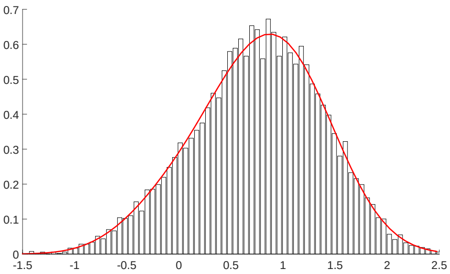
Figure 2: Maximum pdf $f_{3:3}$ and histogram for $p = 1$, increasing sample sizes and dg $g^{(3)}_{PE}$.



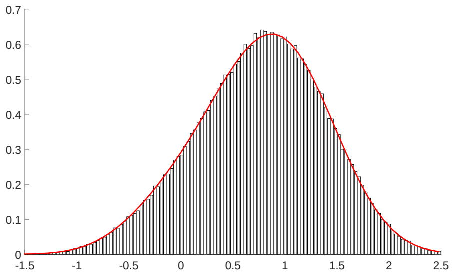
(a) sample size 10^3



(b) sample size 10^4

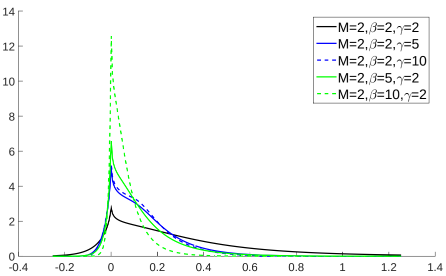


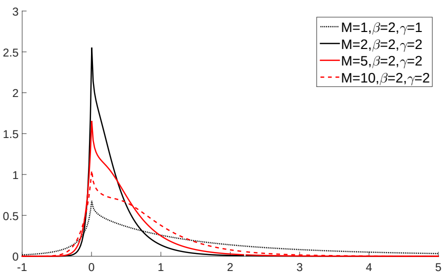
(c) sample size 10^5



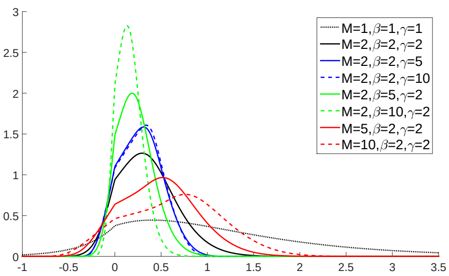
(d) sample size 2.5×10^5

Figure 3: Maximum pdf $f_{3:3}$ and histogram for $p = 3$, increasing sample sizes and dg $g_{PE}^{(3)}$.

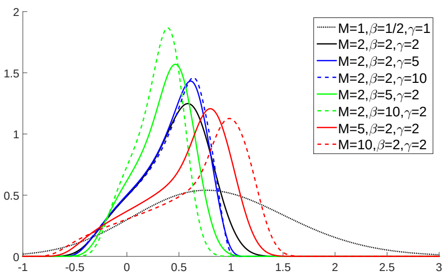




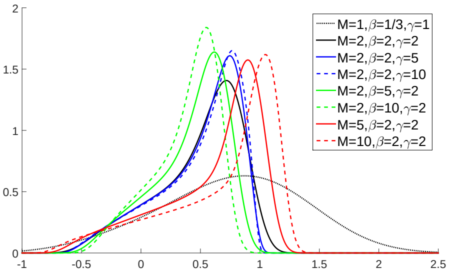
(a) $p = \frac{1}{2}$



(b) $p = 1$

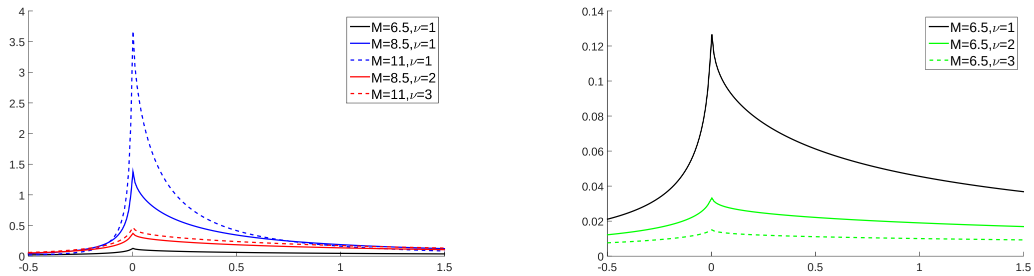


(c) $p = 2$

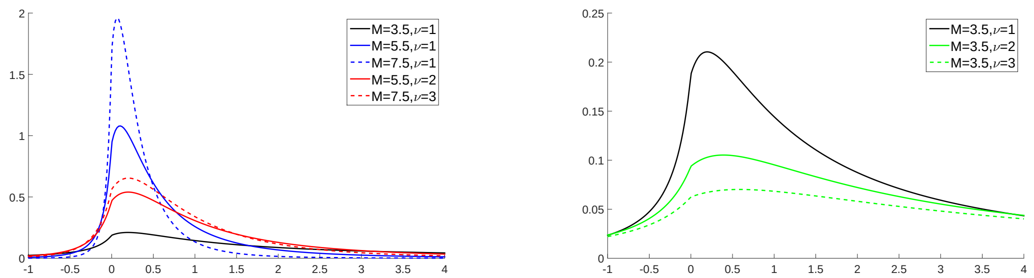


(d) $p = 3$

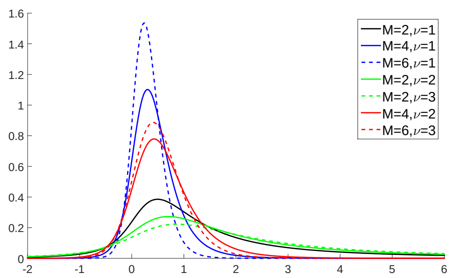
Figure 4: Maximum pdf $f_{3:3}$ for $p \in \left\{\frac{1}{2}, 1, 2, 3\right\}$, dg $g^{(3)}_{Kt;M,\beta,\gamma}$, and several choices of the parameters $M > 1 - \frac{3}{p}$, $\beta > 0$, and $\gamma > 0$ (black dashed: the special case of trivariate p-generalized Gaussian distribution).



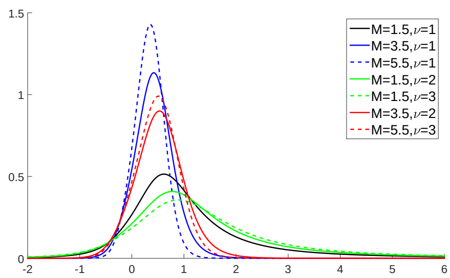
(a) $p = \frac{1}{2}$



(b) $p = 1$



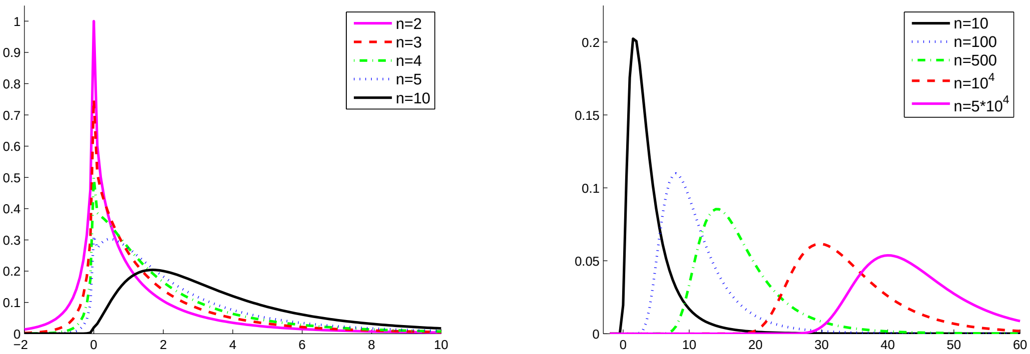
(c) $p = 2$



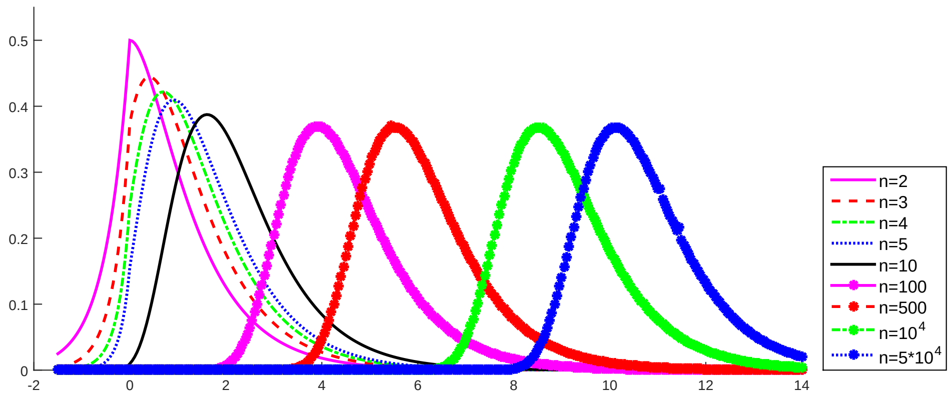
(d) $p = 3$

Figure 5: Maximum pdf $f_{3:3}$ for $p \in \left\{\frac{1}{2}, 1, 2, 3\right\}$, dg $g^{(3)}_{PT7;M,\nu}$, and several choices of the parameters $M > \frac{3}{p}$ and $\nu > 0$.

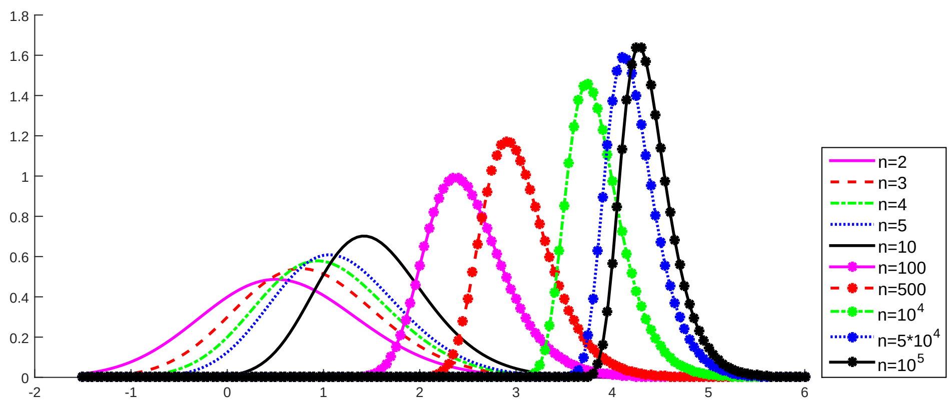
of three dependent rvs is discussed in Section 4 of Müller and Richter (2015a). In the present section, first, an analog study for the case of the maximum distribution in such populations is done. Second, we examine the heaviness of the tails of the maximum distribution from other $l_{3,p}$-symmetric sample distributions with respect to their parameters.



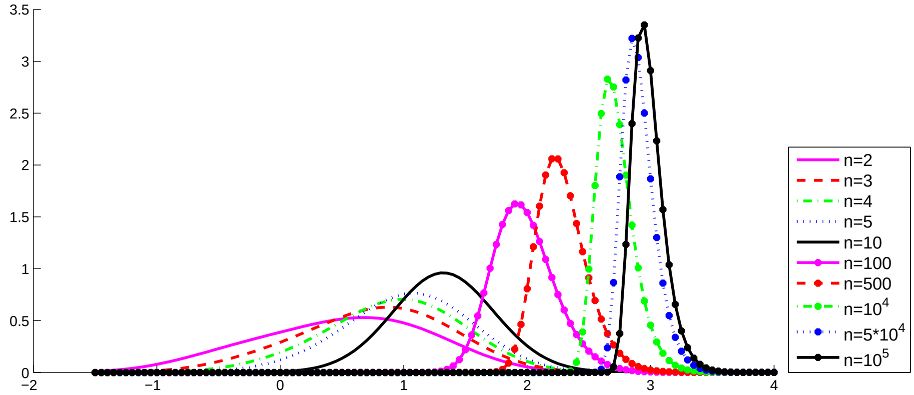
(a) $p = \frac{1}{2}$



(b) $p = 1$



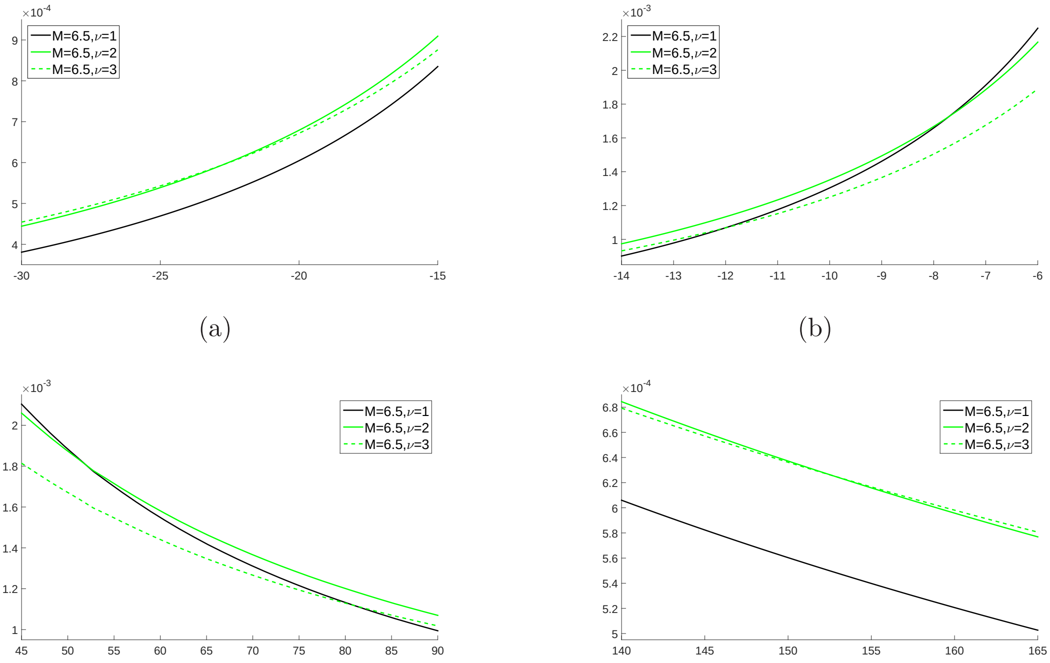
(c) $p = 2$



(d) $p = 3$

Figure 6: Maximum pdf $f_{n:n}$ for different n and dg $g_{PE}^{(n)}$ if $p \in \left\{\frac{1}{2}, 1, 2, 3\right\}$.

