Exact inference on scaling parameters in norm and antinorm contoured sample distributions

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Abstract

Exact distributions of generalized Chi-square and Fisher statistics are used to derive confidence intervals and significance tests for inferring on one or two scaling parameters, respectively, under non-standard assumptions w.r.t. the multivariate sample distribution. The latter may have convex or radially concave density level sets and heavy or light distribution centers and tails. Independent and $l_{n,p}$-dependent sample variables are considered.

Keywords: estimating and testing scaling parameters, non-standard model assumptions, generalized Chi-square and Fisher distributions, heavy and light distribution tails, heavy and light distribution centers, independence sampling, dependence sampling

1 Introduction

The well known Chi-square and Fisher distributions were introduced already a long time ago in [12], [22] and [7, 9, 10], respectively. Since then, methods of statistical inference on variances are extensively studied under various aspects in numerous papers, see e.g. [24], [19], [38] and the recent survey in [3], for only to mention a few of them.

It is often natural to assume that expectations are unknown and thus to additionally be estimated when studying variances. In some cases, however, it may be reasonable to assume that expectations are known. Assume, for example, a company produces tubes of a certain norm length. The production dependent deviations from this length may according to [4], p. 162 be considered as normally distributed with expectation zero and unknown variance. If the company indicates to customers a certain variance of tube length then one might be interested in proving this.

The successful fulfillment of manufacturing orders requires according to [11] machine tools with high thermal stability. Rapidly changing operating conditions for machine tools, however, may lead to unknown variances of technical features having a known expectation and make it difficult to constantly ensure accuracy in production processes.

Moreover, comparing in another situation instrumental accuracy by measuring a certain standard several times with two gages may generate random samples having the same known mean and unknown variances which one might be interested then to compare with each other.

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If in such situations independent random variables $X_1, ..., X_n$ and $Y_1, ..., Y_m$ follow Gaussian laws with known expectations μ_1, μ_2 and unknown standard deviations σ_1, σ_2, respectively, then, with $\hat{\sigma}_1^2 = \frac{1}{n} \sum_1^n (X_i - \mu_1)^2$ being the maximum likelihood estimator of the variance σ_1^2, $n(\frac{\hat{\sigma}_1}{\sigma_1})^2$ follows a Chi-square density with n d.f., and $\frac{n(\hat{\sigma}_1/\sigma_1)^2}{m(\hat{\sigma}_2/\sigma_2)^2}$ follows Fisher's F-distribution with (n, m) d.f..

The aim of the present paper is to allow more general distributions of the sample vector, and to study exact distributions of generalized Chi-square and Fisher statistics being based upon the maximum likelihood estimators of the model's scaling parameters. The multivariate sample distributions considered to this end are convex or radially concave contoured. These two distribution classes are subclasses of the family of star-shaped distributions.

The paper is organized as follows. Star-shaped distributions are introduced in Section 2. A one sample non-standard scaling model for i.i.d. power exponentially distributed random variables and a certain extension of it are introduced in Section 3.1. Confidence estimation of and significance testing of hypotheses on the scaling parameter are discussed in Sections 3.2 and 3.3, respectively. A two sample non-standard scaling parameter model ist studied then in Section 3.4 where estimation and testing are dealt with. A brief comparison of norms and antinorms, generating convex or radially concave level sets of the multivariate sample distribution, respectively, is given in the Appendix.

Finally, we remark that an introduction to non-standard models and exact distribution theory is given in [30, 32]. For closely related studies on expectations and variances based upon $l_{n,p}$-norms and goodness of fitting using $l_{n,p}$-norms for data sets that follow a power exponential distribution, we refer to [16]-[18]. For a comparison of the variances of two dependent random variables, see [39].