Note that the graphs of the right hand side of Figure 5(a) convey the impression that the visualized densities build a monotonically decreasing sequence of functions. The more detailed views in Figure 7, however, show the regions of intersection between the black and the green solid, the black and the green dashed, and the green solid and the green dashed graphs, being approximately $\{-8; 52\}$, $\{-12; 81\}$, and $\{-23; 153\}$, respectively.



(a)

(b)

(c)

(d)

Figure 7: Some detailed views on the graphs of the right hand side of Figure 5(a).

Correcting a potentially misleading impression coming from considering just the restricted central part of the densities in Figure 5 thus makes it necessary to study the far tails of the same distributions. Particularly, Figure 7 emphasizes the increasing heaviness of tails of the distribution of the maximum statistic of three dependent rvs following a joint $l_{3,\frac{1}{2}}$-symmetric distribution with dg $g^{(3)}_{PT7;\frac{13}{2},\nu}$ if ν increases. The same tendency can be seen in the case of $l_{3,1}$-symmetric Pearson Type VII sample distribution with constant parameter $M = \frac{7}{2}$ and increasing parameter ν. In Table 1, the integral $A^p_{M,\nu}(z_1, z_2) = \int\limits_{z_1}^{z_2} f_{3:3}(t)\, dt$ of the maximum pdf $f_{3:3}$ over some asymmetric intervals $[z_1, z_2]$ is numerically computed for $l_{3,\frac{1}{2}}$-symmetric Pearson Type VII distributed populations with parameters $(p, M) \in \left\{\left(\frac{1}{2}, \frac{13}{2}\right), \left(1, \frac{7}{2}\right)\right\}$ and $\nu \in \{1, 2, 3\}$ to get a more detailed numerical impression of the tendency of heaviness of tails.

$A^p_{M,\nu}(z_1, z_2)$	z_1 z_2	$-\frac{1}{2}$ $\frac{3}{2}$	$-\frac{5}{2}$ 10	-15 100	-50 10^3	-500 10^4	-5×10^3 10^5	-10^4 5×10^5	-5×10^4 10^6
$p = \frac{1}{2}$	$\nu = 1$	0.1070	0.2704	0.5109	0.6968	0.8255	0.9011	0.9311	0.9443
$M =$	$\nu = 2$	0.0407	0.1407	0.3628	0.5849	0.7559	0.8607	0.9027	0.9213
$\frac{13}{2}$	$\nu = 3$	0.0211	0.0858	0.2747	0.5071	0.7042	0.8300	0.8811	0.9037
$p = 1$	$\nu = 1$	0.2940	0.6461	0.8780	0.9569	0.9864	0.9957	0.9978	0.9986
$M =$	$\nu = 2$	0.1766	0.5252	0.8286	0.9391	0.9807	0.9939	0.9969	0.9981
$\frac{7}{2}$	$\nu = 3$	0.1245	0.4458	0.7915	0.9255	0.9764	0.9925	0.9962	0.9976

Table 1: Heaviness of the extremely far reaching tails of maximum distribution in jointly $l_{3,p}$-symmetrically Pearson Type VII distributed populations with $(p, M) \in \left\{\left(\frac{1}{2}, \frac{13}{2}\right), \left(1, \frac{7}{2}\right)\right\}$ and $\nu \in \{1, 2, 3\}$.

For the sake of comparison of heaviness of the tails of the median and the maximum distribution, our present consideration is for the class of $l_{3,\frac{1}{2}}$-symmetric Pearson Type VII distributions with the same different values of $\nu > 0$ and $M = \frac{13}{2}$ as in Section 4 in Müller and Richter (2015a). Equally, interpreting the values in Table 1, one can concentrate on one of Figures 5(c) and 5(d) and check the same effect of increase of heaviness of tails if the parameter $M > \frac{3}{p}$ is constant and the parameter $\nu > 0$ increases.

According to Table 1, all the three cases in Figure 5(a) cover only a small part of the entire probability mass. Therefore, the behavior of the graphs outside the considered interval $[-0.5; 1.5]$ is not clearly predictable. However, for the case of $\nu = 3$, Figure 8 shows the pdf of the maximum statistic of rvs following a joint $l_{3,\frac{1}{2}}$-symmetric distribution with dg $g^{(3)}_{PT7;\frac{13}{2},3}$ over the interval $[-200; 1000]$, suggesting a monotonically increasing behavior over the negative real line and a monotonically decreasing one over the positive real line. Note that only an extremely small proportion of probability mass generates a peak of the density function close right to the zero point and that the overwhelming part of probability mass is seemingly uniformly distributed on an extremely long interval.

Additionally, in Table 2, the integral $A^p_{M,1}(z_1, z_2)$ is numerically evaluated for $p = 2$ and $M \in \{2, 4, 6\}$ and $p = 3$ and $M \in \left\{\frac{3}{2}, \frac{7}{2}, \frac{11}{2}\right\}$, respectively and several asymmetric real intervals $[z_1, z_2]$ where "≈ 1" denotes the case that the value of $A^p_{M,1}(z_1, z_2)$ rounded

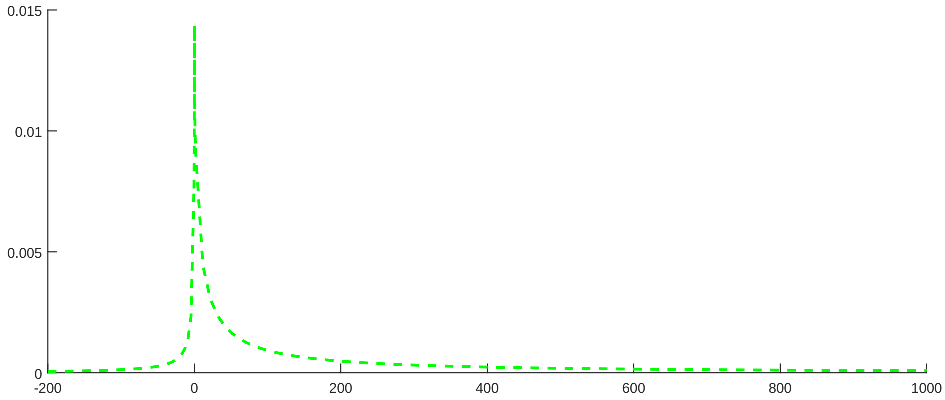


Figure 8: Maximum pdf $f_{3:3}$ for dg $g^{(3)}_{PT7;\frac{13}{2},3}$ over the interval $[-200; 1000]$.

to the sixth decimal place equals 1. These values emphasize a decreasing heaviness of the tails of the maximum distribution in jointly $l_{3,p}$-symmetrically Pearson Type VII distributed samples if the parameter $M > \frac{3}{p}$ increases and the parameter $\nu > 0$ is constant.

$A^p_{M,1}(z_1, z_2)$		$z_1 = -\frac{1}{4}$, $z_2 = 1$	$z_1 = -\frac{1}{2}$, $z_2 = \frac{3}{2}$	$z_1 = -\frac{3}{4}$, $z_2 = 2$	$z_1 = -1$, $z_2 = \frac{5}{2}$	$z_1 = -\frac{3}{2}$, $z_2 = \frac{9}{2}$	$z_1 = -2$, $z_2 = 6$	$z_1 = -3$, $z_2 = 9$
$p = 2$	$M = 2$	0.389589	0.539507	0.636473	0.701791	0.823962	0.866781	0.910605
	$M = 4$	0.867520	0.964969	0.988921	0.995823	0.999661	0.999914	0.999988
	$M = 6$	0.963251	0.996455	0.999557	0.999925	0.999999	≈ 1	
$p = 3$	$M = \frac{3}{2}$	0.492762	0.681533	0.785176	0.844627	0.932018	0.955714	0.975857
	$M = \frac{7}{2}$	0.900511	0.985822	0.997757	0.999554	0.999984	0.999998	≈ 1
	$M = \frac{11}{2}$	0.965001	0.997886	0.999897	0.999994	≈ 1		

Table 2: Heaviness of the tails of maximum distribution in jointly $l_{3,p}$-symmetrically Pearson Type VII distributed populations with $(p, M) \in \left\{(2,2), (2,4), (2,6), \left(3, \frac{3}{2}\right), \left(3, \frac{7}{2}\right), \left(3, \frac{11}{2}\right)\right\}$ and $\nu = 1$.

Finally, Table 3, shows different heaviness of the tails of the distribution of the maximum statistic of three jointly $l_{3,1}$-symmetric Kotz type distributed rvs for the choices of parameters from Figure 4(b) by numerically computing the integral $B_{M,\beta,\gamma}(z_1, z_2) = \int\limits_{z_1}^{z_2} f_{3:3}(t)\, dt$ of the maximum pdf in such populations. This suggests that the heaviness of the tails of the maximum distribution decreases if either γ or β increases and increases if M increases where, in each of the cases, the other two parameters are constant.

4.2 Tail indices

While the tail of the distribution of the univariate maximum statistic in jointly $l_{3,p}$-symmetrically Pearson Type VII or Kotz type distributed samples is explored in Section

$B_{M,\beta,\gamma}(z_1,z_2)$	$z_1=-0.1$ $z_2=0.5$	$z_1=-0.15$ $z_2=0.75$	$z_1=-0.2$ $z_2=1$	$z_1=-0.35$ $z_2=1.5$	$z_1=-0.5$ $z_2=2$	$z_1=-1$ $z_2=3.5$
$M=1,\ \beta=1,\ \gamma=1$	0.245620	0.365919	0.474857	0.657514	0.782533	0.949161
$M=2,\ \beta=2,\ \gamma=2$	0.649731	0.859058	0.950510	0.996536	0.999853	≈ 1
$M=2,\ \beta=2,\ \gamma=5$	0.777509	0.954045	0.991331	0.999968	≈ 1	
$M=2,\ \beta=2,\ \gamma=10$	0.780351	0.961325	0.992789	0.999999	≈ 1	
$M=2,\ \beta=5,\ \gamma=2$	0.880293	0.977946	0.995712	0.999962	≈ 1	
$M=2,\ \beta=10,\ \gamma=2$	0.970380	0.997038	0.999646	≈ 1		
$M=5,\ \beta=2,\ \gamma=2$	0.456971	0.703103	0.865723	0.984454	0.998985	≈ 1
$M=10,\ \beta=2,\ \gamma=2$	0.312513	0.507916	0.702702	0.935478	0.992512	≈ 1

Table 3: Heaviness of the tails of maximum distribution in jointly $l_{3,p}$-symmetrically Kotz type distributed populations for $p=1$ and the choices of parameters $M > 1-\frac{3}{p}$, $\beta > 0$ and $\gamma > 0$ from Figure 4(b).

4.1, here, we consider the tail of the multivariate $l_{n,p}$-symmetric distribution itself. To this end, we restrict our considerations to Pearson Type VII, Pearson Type II, and Kotz type distributions, see Appendix A. We determine the tail index of regularly varying distributions in Section 4.2.1 and study the multivariate tail behavior for light tails and bounded supports in Sections 4.2.2-4.2.3, respectively.

4.2.1 Heavy Tails

Let us call a random variable regularly varying with tail index $\alpha > 0$ w.r.t. the p-functional $|\cdot|_p$, $p > 0$, if there exist a positive constant α and a probability law $\mathfrak{S}$ on the Borel-σ-field $\mathfrak{B}^n \cap S_{n,p}$ of subsets of the $|\cdot|_p$-unit sphere such that for every $x > 0$

$$x^\alpha \mu_z\left(\cdot; |\cdot|_p\right) \Rightarrow \mathfrak{S} \qquad \text{as } z \to \infty$$

where the symbol $\Rightarrow$ means weak convergence, and

$$\mu_z\left(M; |\cdot|_p\right) = \frac{P\left(|X|_p > xz\,,\ \frac{X}{|X|_p} \in M\right)}{P\left(|X|_p > z\right)}, \quad M \in \mathfrak{B}^n \cap S_{n,p}.$$

Furthermore, $\mathfrak{S}$ is called the spectral measure w.r.t. $|\cdot|_p$. Note that this complies with the common notion of a regularly varying distribution w.r.t. the p-norm if $p \geq 1$.

Now, let $X \sim \Phi_{g^{(n)},p}$ with dg $g^{(n)}$. Thus, because of the stochastic representation (3),

$$\mu_z\left(M; |\cdot|_p\right) = \frac{P\left(R > xz\,,\ U_p^{(n)} \in M\right)}{P\left(R > z\right)}, \quad M \in \mathfrak{B}^n \cap S_{n,p},$$

with R and $U_p^{(n)}$ being independent. If

$$\lim_{z\to\infty}\frac{P\left(R>xz\right)}{P\left(R>z\right)}=x^{-\alpha},\quad x>0,$$

i.e., according to the notion in Resnick (1987), if the survival function of the univariate random variable R is regularly varying at ∞ with index $-\alpha$, then the n-dimensional p-generalized uniform distribution on $S_{n,p}$ is the spectral measure of $\Phi_{g^{(n)},p}$ w.r.t. $|\cdot|_p$,

$$\mathfrak{S}(M)=P\left(U_p^{(n)}\in M\right),\quad M\in\mathfrak{B}^n\cap S_{n,p}.$$

Example 1. In the particular case $g^{(n)}=g^{(n)}_{PT7;M,\nu}$ with $M>\frac{n}{p}$ and $\nu>0$, the application of L'Hôpital's rule yields

$$\lim_{z\to\infty}\frac{P\left(R>xz\right)}{P\left(R>z\right)}=\lim_{z\to\infty}\frac{\int\limits_{xz}^{\infty}r^{n-1}g^{(n)}_{PT7;M,\nu}(r)\,dr}{\int\limits_{z}^{\infty}r^{n-1}g^{(n)}_{PT7;M,\nu}(r)\,dr}=x^{-(Mp-n)},\quad x>0.$$

Consequently, the $l_{n,p}$-symmetric Pearson Type VII distribution with parameters $M>\frac{n}{p}$ and $\nu>0$ has tail index $Mp-n$. In the case $n=2$ and $p\geq 1$, this result is already covered by Example 6 in Richter (2015b).

Remark 6. If $g^{(n)}=g^{(n)}_{Kt;M,\beta,\gamma}$ with $M>1-\frac{n}{p}$, $\beta>0$ and $\gamma>0$, then

$$\lim_{z\to\infty}\frac{P\left(R>xz\right)}{P\left(R>z\right)}=x^{n+p(M-1)}\lim_{z\to\infty}e^{-\beta(x^{p\gamma}-1)z^{p\gamma}}=\begin{cases}\infty & ,0<x<1\\ 1 & ,x=1\\ 0 & ,x>1\end{cases}.$$

Hence, the tail index of $l_{n,p}$-symmetric Kotz type distribution does not exist.

4.2.2 Light Tails

Adopting de Haan's notion of Γ-variation, see Resnick (1987) and original references cited therein, let us call a random vector Γ-varying w.r.t. the p-functional $|\cdot|_p$, $p>0$, if there exist a positive function f and a probability law $\mathfrak{S}$ on $\mathfrak{B}^n\cap S_{n,p}$ such that for every $x\in\mathbb{R}$

$$e^x\mu_z^{\Gamma}\left(\cdot;|\cdot|_p\right)\Rightarrow\mathfrak{S}\qquad\text{as } z\to\infty$$

where

$$\mu_z^{\Gamma}\left(M;|\cdot|_p\right)=\frac{P\left(|X|_p>z+xf(z)\,,\,\frac{X}{|X|_p}\in M\right)}{P\left(|X|_p>z\right)},\quad M\in\mathfrak{B}^n\cap S_{n,p}.$$

Here, f is called an auxiliary function. If $X \sim \Phi_{g^{(n)},p}$ then

$$\mu_z^{\Gamma}\left(M; |\cdot|_p\right) = \frac{P\left(R > z + xf(z)\right)}{P\left(R > z\right)}\, P\left(U_p^{(n)} \in M\right)$$

and hence $\mathfrak{S}$ is the p-generalized uniform distribution on the Borel-σ-field over $S_{n,p}$ if

$$\lim_{z\to\infty} \frac{P\left(R > z + xf(z)\right)}{P\left(R > z\right)} = e^{-x}, \quad x \in \mathbb{R},$$

i.e. the survival function of the univariate p-radius variable R of X is Γ-varying with auxiliary function f.

Example 2. We consider the special case of an $l_{n,p}$-symmetric Kotz type distribution with parameters $M > 1 - \frac{n}{p}$, $\beta > 0$ and $\gamma > 0$, i.e. $g^{(n)} = g^{(n)}_{Kt;M,\beta,\gamma}$, and denote the pdf and the cdf of the Radius variable R by f_{Kt} and F_{Kt}, respectively. For $f(z) = \frac{1}{\beta\gamma p\, z^{p\gamma-1}}$, $z > 0$, and for every $x \in \mathbb{R}$, from of the asymptotic equivalence relations

$$1 - F_{Kt}(r) \sim \frac{f_{Kt}(r)}{\beta\gamma p\, r^{p\gamma-1}} \qquad \text{as } r \to \infty$$

and $\lim\limits_{z\to\infty} -\beta\left(z + xf(z)\right)^{p\gamma} + \beta z^{p\gamma} = -x$, it follows that

$$\lim_{z\to\infty} \frac{1 - F_{Kt}\left(z + xf(z)\right)}{1 - F_{Kt}\left(z\right)} = e^{-x}.$$

Thus, a random vector following an $l_{n,p}$-symmetric Kotz type distribution with parameters $M > 1 - \frac{n}{p}$, $\beta > 0$ and $\gamma > 0$ is Γ-varying w.r.t. the p-functional and with auxiliary function $f(z) = \frac{z^{1-p\gamma}}{\beta\gamma p}$, $z > 0$.

Remark 7. For a random vector being $l_{n,p}$-symmetrically Pearson Type II distributed with parameter $\nu > 0$ one can verify neither the property of regular variation nor that of Γ-variation w.r.t. the p-functional, $p > 0$, since the support of pdf of the corresponding radius variable is bounded.

4.2.3 Bounded supports

Let X be an $l_{n,p}$-symmetrically contoured random vector such that the distribution of $|X|_p$ has a bounded support, and let us denote the right endpoint of if by x_E. If there exist a positive constant α and a probability law $\mathfrak{S}$ on $\mathfrak{B}^n \cap S_{n,p}$ such that for every $x > 0$

$$x^{\alpha}\mu_z^{b}\left(\cdot; |\cdot|_p\right) \Rightarrow \mathfrak{S} \qquad \text{as } z \to \infty$$

where

$$\mu_z^b\left(M; |\cdot|_p\right) = \frac{P\left(|X|_p > x_E - \frac{1}{xz}\,,\ \frac{X}{|X|_p} \in M\right)}{P\left(|X|_p > x_E - \frac{1}{z}\right)}, \quad M \in \mathfrak{B}^n \cap S_{n,p},$$

then we call the random vector X bounded regularly varying with tail index $\alpha > 0$ w.r.t. $|\cdot|_p$, $p > 0$. In this case,

$$\mu_z^b\Gamma\left(\cdot; |\cdot|_p\right) = \frac{P\left(R > x_E - \frac{1}{xz}\right)}{P\left(R > x_E - \frac{1}{z}\right)}\, P\left(U_p^{(n)} \in \cdot\right)$$

and $\mathfrak{S}$ is the n-dimensional p-generalized uniform measure on $S_{n,p}$ if for every $x > 0$

$$\lim_{z\to\infty} \frac{P\left(R > x_E - \frac{1}{xz}\right)}{P\left(R > x_E - \frac{1}{z}\right)} = x^{-\alpha},$$

i.e. if the survival function of the univariate random variable $\frac{1}{x_E - R}$ is regularly varying at ∞ with index $-\alpha$.

Example 3. In the special case $g^{(n)} = g^{(n)}_{PT2;\nu}$ with $\nu > 0$, on the one hand, $x_E = 1$ and, on the other hand,

$$\lim_{z\to\infty} \frac{P\left(R > 1 - \frac{1}{xz}\right)}{P\left(R > 1 - \frac{1}{z}\right)} = \frac{1}{x} \lim_{z\to\infty} \left[\left(\frac{1 - \frac{1}{xz}}{1 - \frac{1}{z}}\right)^{n-1} \left(\frac{1 - \left(1 - \frac{1}{xz}\right)^p}{1 - \left(1 - \frac{1}{z}\right)^p}\right)^{\nu} \right] = x^{-(\nu+1)}$$

for all $x > 0$. Therefore, the $l_{n,p}$-symmetric Pearson Type II distribution with parameter $\nu > 0$ or a random vector following that distribution is bounded regularly varying with tail index $\nu + 1$ w.r.t. $|\cdot|_p$.

4.3 Light and heavy distribution centers

While extremely long concentration intervals and extremely far tails of probability distributions were studied in Sections 4.1 and 4.2, here the focus is on the centers of $l_{3,5}$-symmetric Kotz type distributions for certain choices of parameters.

Nevertheless, it is worthwhile to mention that $l_{n,p}$-symmetric Kotz type distributions have relatively light tails caused by the exponential part of their dgs. The monomial part of the dg ensures that the heaviness of the distribution center can be modeled with the help of the parameter M. It can be seen from Figure 9(a), that the choice of $M = 1$ is the decisive factor to have a heavy distribution center. Standard examples of this distributional type are power exponential and, particularly, Gaussian and Laplace distributions, and their $l_{n,p}$-generalizations. Additionally, in the case $M = 1$, Figure 9(a) shows that the parameter $\beta > 0$ controls mainly the height and the parameter $\gamma > 0$

mainly the decay behavior of the dg. In the case $M = 2$, the parameter β as well as γ regulate the height of the dg. Furthermore, they induce a shift of the probability mass. The effect of these choices of parameters on the shape of the pdf of the maximum distribution of three dependent rvs following a joint $l_{3,5}$-symmetric Kotz type distribution can be seen in Figure 9(b). Finally, one can compare Figure 9(b) with Figure 4 to get an impression of the impact of a further increase of the parameter p.

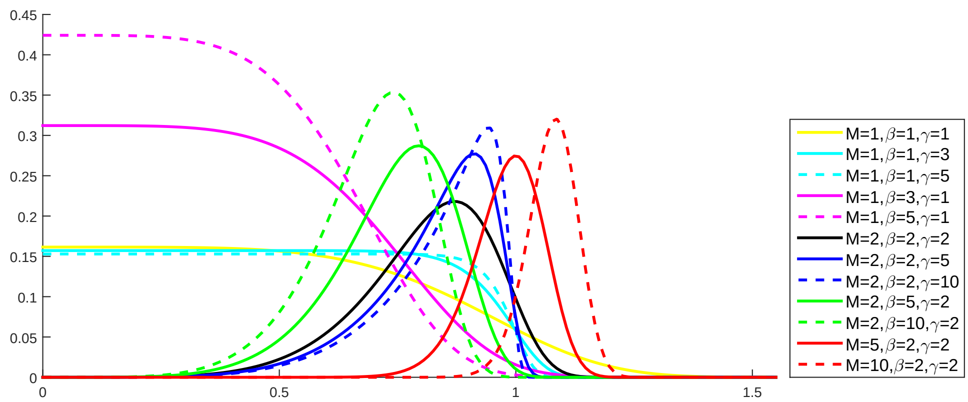
5 Proofs

5.1 Measure-of-cone representations of skewed $l_{k,p}$-symmetric distributions

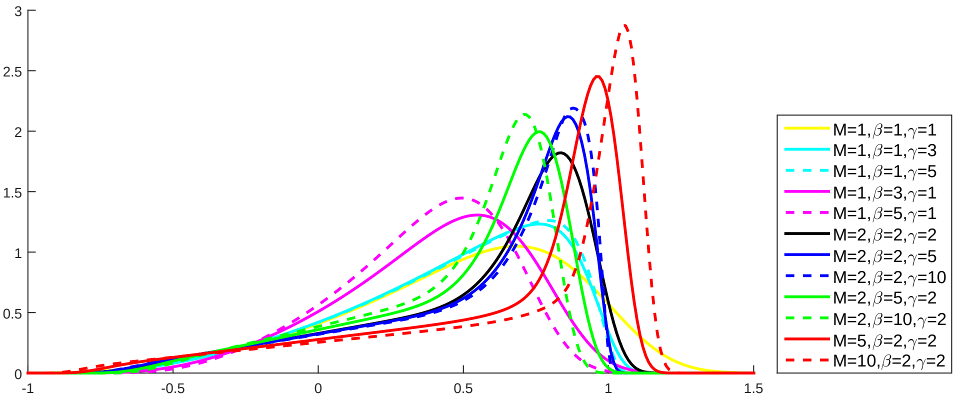
An initial step of the proof of Theorem 1 deals with deriving a representation of the cdfs of skewed $l_{k,p}$-symmetric distributions with dimensionality parameter m in terms of specific $l_{n,p}$-symmetric measures of cones by analogy to what was done in Richter and Venz (2014) for skewed elliptically contoured distributions with dimensionality parameter $m = 1$. Afterwards, this measure-of-cone representation is used to prove Remark 4.