2 Star-shaped sample distributions

2.1 The general class

The most general non-standard distributions of the multivariate sample vector we are considering in the present paper are star-shaped distributions. More specific distributions are $l_{n,p}$-symmetric and p-generalized elliptically contoured distributions considered, e.g., in [30] and [32], respectively. Let $K \subset \mathbb{R}^n$ denote a star body with the origin being an inner point, and let S be its topological boundary which will be called a star sphere. The Minkowski functional of K is defined by $h_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}$, $x \in \mathbb{R}^n$ and satisfies $h_K(x) = 1, x \in S$. Assume that a random vector X defined on a probability space $(\Omega, \mathfrak{A}, P)$ allows the stochastic representation

$$X \stackrel{d}{=} R \cdot U_S$$

where R is a non-negative random variable and U_S follows independently of R the S-generalized uniform distribution ω_S on S. Then X is called origin-including star-shaped distributed. Note that $X \stackrel{d}{=} Y$ means that X and Y follow the same distribution law, and

$$\omega_S(A) = \frac{O_S(A)}{O_S(S)}, A \in \mathfrak{B}_S = \mathfrak{B}^n \cap S$$

where $\mathfrak{B}^n$ denotes the Borel σ-field in $\mathbb{R}^n$ and O_S the star generalized surface content measure on $\mathfrak{B}_S$. If μ stands for the Lebesgue measure on $\mathfrak{B}^n$ and $CPC(A) = \{x \in \mathbb{R}^n : x/h_K(x) \in A\}$ is the central projection cone generated by $A \in \mathfrak{B}_S$ then the star generalized surface content of A is formally defined for a particular class of star spheres S in [27] and under the weak technical Assumption 1 for general star spheres in [32] by

$$O_S(A) = f'_A(1) \tag{1}$$

where

$$f'_A(r) = \frac{d}{dr} f_A(r), \quad f_A(r) = \mu(sector(A, r)), \quad sector(A, r) = CPC(A) \cap K(r)$$

and $K(r) = r \cdot K = \{(rx_1, ..., rx_n)^T \in \mathbb{R}^n : (x_1, ..., x_n)^T \in K\}$. Integral, or differential geometric, characterizations of O_S which can equivalently be taken as a definition of O_S are given for particular classes of star bodies in [28, 31] and for arbitrary two-dimensional convex star bodies in [34]. Note that, e.g., the well known ([27]) $l_{n,p}$-generalized (non-Euclidean) surface content of the $l_{n,p}$-unit sphere,

$$\omega_{n,p} = \frac{2^n \Gamma(\frac{1}{p})^n}{p^n \Gamma(\frac{n}{p})}, \, p > 0,$$

is equal to the Euclidean surface content, if and only if $p = 2$.

There are widespread miss-understandings w.r.t. this circumstance in the literature. The reader should be aware that there is some literature where formulas are derived which are called surface content and uniform distribution, respectively, however neither defining in advance the notion of surface content nor referring to literature where such definition is given.

The star generalized surface measure on the boundary of n-dimensional convex or radially concave star bodies is dealt with under different aspects in [33, 35, 36]. A general result in [36] says that

$$O_S(A) = \int\limits_{G(A)} \hat{h}_{\hat{K}}(N(x)) dx$$

where

$$\hat{K} = \{\lambda x : x \in Grad(S), 0 \le \lambda \le 1\}, \quad Grad(S) = \{\nabla h_K(x), x \in S\}$$

and

$$\hat{h}_{\hat{K}}(t) = \inf\{\lambda > 0 : t \in \lambda \hat{K}\}, \, t \in pos \, \hat{K}$$

is a certain modification of the Minkowski functional of $\hat{K}$. Here, pos $x = \{\lambda x : \lambda \ge 0\}, x \in \mathbb{R}^n$ and pos $M = \bigcup\limits_{x \in M} pos \, x$.

If X is origin-including star-shaped distributed and $\nu \in st(K)$ where $st(K)$ denotes the set of all points w.r.t. which K is star-shaped then $X - \nu$ is also origin-including star-shaped distributed. If, however, ν does not belong to $st(K)$ then $X - \nu$ will be called just star-shaped distributed. The first of these two cases is dealt with in [32], Section 5.1 where non-concentric elliptically contoured distributions are introduced. Both cases are considered when studying non-central Chi-square distributions in [14]. For a density having non-concentric circular contours with centers on a straight line, see [2].