Let a random vector Z follow the skewed $l_{k,p}$-symmetric distribution with dimensionality parameter m, dg $g^{(k+m)}$, and matrix-parameter $\Lambda \in \mathbb{R}^{m\times k}$, and let $e_j^{(n)}$, $j = 1, \ldots, n$, still denote the jth standard unit vector of $\mathbb{R}^n$. For arbitrary $z \in \mathbb{R}^k$, we consider the cone

$$\begin{aligned} A_0(z) &= C_m\left(a_{0,1}, \ldots, a_{0,m}, a_1, \ldots, a_k; z\right) \\ &= \left(\bigcap_{l=1}^{m} \left\{x \in \mathbb{R}^{k+m} \colon a_{0,l}^{\mathsf{T}} x < 0\right\}\right) \cap \left(\bigcap_{i=1}^{k} \left\{x \in \mathbb{R}^{k+m} \colon a_i^{\mathsf{T}} x < z_i\right\}\right) \end{aligned}$$



(a) The dg $g^{(3)}_{Kt;M,\beta,\gamma}$ for certain parameters $M > 1 - \frac{3}{5}$, $\beta > 0$ and $\gamma > 0$.



(b) The pdf of the maximum statistic for the same parameters as in Figure 9(a).

Figure 9: The $l_{3,5}$-symmetric Kotz type sample distribution.

where the quantities $a_{0,l} = -\Gamma^{\mathsf{T}} e_l^{(m)}$, $l = 1, \ldots, m$, $a_i = e_i^{(k+m)}$, $i = 1, \ldots, k$, and $\Gamma = (\Lambda, -I_m)$ are as in Section 2. This cone generalizes that in Richter and Venz (2014) where the case $m = 1$ is dealt with. Note that $A_0(z)$ has its vertex at $(z^{\mathsf{T}}, (\Lambda z)^{\mathsf{T}})^{\mathsf{T}}$. Furthermore, $A_0(z)$ is the intersection of $k + m$ half spaces, m of whom containing the origin in its boundary.

Lemma 1 (**A specific measure-of-cone representation of the cdf** $F_{k,m,p}\left(\cdot; \Lambda, g^{(k+m)}\right)$**).** *If $Z \sim SS_{k,m,p}\left(\Lambda, g^{(k+m)}\right)$, then the cdf of Z allows the representation*

$$F_{k,m,p}\left(z; \Lambda, g^{(k+m)}\right) = \frac{1}{F_{m,p}^{(2)}\left(0_m; I_m + \Lambda\Lambda^{\mathsf{T}}, g_{(k+m)}^{(m)}\right)} \, \Phi_{g^{(k+m)},p}(A_0(z)), \quad z \in \mathbb{R}^k,$$

where $F_{m,p}^{(2)}\left(\cdot; I_m + \Lambda\Lambda^{\mathsf{T}}, g_{(k+m)}^{(m)}\right)$ is the normalizing constant from (8)*.*

Proof. Let $C_{m,p} = \frac{1}{F_{m,p}^{(2)}\left(0_m; I_m + \Lambda\Lambda^{\mathsf{T}}, g_{(k+m)}^{(m)}\right)}$. Then

$$\begin{aligned}
F_{k,m,p}\left(z; \Lambda, g^{(k+m)}\right) &= C_{m,p} \int\limits_{-\infty}^{z} g_{(k+m)}^{(k)}\left(|\zeta|_p\right) F_{m,p}^{(1)}\left(\Lambda\zeta; g_{[|\zeta|_p]}^{(m)}\right) d\zeta \\
&= C_{m,p} \int\limits_{-\infty}^{z} g_{(k+m)}^{(k)}\left(|\zeta|_p\right) \int\limits_{\mathbb{R}_+^m} g_{[|\zeta|_p]}^{(m)}\left(|\Lambda\zeta - \xi|_p\right) d\xi \, d\zeta \\
&= C_{m,p} \int\limits_{-\infty}^{z} \int\limits_{\mathbb{R}_+^m} g^{(k+m)}\left(\sqrt[p]{|\zeta|_p^p + |\Lambda\zeta - \xi|_p^p}\right) d\xi \, d\zeta, \quad z \in \mathbb{R}^k,
\end{aligned}$$

where the integral $\int\limits_{-\infty}^{z} h(\zeta)\, d\zeta$, $z \in \mathbb{R}^k$, is defined as a k-fold one. Further, using the notation $X = \left(X^{(1)}, X^{(2)}\right)^{\mathsf{T}}$ where the vectors $X^{(1)}$ and $X^{(2)}$ take their values in $\mathbb{R}^k$ and

$\mathbb{R}^m$, respectively,

$$\begin{aligned} F_{k,m,p}\left(z;\Lambda,g^{(k+m)}\right) &= C_{m,p}\int\limits_{-\infty}^{z}\int\limits_{\mathbb{R}_+^m}\varphi_{g^{(k+m)},p}\left(\zeta,\Lambda\zeta-\xi\right)\,d\xi\,d\zeta \\ &= C_{m,p}\cdot P(X^{(1)}<z,X^{(2)}<\Lambda X^{(1)}),\quad z\in\mathbb{R}^k. \end{aligned}$$

Here, $\varphi_{g^{(k+m)},p}$ is the pdf of X, and

$$\begin{aligned} &\left\{\left(x^{(1)},x^{(2)}\right)^{\mathtt{T}}\in\mathbb{R}^{k+m}\colon x^{(1)}<z,x^{(2)}<\Lambda x^{(1)}\right\} \\ =&\left\{x=\left(x^{(1)},x^{(2)}\right)^{\mathtt{T}}\in\mathbb{R}^{k+m}\colon x^{(1)}<z,-\Gamma x<0\right\} \\ =&\left\{x\in\mathbb{R}^{k+m}\colon e_i^{(k+m)\mathtt{T}}x<z_i,i=1,\ldots,k,\left(-\Gamma^{\mathtt{T}}e_l^{(m)}\right)^{\mathtt{T}}x<0,l=1,\ldots,m\right\}=A_0(z). \end{aligned}$$

Thus,

$$\begin{aligned} F_{k,m,p}\left(z;\Lambda,g^{(k+m)}\right) &= C_{m,p}\cdot P(X^{(1)}<z,X^{(2)}<\Lambda X^{(1)}) \\ &= C_{m,p}\cdot P(X\in A_0(z)) \\ &= C_{m,p}\cdot\Phi_{g^{(k+m)},p}(A_0(z)),\quad z\in\mathbb{R}^k. \end{aligned}$$

□

Note that Theorem 1 in Richter and Venz (2014) follows from Lemma 1 for $m=1$ and $p=2$.

Let $\mathfrak{O}_p$ denote the $l_{n,p}$-generalized surface content on $\mathfrak{B}^n\cap S_{n,p}$ and $\mathfrak{F}_p\colon(0,\infty)\to[0,\infty)$ the $l_{n,p}$-sphere intersection-proportion function (ipf) defined by

$$r\mapsto\mathfrak{F}_p\left(A,r\right)=\frac{\mathfrak{O}_p\left(\frac{1}{r}A\cap S_{n,p}\right)}{\mathfrak{O}_p\left(S_{n,p}\right)}=\frac{\mathfrak{O}_p\left(\frac{1}{r}A\cap S_{n,p}\right)}{\omega_{n,p}},\quad A\in\mathfrak{B}^n. \tag{14}$$

According to Richter (2009), but with suitably adapted notations as in Richter (2014, 2015a) and Müller and Richter (2015a), for arbitrary $p>0$ and $n\in\mathbb{N}$, the continuous $l_{n,p}$-symmetric distribution $\Phi_{g^{(n)},p}$ with dg $g^{(n)}$ satisfies the geometric measure representation

$$\Phi_{g^{(n)},p}\left(A\right)=\frac{1}{I\left(g^{(n)}\right)}\int\limits_0^\infty\mathfrak{F}_p(A,r)\,r^{n-1}\,g^{(n)}(r)\,dr,\quad A\in\mathfrak{B}^n. \tag{15}$$

Note that this formula was first proved for $p=2$ and $g^{(n)}=g_{PE}^{(n)}$ in Richter (1985), and for $p=2$ and arbitrary dgf g in Richter (1991).

Using formula (15) and the symmetry of both $\mathfrak{O}_p$ and the $l_{n,p}$-unit sphere $S_{n,p}$, the following lemma provides some invariance properties of the $l_{n,p}$-symmetric probability measure.

Lemma 2. *The $l_{n,p}$-symmetric probability measure $\Phi_{g^{(n)},p}$ is permutation and sign invariant.*

Proof. Let $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ be a permutation defined by $i \mapsto \sigma(i)$, $i = 1,\dots,n$, and M the corresponding permutation matrix, i.e. $M = \left(e^{(n)}_{\sigma(1)} e^{(n)}_{\sigma(2)} \cdots e^{(n)}_{\sigma(n)}\right)$. Since permutation matrices are orthogonal matrices, M is invertible and $M^{-1} = M^{\mathsf{T}}$. Further, let $X \sim \Phi_{g^{(n)},p}$. By the geometric measure representation,

$$P(MX \in A) = \Phi_{g^{(n)},p}(M^{\mathsf{T}}A) = \frac{1}{I\left(g^{(n)}\right)} \int\limits_0^\infty \mathfrak{F}_p \left(M^{\mathsf{T}}A, r\right) r^{n-1}\, g^{(n)}(r)\, dr, \quad A \in \mathfrak{B}^n,$$

where, because of $M^{\mathsf{T}} S_{n,p} = S_{n,p}$ and $\mathfrak{O}_p \left(M^{\mathsf{T}}A \cap S_{n,p}(r)\right) = \mathfrak{O}_p \left(M^{\mathsf{T}}(A \cap S_{n,p}(r))\right) = \mathfrak{O}_p \left(A \cap S_{n,p}(r)\right)$, the ipf satisfies $\mathfrak{F}_p \left(M^{\mathsf{T}}A, r\right) = \mathfrak{F}_p \left(A, r\right)$, $A \in \mathfrak{B}^n$. Thus,

$$P(MX \in A) = \Phi_{g^{(n)},p}(A) = P(X \in A), \quad A \in \mathfrak{B}^n.$$

Let D be an $n \times n$ sign matrix, i.e. $D = \operatorname{diag}\{d_1,\dots,d_n\}$ with $d_i \in \{1,-1\}$, $i = 1,\dots,n$. D is an orthogonal matrix and, by analogous considerations as before,

$$P(DX \in A) = P(X \in DA) = \Phi_{g^{(n)},p}(DA) = \Phi_{g^{(n)},p}(A) = P(X \in A), \quad A \in \mathfrak{B}^n.$$

□

Consequently, $\Phi_{g^{(n)},p}$ is a member of the class $\mathcal{SI}$ of sign invariant distributions, considered in Arellano-Valle and del Pino (2004). Note that one can prove Lemma 2 alternatively without using formula (15), starting from $\Phi_{g^{(n)},p}(A) = \int\limits_A \varphi_{g^{(n)},p}(x)\, dx = \int\limits_A g^{(n)}(|x|_p)\, dx$ and using the invariance properties of the p-functional $|\cdot|_p$.

Next, we are going to generalize the specific measure-of-cone representation formula for the cdf of a skewed $l_{k,p}$-symmetrically distributed random vector given in Lemma 1. Doing this, first, a class of cones is introduced and, then, the permutation and sign invariance of the measure $\Phi_{g^{(k+m)},p}$ is basically utilized. We recall that the invariance of $\Phi_{g^{(n)},2}$ with respect to all orthogonal transformations was used to construct general geometric measure representations for $n = 2$ in Günzel et al. (2012) and for arbitrary n in Richter and Venz (2014).

Let $\Lambda\colon \mathbb{R}^k \to \mathbb{R}^m$ be a matrix with entries $\Lambda_{j,i}$, $j = 1,\dots,m$, $i = 1,\dots,k$. Further, let $I = \{i_1,\dots,i_k\} \subset \{1,\dots,k+m\}$ denote a set of indices with $|I| = k$ and D a $(k+m) \times (k+m)$ sign matrix. Moreover, let $V_{I,D}(\Lambda; z) = D \cdot (v_1,\dots,v_{k+m})^{\mathsf{T}}$ with $v_{i_l} = z_l$ for $l = 1,\dots,k$ and $v_{j_\nu} = (\Lambda z)_\nu = \sum\limits_{l=1}^{k} \Lambda_{\nu,l} z_l$ for $j_\nu \in \{1,\dots,k+m\} \backslash I$ and

$\nu = 1, \ldots, m$. For every $z \in \mathbb{R}^k$, we define a cone with vertex at $V_{I,D}(\Lambda; z)$ by

$$C_{I,D}(\Lambda; z) = \{Dx \in \mathbb{R}^{k+m} \colon x_{i_l} < z_l, l = 1, \ldots, k \\ x_{j_\nu} < \sum_{l=1}^{k} \Lambda_{\nu,l} x_{i_l}, j_\nu \in \{1, \ldots, k+m\} \backslash I, \nu = 1, \ldots, m\}.$$

From here on, the collection of such cones will be denoted by $\mathcal{C}_{k,m}(\Lambda; z)$. Particularly, $A_0(z) = C_{\{1,\ldots,k\}, I_{k+m}}(\Lambda; z)$ is an element from the class $\mathcal{C}_{k,m}(\Lambda; z)$ of cones. Consequently, $\mathcal{C}_{k,m}(\Lambda; z) = \{MDA_0(z) \colon M \in \Pi_{k+m}, D \in \mathcal{S}_{k+m}\}$ where Π_n and $\mathcal{S}_n$ denote the sets of $n \times n$ permutation and sign matrices, respectively.

Corollary 2 (A general measure-of-cone representation of the cdf $F_{k,m,p}\left(\cdot; \Lambda, g^{(k+m)}\right)$). *The skewed distribution in Lemma 1 allows each of the representations*

$$F_{k,m,p}\Big(z; \Lambda, g^{(k+m)}\Big) = \frac{1}{F^{(2)}_{m,p}\left(0_m; I_m + \Lambda\Lambda^{\mathsf{T}}, g^{(m)}_{(k+m)}\right)} \Phi_{g^{(k+m)},p}\left(C_{\{i_1,\ldots,i_k\},D}\left(\Lambda; z\right)\right), \quad z \in \mathbb{R}^k,$$

where $C_{\{i_1,\ldots,i_k\},D}(\Lambda; z)$ is an arbitrary element of the class $\mathcal{C}_{k,m}(\Lambda; z)$.

Proof. By definition of $\mathcal{C}_{k,m}(\Lambda; z)$, there is a permutation matrix M such that

$$C_{\{i_1,\ldots,i_k\},D}(\Lambda; z) = M \cdot C_{\{1,\ldots,k\},D}(\Lambda; z) = M \cdot D \cdot A_0(z).$$

Then M defines a permutation σ with $\sigma(l) = i_l$, $l = 1, \ldots, k$, and $\sigma(k+1), \ldots, \sigma(k+m) \in \{1, \ldots, k+m\} \backslash \{i_1, \ldots, i_k\}$ so that the bijectivity of σ is ensured. By Lemma 1 and Lemma 2, and with $C_{m,p} = \frac{1}{F^{(2)}_{m,p}\left(0_m; I_m + \Lambda\Lambda^{\mathsf{T}}, g^{(m)}_{(k+m)}\right)}$ as in the proof of Lemma 1,

$$\begin{aligned} F_{k,m,p}\Big(z; \Lambda, g^{(k+m)}\Big) &= C_{m,p} \cdot \Phi_{g^{(k+m)},p}\left(A_0(z)\right) \\ &= C_{m,p} \cdot \Phi_{g^{(k+m)},p}\left(M \cdot D \cdot A_0(z)\right) \\ &= C_{m,p} \cdot \Phi_{g^{(k+m)},p}\left(C_{\{i_1,\ldots,i_k\},D}\left(\Lambda; z\right)\right), \quad z \in \mathbb{R}^k. \end{aligned}$$

□

Proof of Remark 4. Initializing the proof of part a), let M_1 be a $k \times k$ permutation matrix and $Z \sim SS_{k,m,p}\Big(\Lambda, g^{(k+m)}\Big)$. Since $|M_1\xi|_p = |\xi|_p$, $\xi \in \mathbb{R}^k$, and $|\det(M_1)| = 1$,

$$\begin{aligned} P(M_1 Z < z) = P(Z < M_1^{\mathsf{T}} z) &= \int_{-\infty}^{M_1^{\mathsf{T}} z} f_Z(\zeta)\, d\zeta = \int_{-\infty}^{z} f_Z(M^{\mathsf{T}}\xi)\, d\xi \\ &= \frac{1}{F^{(2)}_{m,p}\Big(0_m; I_m + \Lambda_1\Lambda_1^{\mathsf{T}}, g^{(m)}_{(k+m)}\Big)} \int_{-\infty}^{z} g^{(k)}_{(k+m)}\left(|\xi|_p\right) F^{(1)}_{m,p}\left(\Lambda_1\xi; g^{(m)}_{[|\xi|_p]}\right)\, d\xi, \end{aligned}$$

$z \in \mathbb{R}^k$, where $\Lambda_1 = \Lambda M_1^{\intercal}$. Thus, $M_1 Z \sim SS_{k,m,p}\Big(\Lambda M_1^{\intercal}, g^{(k+m)}\Big)$.

To start the proof of part b), let M_2 be a $m \times m$ permutation matrix in $\mathbb{R}^m$ and $Z \sim SS_{k,m,p}\Big(M_2\Lambda, g^{(k+m)}\Big)$. As in the proof of Lemma 1, but with $X = \Big(X^{(1)}, X^{(2)}\Big)^{\intercal} \sim \Phi_{g^{(k+m)},p}$,

$$F_{k,m,p}\Big(z; M_2\Lambda, g^{(k+m)}\Big) = \frac{P(X^{(1)} < z, X^{(2)} < M_2\Lambda X^{(1)})}{F^{(2)}_{m,p}\Big(0_m; I_m + (M_2\Lambda)(M_2\Lambda)^{\intercal}, g^{(m)}_{(k+m)}\Big)}, \quad z \in \mathbb{R}^k.$$

Note that

$$\begin{aligned}
&\Big\{\Big(x^{(1)}, x^{(2)}\Big)^{\intercal} \in \mathbb{R}^{k+m} \colon x^{(1)} < z, x^{(2)} < M_2\Lambda x^{(1)}\Big\} \\
=&\Big\{\Big(x^{(1)}, x^{(2)}\Big)^{\intercal} \in \mathbb{R}^{k+m} \colon x^{(1)} < z, M_2^{\intercal} x^{(2)} < \Lambda x^{(1)}\Big\} \\
=&\Big\{\Big(x^{(1)}, M_2\tilde{x}^{(2)}\Big)^{\intercal} \in \mathbb{R}^{k+m} \colon x^{(1)} < z, \tilde{x}^{(2)} < \Lambda x^{(1)}\Big\} = \tilde{M}_2 \cdot A_0(z)
\end{aligned}$$

with $\tilde{M}_2 = \operatorname{diag}[I_k, M_2] \in \mathbb{R}^{(k+m)\times(k+m)}$ being a permutation matrix. The set $A_1(z) = \tilde{M}_2 \cdot A_0(z)$ is an element of the class $\mathcal{C}_{k,m}(\Lambda; z)$. By Lemma 2,

$$P(X^{(1)} < z, X^{(2)} < M_2\Lambda X^{(1)}) = P(X \in A_1(z)) = P(X \in A_0(z)).$$

Additionally, let $\Gamma = (\Lambda, -I_m)$ and $\Gamma_1 = (M_2\Lambda, -I_m)$, then

$$\begin{aligned}
&F^{(2)}_{m,p}\Big(0_m; I_m + (M_2\Lambda)(M_2\Lambda)^{\intercal}, g^{(m)}_{(k+m)}\Big) = P(\Gamma_1 X < 0^{(m)}) \\
=& \int\limits_{\Big\{x=\big(x^{(1)},x^{(2)}\big)^{\intercal} \in \mathbb{R}^{k+m} \colon M_2\Lambda x^{(1)} < x^{(2)}\Big\}} g^{(k+m)}\Big(|x|_p\Big)\, dx \\
=& \int\limits_{\Big\{y=\big(y^{(1)},y^{(2)}\big)^{\intercal} \in \mathbb{R}^{k+m} \colon \Lambda y^{(1)} < y^{(2)}\Big\}} g^{(k+m)}\Big(\Big|\tilde{M}_2 y\Big|_p\Big)\, dy = P(\Gamma X < 0^{(m)}) \\
=& F^{(2)}_{m,p}\Big(0_m; I_m + \Lambda\Lambda^{\intercal}, g^{(m)}_{(k+m)}\Big).
\end{aligned}$$

Finally,

$$\begin{aligned}
F_{k,m,p}\Big(z; M_2\Lambda, g^{(k+m)}\Big) &= \frac{1}{F^{(2)}_{m,p}\Big(0_m; I_m + (M_2\Lambda)(M_2\Lambda)^{\intercal}, g^{(m)}_{(k+m)}\Big)} P(X \in A_1(z)) \\
&= \frac{1}{F^{(2)}_{m,p}\Big(0_m; I_m + \Lambda\Lambda^{\intercal}, g^{(m)}_{(k+m)}\Big)} P(X \in A_0(z)) \\
&= F_{k,m,p}\Big(z; \Lambda, g^{(k+m)}\Big), \quad z \in \mathbb{R}^k.
\end{aligned}$$

$\square$

5.2 Applying the advanced geometric method to extremes

We recall that advanced applications of the geometric measure representation (15) make use of types of intersection percentage functions (14) being valid for whole classes of random events. The classes of events considered here are cones generated by intersecting half spaces. In this section, the measure-of-cone representations from Corollary 2 are mainly used to prove Theorem 1 which, in turn, is basic to establish Corollary 1 and Remark 5. Moreover, the direct and the advanced geometric methods considered in this paper are briefly compared.

Let $A_n^n(t) = \Big\{(x_1,\dots,x_n)^{\mathsf{T}} \in \mathbb{R}^n \colon x_1 < t,\dots,x_n < t\Big\}$, $t \in \mathbb{R}$, be a sublevel set generated by the maximum statistic of an n-dimensional random vector $X = (X_1,\dots,X_n)^{\mathsf{T}}$. Then,

$$P\Big(\max\{X_1,\dots,X_n\} < t\Big) = P\Big(X \in A_n^n(t)\Big), t \in \mathbb{R}.$$

Moreover, illustrations of the set $A_n^n(t)$ may be found in the two earlier papers of the authors for $n \in \{2,3\}$.

Proof of Theorem 1. Let $\nu \in \{1,\dots,n-1\}$ be fixed. Further, let $\tilde{B}_{1,0}^{(1)}(t) = \{x \in \mathbb{R}^2 \colon x_1 < t, x_2 < x_1\}$ and $\tilde{B}_{1,1}^{(1)}(t) = \{x \in \mathbb{R}^2 \colon x_2 < t, x_1 \le x_2\}$. Then $A_2^2(t) = \tilde{B}_{1,0}^{(1)}(t) \cup \tilde{B}_{1,1}^{(1)}(t)$ is a disjoint decomposition of the cone $A_2^2(t)$ with vertex at $(t,t)^{\mathsf{T}}$, see Figure 10. Similarly, on the one hand, one has that $A_3^3(t) = B_2^{(1)}(t) \cup \tilde{B}_{2,1}^{(1)}(t)$ where

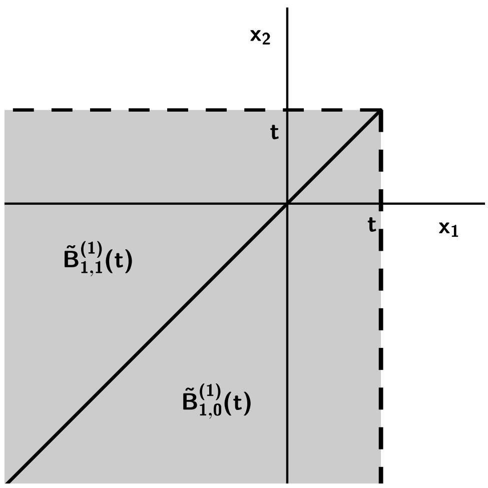


Figure 10: Disjoint decomposition of $A_2^2(t)$ for $t > 0$.

$B_2^{(1)}(t) = \{x_1 < t, x_2 < t, x_3 < x_2\}$ and $\tilde{B}_{2,1}^{(1)}(t) = \{x_1 < t, x_3 < t, x_2 \le x_3\}$. On the other hand, using this and the decomposition of $A_2^2(t)$ again,

$$\begin{aligned} A_3^3(t) &= \{x_1 < t, x_2 < t, x_3 < x_2\} \cup \{x_1 < t, x_3 < t, x_2 \le x_3\} \\ &= \{x_1 < t, x_2 < x_1, x_3 < x_2\} \cup \{x_1 < t, x_3 < x_1, x_2 \le x_3\} \cup \tilde{B}_{1,1}^{(2)}(t) \cup \tilde{B}_{1,2}^{(2)}(t) \\ &= \{x_1 < t, (x_2,x_3)^{\mathsf{T}} \in A_2^2(x_1)\} \cup \tilde{B}_{1,1}^{(2)}(t) \cup \tilde{B}_{1,2}^{(2)}(t) \end{aligned}$$

$$= B_1^{(2)}(t) \cup \tilde{B}_{1,1}^{(2)}(t) \cup \tilde{B}_{1,2}^{(2)}(t)$$

where, for every $i \in \{1, \ldots, n-\nu\}$, $B_i^{(\nu)}(t) = \tilde{B}_{i,0}^{(\nu)}(t)$ and $\tilde{B}_{i,j}^{(\nu)}(t) = \{x \in \mathbb{R}^n \colon x_{i+j} < t,\ x_{l_1} \leq x_{i+j}, \forall\, l_1 \in [i, i+j] \backslash \{i+j\}, x_{l_2} < x_{i+j}, \forall\, l_2 \in [i+j, i+\nu] \backslash \{i+j\},\ x_{l_3} < t, \forall\, l_3 \in I_{i,\nu}\}$, $j = 0, 1, \ldots, \nu$, are cones with vertices at $(t, \ldots, t)^{\mathsf{T}} \in \mathbb{R}^n$ and where $[i, i+j] = \{i, \ldots, i+j\}$ and $I_{i,\nu} = [1, n] \backslash [i, i+\nu]$. Further, one can inductively prove the following disjoint decomposition

$$A_n^n(t) = B_i^{(\nu)}(t) \cup \tilde{B}_{i,1}^{(\nu)}(t) \cup \tilde{B}_{i,2}^{(\nu)}(t) \cup \cdots \cup \tilde{B}_{i,\nu}^{(\nu)}(t)$$

into $\nu + 1$ cones so that each of them, which is an intersection of n half spaces from $\mathbb{R}^n$, contains the origin in the boundary of ν of its n intersecting half spaces. In the spherical case $p = 2$, this idea, that at least one hyperplane contains the origin, arises in Günzel et al. (2012) for $n = 2$, and in Richter and Venz (2014) for an arbitrary n. Indicating the topological interior of the set $A \subseteq \mathbb{R}^n$ by $\mathrm{int}(A)$,

$$\mathrm{int}\left(\tilde{B}_{i,j}^{(\nu)}(t)\right) = \mathrm{int}\left(M_{i,j} \cdot B_i^{(\nu)}(t)\right), \quad j = 1, \ldots, \nu,$$

i.e. one can transform each of the cones $\tilde{B}_{i,j}^{(\nu)}(t)$, $j = 1, \ldots, \nu$, by permutation into the cone $B_i^{(\nu)}(t)$, where the matrix $M_{i,j} \in \mathbb{R}^{n \times n}$ defines the transposition $\sigma_{i,j}$ with $\sigma_{i,j}(i) = j$, $\sigma_{i,j}(j) = i$ and $\sigma_{i,j}(l) = l$ for all $l \in [1, n] \backslash \{i, j\}$. By Lemma 2, for every $i \in \{1, \ldots, n-\nu\}$, the maximum cdf $F_{n:n}$ of the components of a continuous $l_{n,p}$-symmetrically distributed random vector X with dg $g^{(n)}$ satisfies

$$\begin{aligned} F_{n:n}(t) = P(X \in A_n^n(t)) =& \Phi_{g^{(n)},p}(A_n^n(t)) \\ =& \Phi_{g^{(n)},p}\Big(B_i^{(\nu)}(t)\Big) + \sum_{j=1}^{\nu} \Phi_{g^{(n)},p}\Big(M_{i,j} B_i^{(\nu)}(t)\Big) \\ =& (\nu+1)\, \Phi_{g^{(n)},p}\Big(B_i^{(\nu)}(t)\Big), \quad t \in \mathbb{R}. \end{aligned}$$

Further,

$$B_i^{(\nu)}(t) \in \mathcal{C}_{n-\nu,\nu}\Big(E_i^{(\nu)}; t1_{n-\nu}\Big)$$

where $1_{n-\nu} = (1, \ldots, 1)^{\mathsf{T}} \in \mathbb{R}^{n-\nu}$ and $E_i^{(\nu)} \in \mathbb{R}^{\nu \times (n-\nu)}$ is a matrix whose ith column is 1_ν and all the others are ν-dimensional zero vectors. Then, Corollary 2 implies for every $\nu \in \{1, \ldots, n-1\}$ and $i \in \{1, \ldots, n-\nu\}$ that

$$F_{n:n}(t) = (\nu+1) F_{\nu,p}^{(2)}\left(0_\nu; I_\nu + E_i^{(\nu)} E_i^{(\nu)\mathsf{T}}, g_{(n)}^{(\nu)}\right) \cdot F_{n-\nu,\nu,p}\left(t1_{n-\nu}; E_i^{(\nu)}, g^{(n)}\right), \quad t \in \mathbb{R}. \quad (16)$$

Furthermore, on the one hand, $I_\nu + E_i^{(\nu)} E_i^{(\nu)\mathsf{T}} = I_\nu + 1_\nu 1_\nu^{\mathsf{T}}$ for every $i \in \{1. \ldots, n-\nu\}$.

On the other hand, as the above cdf $F_{n-\nu,\nu,p}(\,\cdot\,; E_i^{(\nu)}, g^{(n)})$ is evaluated at a point whose components are all equal to each other, Remark 4 a) yields

$$\begin{aligned}F_{n-\nu,\nu,p}(t1_{n-\nu}; E_{i_1}^{(\nu)}, g^{(n)}) &= P(Z < t1_{n-\nu}) = P(Z < M_{i_1,i_2}^{\mathtt{T}} t1_{n-\nu}) \\ &= P(M_{i_1,i_2} Z < t1_{n-\nu}) = F_{n-\nu,\nu,p}(t1_{n-\nu}; E_{i_2}^{(\nu)}, g^{(n)}),\end{aligned}$$

for every $i_1, i_2 \in \{1, \ldots, n-\nu\}$, where $Z \sim SS_{n-\nu,\nu,p}(E_{i_1}^{(\nu)}, g^{(n)})$ and the matrix $M_{i_1,i_2} \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ is the transposition matrix as before. Hence, without any loss of generality, the parameter i in (16) can be chosen as $i = 1$. □

The possibility of choosing (without any loss of generality) $i = 1$ in equation (16) may equivalently be established making use of the exchangeability of the components of an $l_{n,p}$-symmetrically distributed random vector directly. Indeed, since $I_\nu + E_i^{(\nu)} E_i^{(\nu)\mathtt{T}} = I_\nu + 1_\nu 1_\nu^{\mathtt{T}}$, the impact of the parameter i onto the cdf $F_{n-\nu,\nu,p}\left(t1_{n-\nu}; E_i^{(\nu)}, g^{(n)}\right)$, $t \in \mathbb{R}$, is confined to the matrix $E_i^{(\nu)}$. Now, considering the construction formula (9) of the corresponding distribution, for and $X^{(2)} = \left(X_{j_{n-\nu+1}}, \ldots, X_{j_n}\right)\mathtt{T}\colon \Omega \to \mathbb{R}^\nu$, the parameter $i \in \{1, \ldots, n-\nu\}$ in equation (16) regulates which component of $X^{(1)}$ is used in the condition $X^{(2)} < \Lambda X^{(1)}$ since

$$SS_{n-\nu,\nu,p}\left(E_i^{(\nu)}, g^{(n)}\right) = \mathfrak{L}\left(X^{(1)} \,\middle|\, X^{(2)} < X_{j_i} 1_{n-\nu}\right)$$

where $X^{(1)} = \left(X_{j_1}, \ldots, X_{j_{n-\nu}}\right)^{\mathtt{T}} : \Omega \to \mathbb{R}^{n-\nu}$ and $X^{(2)}\colon \Omega \to \mathbb{R}^\nu$ are subvectors of $X = (X_1, \ldots, X_n)$, $X \sim \Phi_{g^{(n)},p}$. By evaluating the cdf of $SS_{n-\nu,\nu,p}\left(E_i^{(\nu)}, g^{(n)}\right)$ at a point whose components equal each other and using the exchangeability of the components of $X^{(1)}$, the alternative proof of choosing $i = 1$ without loss of generality is finished.

Summarizing the methods used in the present paper and that used in Müller and Richter (2015a), there were two ways of applying the geometric measure representation (15) to get exact distribution formulae: the direct and the advanced one. The latter opens the possibility to turn over to considering conditional distributions and their densities. Following the first approach and using the permutation and sign invariance from Lemma 2 and the disjoint decompositions of $A_n^n(t)$ given in the proof of Theorem 1, for every $\nu \in \{1, \ldots, n-1\}$ results of the form

$$F_{n:n}(t) = (\nu + 1) \int_0^\infty \tilde{f}_\nu(t, r)\, r^{n-1}\, g^{(n)}(r)\, dr$$

can be achieved where $\tilde{f}_\nu(t, r) = \mathfrak{O}_p\left(\left[\frac{1}{r} B_j^{(\nu)}(t)\right] \cap S_{n,p}\right)$, $j = 1, \ldots, n-\nu$. The case $\nu = 0$ covers formula (1) in Müller and Richter (2015a) with $k = n$ and $f = \tilde{f}_0$, if $B^{(0)} = A_n^n(t)$ is considered as the zero-fold decomposition of $A_n^n(t)$. Furthermore, details of computing $\tilde{f}_\nu$ are provided in the same paper. If the structure of $\tilde{f}_\nu$ and of the argument $\left[\frac{1}{r}B_j^{(\nu)}(t)\right] \cap S_{n,p}$ of $\mathfrak{O}_p$, respectively, will be too involved, the second approach is preferred here which arises from a combination of the methods studied in Arellano-Valle and Richter (2012) and Richter and Venz (2014). This advanced geometric method of using the measure-of-cone representations from Corollary 2 is more effective than the direct application of geometric measure representation (15). In the case of the exact distribution of the maximum statistic, the second approach results in Theorem 1.

Remark 8. Let $Z^{(n)}$ be an n-dimensional random vector with pdf $f_{Z^{(n)}}$. Then, for every $t \in \mathbb{R}$, $\frac{d}{dt}\left(P(Z^{(1)} < t)\right) = f_{Z^{(1)}}(t)$, and, if $n \geq 2$,

$$\frac{d}{dt} P\left(Z^{(n)} < t1_n\right) = \mathcal{D}_n(f_{Z^{(n)}}, t)$$

where

$$\begin{aligned}
&\mathcal{D}_n(f_{Z^{(n)}}, t) \\
&= \sum_{i=1}^{n} \int\limits_{D_i^{(n)}(t)} f_{Z^{(n)}}(z_1, \ldots, z_{i-1}, t, z_{i+1}, \ldots, z_n)\, dz \\
&= \sum_{i=1}^{n} \int\limits_{-\infty}^{t} \cdots \int\limits_{-\infty}^{t} \int\limits_{-\infty}^{t} \cdots \int\limits_{-\infty}^{t} f_{Z^{(n)}}(z_1, \ldots, z_{i-1}, t, z_{i+1}, \ldots, z_n)\, dz_1 \cdots dz_{i-1}\, dz_{i+1} \cdots dz_n
\end{aligned}$$

and $D_i^{(n)}(t) = \{z = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)^{\mathtt{T}} \in \mathbb{R}^{n-1} \colon z < t1_{n-1}\}$, $i = 1, \ldots, n$.

Sketch of proof. If $n = 2$, the Leibniz integral rule applies,

$$\begin{aligned}
\frac{d}{dt} P\left(Z^{(2)} < t1_2\right) &= \frac{d}{dt} \int\limits_{-\infty}^{t} \int\limits_{-\infty}^{t} f_{Z^{(2)}}(z_1, z_2)\, dz_1\, dz_2 \\
&= \int\limits_{-\infty}^{t} \frac{d}{dt} \left(\int\limits_{-\infty}^{t} f_{Z^{(2)}}(z_1, z_2)\, dz_1 \right) dz_2 + \int\limits_{-\infty}^{t} f_{Z^{(2)}}(z_1, t)\, dz_1 \\
&= \int\limits_{-\infty}^{t} f_{Z^{(2)}}(t, z_2)\, dz_2 + \int\limits_{-\infty}^{t} f_{Z^{(2)}}(z_1, t)\, dz_1 = \mathcal{D}_2(f_{Z^{(2)}}, t), \quad t \in \mathbb{R}.
\end{aligned}$$

In the other cases, the assertion follows by induction with respect to n. □

Proof of Corollary 1. According to Theorem 1 and Remark 8, and with the random vector $Z \sim SS_{n-\nu,\nu,p}\left(E^{(\nu)}, g^{(n)}\right)$,

$$\begin{aligned}
f_{n:n}(t) =& (\nu + 1) F_{\nu,p}^{(2)}\left(0_\nu; I_\nu + E^{(\nu)} E^{(\nu)\mathtt{T}}, g_{(n)}^{(\nu)}\right) \cdot \frac{d}{dt} P\left(Z < t1_{n-\nu}\right) \\
=& (\nu + 1) F_{\nu,p}^{(2)}\left(0_\nu; I_\nu + 1_\nu 1_\nu^{\mathtt{T}}, g_{(n)}^{(\nu)}\right) \cdot \mathcal{D}_{n-\nu}(f_Z, t)
\end{aligned}$$

$$
\begin{aligned}
=&(\nu+1)\sum_{j=2}^{n-\nu}\int\limits_{D_j^{(n-\nu)}(t)} g_{(n)}^{(n-\nu)}\left(\left|z_{(j)}\right|_p\right) F_{\nu,p}^{(1)}\left(z_1 1_\nu; g_{[|z_{(j)}|_p]}^{(\nu)}\right)\, dz\\
&+(\nu+1)\int\limits_{D_1^{(n-\nu)}(t)} g_{(n)}^{(n-\nu)}\left(\left|z_{(1)}\right|_p\right) F_{\nu,p}^{(1)}\left(t 1_\nu; g_{[|z_{(1)}|_p]}^{(\nu)}\right)\, dz
\end{aligned}
$$

for $\nu \in \{1,\dots,n-2\}$, where $z_{(j)} = (z_1,\dots,z_{j-1},t,z_{j+1},\dots,z_{n-\nu})^{\mathrm{T}}$ and

$$
D_j^{(n-\nu)}(t) = \{z = (z_1,\dots,z_{i-1},z_{i+1},\dots,z_{n-\nu}) \in \mathbb{R}^{n-\nu-1} \colon z^{\mathrm{T}} < t1_{n-\nu-1}\}.
$$

Let $M_{l,j} \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ be the matrix defining the transposition $\sigma\colon \{1,\dots,n-\nu\} \to \{1,\dots,n-\nu\}$ with $\sigma(l)=j$, $\sigma(j)=l$ and $\sigma(\kappa)=\kappa$ for all $\kappa \in \{1,\dots,n-\nu\}\backslash\{l,j\}$. Using this notation, because of $\left|z_{(j)}\right|_p = \left|M_{l,j}z_{(l)}\right|_p$ and $|\det(M_{l,j})| = 1$, any two integrals being summands of the above sum with summation index j from $\{2,\dots,n-\nu\}$ can be transformed into each other. Because of this, one can also choose $j = n-\nu$ without any loss of generality. Then,

$$
\begin{aligned}
f_{n:n}(t) =&(\nu+1)(n-\nu-1)\int\limits_{D_{n-\nu}^{(n-\nu)}(t)} g_{(n)}^{(n-\nu)}\left(\left|z_{(n-\nu)}\right|_p\right) F_{\nu,p}^{(1)}\left(z_1 1_\nu; g_{[|z_{(n-\nu)}|_p]}^{(\nu)}\right)\, dz\\
&+(\nu+1)\int\limits_{D_1^{(n-\nu)}(t)} g_{(n)}^{(n-\nu)}\left(\left|z_{(1)}\right|_p\right) F_{\nu,p}^{(1)}\left(t 1_\nu; g_{[|z_{(1)}|_p]}^{(\nu)}\right)\, dz\\
=&(\nu+1)(n-\nu-1)\int\limits_{z\in D(t)} g_{(n)}^{(n-\nu)}\left(\sqrt[p]{|t|^p+|z|_p^p}\right) F_{\nu,p}^{(1)}\left(z_1 1_\nu; g_{[\sqrt[p]{|t|^p+|z|_p^p}]}^{(\nu)}\right)\, dz\\
&+(\nu+1)\int\limits_{z\in D(t)} g_{(n)}^{(n-\nu)}\left(\sqrt[p]{|t|^p+|z|_p^p}\right) F_{\nu,p}^{(1)}\left(t 1_\nu; g_{[\sqrt[p]{|t|^p+|z|_p^p}]}^{(\nu)}\right)\, dz
\end{aligned}
$$

for every $\nu \in \{1,\dots,n-2\}$ where $D(t) = D_{n-\nu}^{(n-\nu)}(t) = \{z \in \mathbb{R}^{n-\nu-1} \colon z < t1_{n-\nu-1}\}$. The case $\nu = n-1$ can be dealt with in an analogous way. $\square$

Proof of Remark 5. If $\nu \in \{1,\dots,n-1\}$ and $a = |y|_p$ with $y \in \mathbb{R}^{n-\nu}$, it follows from Fubini's theorem that

$$
\begin{aligned}
F_{\nu,p}^{(1)}\left(x; g_{PE[a]}^{(\nu)}\right) &= \int_{\mathbb{R}_+^\nu} g_{PE[a]}^{(\nu)}\left(|x-v|_p\right)\, dv = \int_{\mathbb{R}_+^\nu} g_{PE}^{(\nu)}(|x-v|_p)\, dv\\
&= \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_\nu}\left(\prod_{j=1}^{\nu} g_{PE}^{(1)}(|\tilde{v}_j|^p)\right) d\tilde{v}_\nu\cdots d\tilde{v}_1 = \prod_{j=1}^{\nu}\Phi_p(x_j), \quad x \in \mathbb{R}^\nu.
\end{aligned}
$$

Using this and Fubini's theorem again, Corollary 1 leads to

$$
f_{n:n}(t) = (\nu+1)\varphi_p(t)\left(\Phi_p(t)\right)^{n-1}
$$

$$+ (\nu+1)(n-\nu-1)\varphi_p(t)\left(\Phi_p(t)\right)^{n-\nu-2} \int\limits_{-\infty}^{t} \varphi_p(s)\left(\Phi_p(s)\right)^{\nu}\, ds, \quad t \in \mathbb{R},$$

for every $\nu \in \{1, \ldots, n-2\}$, and to

$$f_{n:n}(t) = n\varphi_p(t)\left(\Phi_p(t)\right)^{n-1}, \quad t \in \mathbb{R},$$

for $\nu = n-1$. Because of $\int\limits_{-\infty}^{t} \varphi_p(s)\left(\Phi_p(s)\right)^{\nu}\, ds = \frac{1}{\nu+1}\left(\Phi_p(t)\right)^{\nu+1}$, the first $n-2$ representations can be transformed into the claimed form whereby the parameter $\nu \in \{1, \ldots, n-2\}$ does not further have any influence onto these results. $\square$

6 Discussion

In the present paper, we continue our studies on exact distributions of order statistics under non-standard model assumptions, and provide some new results on it. The dependence of the variables considered here is caused in the interplay of the dg and the shape/tail parameter of the multivariate sample distribution. As in the most known spherical case $p = 2$, the uncorrelatedness of the considered rvs leads to their independence if and only if the dg of the sample vector distribution is that of the p-generalized multivariate Gaussian distribution. Because of the possible arbitrary choice of p, $p > 0$, our considerations are not restricted to sample vector densities being convex contoured as in Richter (2015b) but include cases where density level sets are radially concave w.r.t. the standard fan in $\mathbb{R}^n$, see Richter (2014, 2015a). We have established an advanced geometric method using measure-of-cone representations for deriving extreme value distributions of jointly continuous $l_{n,p}$-symmetrically distributed dependent rvs.

7 Acknowledgement

The authors are grateful to the reviewers for their valuable hints leading to various improvements of exposition and making the paper more accessible to a broader readership.

A Subclasses of $l_{n,p}$-symmetric distributions

Let us recall that the well known class of spherical distributions is a subclass of the family of $l_{n,p}$-symmetric distributions. Moreover, if a random vector X follows an $l_{n,p}$-symmetric distribution then it allows the stochastic representation (3) where the variable R follows the density (4). In this Appendix, the dg $g^{(n)}$ attains values from the set of functions $\left\{g^{(n)}_{Kt;M,\beta,\gamma}, g^{(n)}_{PE;\gamma}, g^{(n)}_{PE}, g^{(n)}_{PT7;M,\nu}, g^{(n)}_{St;\nu}, g^{(n)}_{C}, g^{(n)}_{PT2;\nu}\right\}$ explained below and the corresponding generalized radius variable R will be denoted by $R_{Kt;M,\beta,\gamma}$, $R_{PE;\gamma}$, R_{PE}, $R_{PT7;M,\nu}$, $R_{St;\nu}$, R_C, and $R_{PT2;\nu}$, respectively.

A.1 The $l_{n,p}$-symmetric Kotz type distribution

According to Müller and Richter (2015a) and references cited therein, the continuous $l_{n,p}$-symmetrically distribution with dg

$$g^{(n)}_{Kt;M,\beta,\gamma}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\gamma\beta^{\frac{n+p(M-1)}{p\gamma}}\Gamma\left(\frac{n}{p}\right)}{\Gamma\left(\frac{n+p(M-1)}{p\gamma}\right)} r^{p(M-1)} \exp\{-\beta r^{p\gamma}\}, \quad r > 0.$$

is called $l_{n,p}$-symmetric Kotz type distribution with parameters $M > 1 - \frac{n}{p}$, $\beta > 0$ and $\gamma > 0$. In addition, one may establish that both the first and the second moment of this distribution, cf. Remark 3 in the present paper, exist for all choices of the parameters where the expectation of the corresponding radius variable $R^{(n)}_{Kt;M,\beta,\gamma}$ is

$$\mathbb{E}\left(R_{Kt;M,\beta,\gamma}\right) = \int_0^\infty r^n\, g^{(n)}_{Kt;M,\beta,\gamma}(r)\, dr = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(\frac{n+1+p(M-1)}{p\gamma}\right)}{p\beta^{\frac{1}{p\gamma}}\Gamma\left(\frac{n+p(M-1)}{p\gamma}\right)}$$

and univariate variance component satisfies

$$\sigma^2_{p,g^{(n)}_{Kt;M,\beta,\gamma}} = \tau_{n,p}\mathbb{E}\left(R^2_{Kt;M,\beta,\gamma}\right) = \beta^{-\frac{2}{p\gamma}} \frac{\Gamma\left(\frac{3}{p}\right)\Gamma\left(\frac{n}{p}\right)\Gamma\left(\frac{n+2+p(M-1)}{p\gamma}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{n+2}{p}\right)\Gamma\left(\frac{n+p(M-1)}{p\gamma}\right)}$$

with $\tau_{n,p}$ from Remark 3. Specializing the $l_{n,p}$-symmetric Kotz type distribution by the parameter choice $M = 1$ and $\beta = \frac{1}{p}$ yields the $l_{n,p}$-symmetric power exponential distribution with parameter $\gamma > 0$ having dg $g^{(n)}_{PE;\gamma}$ with

$$g^{(n)}_{PE;\gamma}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\gamma\Gamma\left(\frac{n}{p}\right)}{p^{\frac{n}{p\gamma}}\Gamma\left(\frac{n}{p\gamma}\right)} \exp\{-\frac{r^{p\gamma}}{p}\}, \quad r > 0,$$

i.e. $g^{(n)}_{PE;\gamma} = g^{(n)}_{Kt;1,\frac{1}{p},\gamma}$. Therefore, the first and the second order moments of this distribution exist for all $\gamma > 0$ since

$$\mathbb{E}\left(R_{PE;\gamma}\right) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma\left(\frac{n}{p}\right)\Gamma\left(\frac{n+1}{p\gamma}\right)}{p^{1-\frac{1}{p\gamma}}\Gamma\left(\frac{n}{p\gamma}\right)} \qquad \text{and} \qquad \sigma^2_{p,g^{(n)}_{PE;\gamma}} = p^{\frac{2}{p\gamma}} \frac{\Gamma\left(\frac{3}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}.$$

Another important special case of this distribution is the n-dimensional p-power exponential or p-generalized Gaussian or p-generalized Laplace distribution. Its dg is denoted by g_{PE} and satisfies

$$g_{PE}^{(n)}(r) = \left(\frac{p^{1-\frac{1}{p}}}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \exp\left\{-\frac{r^p}{p}\right\}, \quad r > 0,$$

i.e., $g_{PE}^{(n)} = g_{PE;1}^{(n)}$. Furthermore, the first two moments of this distribution exist where

$$\mathbb{E}\left(R_{PE}\right) = \frac{p^{n-1+\frac{1}{p}}}{\left(2\Gamma\left(\frac{1}{p}\right)\right)^n}\Gamma\left(\frac{n+1}{p\gamma}\right) \qquad \text{and} \qquad \sigma^2_{p,g_{PE}^{(n)}} = p^{\frac{2}{p}}\frac{\Gamma\left(\frac{3}{p}\right)}{\Gamma\left(\frac{1}{p}\right)},$$

and the components of an $l_{n,p}$-symmetrically distributed random vector with dg $g^{(n)}$ are not only uncorrelated but also independent if and only if $g^{(n)} = g_{PE}^{(n)}$.

A.2 The $l_{n,p}$-symmetric Pearson Type VII distribution

Another important subclass of the continuous $l_{n,p}$-symmetric distributions are the $l_{n,p}$-symmetric Pearson Type VII distributions with parameters $M > \frac{n}{p}$ and $\nu > 0$ whose dgs $g_{PT7;M,\nu}^{(n)}$ satisfy

$$g_{PT7;M,\nu}^{(n)}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma(M)}{\nu^{\frac{n}{p}}\Gamma\left(M-\frac{n}{p}\right)}\left(1+\frac{r^p}{\nu}\right)^{-M}, \quad r > 0.$$

Furthermore, the first moment of this distribution exists for all $M > \frac{n+1}{p}$ and $\nu > 0$ and the second one exists for all $M > \frac{n+2}{p}$ and $\nu > 0$ where

$$\mathbb{E}\left(R_{PT7;M,\nu}\right) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\nu^{\frac{1}{p}}\Gamma\left(\frac{n+1}{p}\right)\Gamma\left(M-\frac{n+1}{p}\right)}{p\Gamma\left(M-\frac{n}{p}\right)}$$

and univariate variance component satisfies the representation

$$\sigma^2_{p,g_{PT7;M,\nu}^{(n)}} = \nu^{\frac{2}{p}}\frac{\Gamma\left(\frac{3}{p}\right)\Gamma\left(M-\frac{n+2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(M-\frac{n}{p}\right)}.$$

In the sequel, we consider two well known special cases of this distribution. First, an $l_{n,p}$-symmetric Pearson Type VII distribution with parameters $M = \frac{n+\nu}{p}$ and $\nu > 0$ is called $l_{n,p}$-symmetric Student -t distribution with parameter $\nu > 0$. Hence, its dg satisfies

$$g_{St,\nu}^{(n)}(r) = g_{PT7;\frac{n+\nu}{p},\nu}^{(n)}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma(\frac{n+\nu}{p})}{\nu^{\frac{n}{p}}\Gamma\left(\frac{\nu}{p}\right)}\left(1+\frac{r^p}{\nu}\right)^{-\frac{n+\nu}{p}}, \quad r > 0,$$

and the first two moments exist for all $\nu > 1$ and $\nu > 2$, respectively, with

$$\mathbb{E}\left(R_{St;\nu}\right) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\nu^{\frac{1}{p}}\Gamma\left(\frac{n+1}{p}\right)\Gamma\left(\frac{\nu-1}{p}\right)}{p\Gamma\left(\frac{\nu}{p}\right)}$$

and univariate variance component

$$\sigma^2_{p,g^{(n)}_{St,\nu}} = \nu^{\frac{2}{p}} \frac{\Gamma\left(\frac{3}{p}\right)\Gamma\left(\frac{\nu-2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{\nu}{p}\right)}.$$

Second, an $l_{n,p}$-symmetric Student-t distribution with parameter $\nu = 1$ is called $l_{n,p}$-symmetric Cauchy distribution having the dg

$$g^{(n)}_C(r) = g^{(n)}_{St;1}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma(\frac{n+1}{p})}{\Gamma\left(\frac{1}{p}\right)} \left(1 + r^p\right)^{-\frac{n+1}{p}}, \quad r > 0,$$

In particular, both the first and the second order moments of this distribution do not exist.

A.3 The $l_{n,p}$-symmetric Pearson Type II distribution

Let $\mathbb{1}_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{otherwise} \end{cases}$ denote the indicator function of the set A. The third subclass mentioned here is the class of $l_{n,p}$-symmetric Pearson Type II distributions with parameter $\nu > 0$ having the dg

$$g^{(n)}_{PT2;\nu}(r) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma\left(\frac{n}{p} + \nu + 1\right)}{\Gamma(\nu + 1)} \left(1 - r^p\right)^\nu \mathbb{1}_{(0,1)}(r), \quad r > 0,$$

and finite first and second order moments for all parameters $\nu > 0$,

$$\mathbb{E}\left(R_{PT2;\nu}\right) = \left(\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\right)^n \frac{\Gamma\left(\frac{n+1}{p}\right)\Gamma\left(\frac{n}{p} + \nu + 1\right)}{p\Gamma\left(\frac{n+1}{p} + \nu + 1\right)}$$

as well as the univariate variance component

$$\sigma^2_{p,g^{(n)}_{PT2;\nu}} = \frac{\Gamma\left(\frac{3}{p}\right)\Gamma\left(\frac{n}{p} + \nu + 1\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{n+2}{p} + \nu + 1\right)}.$$

B Tables for domain quantiles of $l_{n,p}$-symmetric distributions w.r.t. the $l_{n,p}$-unit ball

Here, we compute values of the quantile function $\mathcal{Q}_{g^{(n)},p}$ at the points $q \in \{0.9, 0.95, 0.99, 0.995, 0.999\}$, see (7), for parameters $p \in \left\{\frac{1}{2}, 1, 3\right\}$ and $n \in \{1, 2, 3\}$ as well as for dgs $g^{(n)}_{PE;\gamma}$ with $\gamma \in \left\{1, \frac{3}{2}, 2\right\}$, $g^{(n)}_{St;\nu}$ with $\nu \in \left\{1, \frac{3}{2}, 2\right\}$, and $g^{(n)}_{PT2;\nu}$ with $\nu \in \left\{1, \frac{3}{2}, 2\right\}$. Remember that the $l_{n,p}$-symmetric power exponential distribution with parameter $\gamma = 1$ is the n-variate p-generalized Gaussian distribution and that the $l_{n,p}$-symmetric Student-t distribution with parameter $\nu = 1$ is the $l_{n,p}$-symmetric Cauchy distribution.

$\mathcal{Q}_{g^{(n)},p}$		$q = 0.9$	$q = 0.95$	$q = 0.99$	$q = 0.995$	$q = 0.999$
$g^{(n)} = g^{(n)}_{PE;1}$	$n = 1$	3.78248	5.62606	11.01693	13.80171	21.31398
	$n = 2$	11.15822	15.02980	25.22610	30.12625	42.65553
	$n = 3$	21.50489	27.63098	42.95809	50.05392	67.68966
$g^{(n)} = g^{(n)}_{PE;\frac{3}{2}}$	$n = 1$	1.61086	2.20110	3.69584	4.38556	6.07842
	$n = 2$	3.26265	4.12764	6.17533	7.07797	9.22476
	$n = 3$	4.99340	6.09025	8.60135	9.68196	12.20816
$g^{(n)} = g^{(n)}_{PE;2}$	$n = 1$	1.15129	1.49787	2.30259	2.64916	3.45388
	$n = 2$	1.94486	2.37193	3.31918	3.71506	4.61671
	$n = 3$	2.66116	3.14790	4.20298	4.63690	5.61444
$g^{(n)} = g^{(n)}_{St;1}$	$n = 1$	16.86950	40.80952	255.26532	536.13094	2855.38783
	$n = 2$	62.56620	145.97687	876.02844	1823.62826	9602.50605
	$n = 3$	136.57617	314.53687	1859.40003	3858.15678	20230.91278
$g^{(n)} = g^{(n)}_{St;\frac{3}{2}}$	$n = 1$	10.11725	20.55423	83.69141	144.66148	480.64167
	$n = 2$	35.59401	68.78394	262.54702	446.54143	1448.61431
	$n = 3$	75.93661	143.99535	536.15393	906.18318	2912.37601
$g^{(n)} = g^{(n)}_{St;2}$	$n = 1$	7.87603	14.72912	49.08511	77.53031	207.11770
	$n = 2$	26.81891	47.28216	145.38910	224.75442	580.38026
	$n = 3$	56.33814	97.05830	289.00536	442.88157	1127.84875
$g^{(n)} = g^{(n)}_{PT2;1}$	$n = 1$	0.64674	0.74762	0.88566	0.91891	0.96360
	$n = 2$	0.78813	0.85296	0.93570	0.95476	0.97990
	$n = 3$	0.84857	0.89610	0.95518	0.96856	0.98609
$g^{(n)} = g^{(n)}_{PT2;\frac{3}{2}}$	$n = 1$	0.54458	0.64958	0.81202	0.85678	0.92418
	$n = 2$	0.70954	0.78354	0.88875	0.91614	0.95628
	$n = 3$	0.78664	0.84311	0.92077	0.94053	0.96919
$g^{(n)} = g^{(n)}_{PT2;2}$	$n = 1$	0.46177	0.56460	0.73811	0.79053	0.87602
	$n = 2$	0.63854	0.71714	0.83772	0.87183	0.92560
	$n = 3$	0.72786	0.79012	0.88202	0.90732	0.94664

Table 4: q-domain quantiles of $\Phi_{g^{(n)},p}$ w.r.t. $B_{n,p}$ are computed for $q \in \{0.9, 0.95, 0.99, 0.995, 0.999\}$, $p = \frac{1}{2}$ and for several dgs.

$\mathcal{Q}_{g^{(n)},p}$		$q=0.9$	$q=0.95$	$q=0.99$	$q=0.995$	$q=0.999$
$g^{(n)}=g^{(n)}_{PE;1}$	$n=1$	2.30259	2.99573	4.60517	5.29832	6.90776
	$n=2$	3.88972	4.74386	6.63835	7.43013	9.23341
	$n=3$	5.32232	6.29579	8.40595	9.27379	11.22887
$g^{(n)}=g^{(n)}_{PE;\frac{3}{2}}$	$n=1$	1.42029	1.74719	2.42300	2.70017	3.28786
	$n=2$	2.01472	2.35508	3.05171	3.32429	3.91365
	$n=3$	2.47331	2.82329	3.53216	3.80771	4.40124
$g^{(n)}=g^{(n)}_{PE;2}$	$n=1$	1.16309	1.38590	1.82139	1.98487	2.32675
	$n=2$	1.51743	1.73082	2.14597	2.30181	2.62826
	$n=3$	1.76796	1.97671	2.38169	2.53359	2.85186
$g^{(n)}=g^{(n)}_{St;1}$	$n=1$	9.00000	19.00000	99.00000	198.00000	999.00000
	$n=2$	18.48683	38.49359	198.49874	398.49937	1998.49987
	$n=3$	27.97659	57.98860	297.99777	597.99889	2997.99978
$g^{(n)}=g^{(n)}_{St;\frac{3}{2}}$	$n=1$	5.46238	9.55209	30.81652	49.79928	148.50000
	$n=2$	10.68529	18.23436	57.41980	92.38924	274.20073
	$n=3$	15.85419	26.82194	83.73197	134.51540	398.54241
$g^{(n)}=g^{(n)}_{St;2}$	$n=1$	4.32456	6.94427	18.00000	26.28427	61.24555
	$n=2$	8.21450	12.77647	31.95405	46.30900	106.87166
	$n=3$	12.02925	18.48940	45.62059	65.92374	151.57507
$g^{(n)}=g^{(n)}_{PT2;1}$	$n=1$	0.68377	0.77639	0.90000	0.92929	0.96838
	$n=2$	0.80420	0.86465	0.94110	0.95860	0.981630
	$n=3$	0.85744	0.90239	0.95800	0.97056	0.98698
$g^{(n)}=g^{(n)}_{PT2;\frac{3}{2}}$	$n=1$	0.60190	0.69829	0.84151	0.87989	0.93690
	$n=2$	0.73796	0.80597	0.90112	0.92562	0.96134
	$n=3$	0.80336	0.85591	0.92757	0.94570	0.97191
$g^{(n)}=g^{(n)}_{PT2;2}$	$n=1$	0.53584	0.63160	0.78456	0.82900	0.90000
	$n=2$	0.67954	0.75140	0.85913	0.88912	0.93596
	$n=3$	0.75336	0.81074	0.89436	0.91717	0.95245

Table 5: q-domain quantiles of $\Phi_{g^{(n)},p}$ w.r.t. $B_{n,p}$ are computed for $q \in \{0.9, 0.95, 0.99, 0.995, 0.999\}$, $p = 1$ and for several dgs.

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$\mathcal{Q}_{g^{(n)},p}$		$q=0.9$	$q=0.95$	$q=0.99$	$q=0.995$	$q=0.999$
$g^{(n)}=g^{(n)}_{PE;1}$	$n=1$	1.42746	1.64110	2.02451	2.15909	2.42742
	$n=2$	1.71881	1.90639	2.24822	2.36993	2.61515
	$n=3$	1.90449	2.07910	2.39951	2.51432	2.74675
$g^{(n)}=g^{(n)}_{PE;\frac{3}{2}}$	$n=1$	1.16826	1.30722	1.53742	1.61338	1.75868
	$n=2$	1.33689	1.45103	1.64672	1.71310	1.84236
	$n=3$	1.43489	1.53747	1.71617	1.77756	1.89815
$g^{(n)}=g^{(n)}_{PE;2}$	$n=1$	1.06993	1.17947	1.35102	1.40527	1.50626
	$n=2$	1.19477	1.28105	1.42285	1.46939	1.55802
	$n=3$	1.26296	1.33895	1.46663	1.50926	1.59137
$g^{(n)}=g^{(n)}_{St;1}$	$n=1$	5.65526	11.31963	56.60462	113.20932	566.04668
	$n=2$	8.26628	16.53895	82.69930	165.39866	826.99334
	$n=3$	9.99667	19.99917	99.99997	199.99999	1000.00000
$g^{(n)}=g^{(n)}_{St;\frac{3}{2}}$	$n=1$	3.21011	5.12738	15.02211	23.84758	69.73213
	$n=2$	4.45601	7.09669	20.77248	32.97533	96.42146
	$n=3$	5.29552	8.42729	24.66130	39.14835	114.47139
$g^{(n)}=g^{(n)}_{St;2}$	$n=1$	2.50051	3.59265	8.09566	11.45457	25.61940
	$n=2$	3.35881	4.79272	10.76385	15.22659	34.05237
	$n=3$	3.94177	5.61346	12.59501	17.81587	39.84178
$g^{(n)}=g^{(n)}_{PT2,1}$	$n=1$	0.75717	0.83269	0.92755	0.94914	0.97747
	$n=2$	0.84247	0.89242	0.95387	0.96768	0.98572
	$n=3$	0.88099	0.91910	0.96549	0.97585	0.98935
$g^{(n)}=g^{(n)}_{PT2;\frac{3}{2}}$	$n=1$	0.70942	0.78726	0.89333	0.92009	0.95872
	$n=2$	0.80051	0.85488	0.92776	0.94597	0.97215
	$n=3$	0.844319	0.88718	0.94410	0.95824	0.97851
$g^{(n)}=g^{(n)}_{PT2;2}$	$n=1$	0.67137	0.74955	0.86175	0.89194	0.93827
	$n=2$	0.76511	0.82179	0.90212	0.92358	0.95642
	$n=3$	0.81223	0.85799	0.92231	0.93940	0.96549

Table 6: q-domain quantiles of $\Phi_{g^{(n)},p}$ w.r.t. $B_{n,p}$ are computed for $q \in \{0.9, 0.95, 0.99, 0.995, 0.999\}$, $p=3$ and for several dgs.

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