Example Let $a = (a_1, ..., a_n)^T \in \mathbb{R}^n$ be a vector having positive components, $p \ge 1$, and $K = \{x \in \mathbb{R}^n : (\sum\limits_{i=1}^{n} |\frac{x_i}{a_i}|^p)^{1/p} \le 1\}$. Then, according to [32], the star-generalized surface content measure on the boundary S of K is

$$O_S(A) = a_n \int\limits_{G(A)} \frac{d(x_1, ..., x_{n-1})}{(1 - \sum\limits_{i=1}^{n} |\frac{x_i}{a_i}|^p)^{1-1/p}}, \, A \in \mathfrak{B}_S. \tag{2}$$

The special case $p = 2$ has been dealt with in [31].

2.2 Norm contoured distributions

Note that a symmetric convex star body having the origin as an inner point can always be generated by a norm, $h_K(.) = ||.||$, and that according to [35]

$$O_S(A) = \int\limits_{G(A)} ||N(x)||^* dx, \ A \in \mathfrak{B}_S \tag{3}$$

where $G(A) = \{\vartheta \in \mathbb{R}^{n-1} : \exists \eta = \eta(\vartheta) \ s.t. \ (\vartheta^T, \eta(\vartheta)^T)^T \in A \}$ and $||.||^*$ denotes the norm dual to $||.||$ and $N(\vartheta) = (\nabla\eta(\theta), -1)^T$ is the outer normal vector to S at $(\vartheta^T, \eta(\vartheta))^T$. If K is the star body from the Example in Section 2.1 then (3) takes the specific form (2).

Let us emphasize that the star generalized surface content of an arbitrary measurable subset A of the star sphere S is a sum of integrals, in general. If S^+ and S^-, however, denote upper and lower half spheres of the norm sphere S, and $K^* = \{y \in \mathbb{R}^n : ||y||^* \leq 1\}$, then

$$O_S(A) = \int\limits_{G(A \cap S^+)} h_{K^*}(N(x))dx + \int\limits_{G(A \cap S^-)} h_{K^*}(N(x))dx, \ A \in \mathfrak{B}_S$$

where

$$G(A \cap S^{+(-)}) = \{\vartheta \in \mathbb{R}^{n-1} : \exists \eta = \eta(\vartheta) \ s.t. \ (\vartheta^T, \eta^T)^T \in A \cap S^{+(-)}\}.$$

For simplicity, and in accordance with Definition 1, Lemma 1 and Corollary 1 in [32], however, $O_S(A)$ will be represented throughout this paper symbolically by a single integral.

Finally note that formula (1) applies whenever it is possible to evaluate the volume of a convex body. W.r.t. this research area we refer, e.g., to [23]. and a discussion on page 11 in [29]. Note that K^* is always a symmetric convex star body and defines therefore a metric geometry in the sense of [5].

2.3 Antinorm contoured distributions

If K is radially concave w.r.t. a fan $\mathcal{F} = \{C_1, C_2, ...\}$ and belongs to the particular class $AN1$ of star bodies introduced in [35] then its Minkowski functional is an antinorm, $h_K(.) = \dagger.\dagger$. For the notion of antinorm, we refer to [20] and the Appendix. Let

$$h_K^{\mathcal{F}}(u) = \sum_i I_{C_i}(u) \inf\{u^T y : y \in S \cap C_i\}, u \in \mathbb{R}^n$$

be the anti-support function of K w.r.t. $\mathcal{F}$ and

$$K^o = \{\lambda(u)u : 0 \leq \lambda(u)h_K^{\mathcal{F}}(u) \leq 1, u \in S_E^{(n-1)}\}$$

the anti-polar set of K where $S_E^{(n-1)}$ denotes the Euclidean unit sphere in $\mathbb{R}^n$ then according to [35]

$$O_S(A) = \int\limits_{G(A)} h_{K^o}(N(x))dx \tag{4}$$

where $N(x)$ is the inner normal vector to S at $x \in S$.

Example Let the antinorm ball and sphere K and S, respectively, be formally defined as in the preceding example but with $p \in (0,1]$, and let a still be a vector with positive components. Then the S-generalized surface content measure $\mathfrak{O}_S$ satisfies representation (2). Notice, however, that we are evaluating star generalized surface contents of subsets of antinorm spheres, here, while star generalized surface contents of subsets of norm spheres are considered in the preceding example. Notice that $1 - 1/p > 0$ if $p > 1$ and $1 - 1/p < 0$ if $0 < p < 1$.

2.4 Continuous distributions

Let a function $g : [0, \infty) \to [0, \infty)$ satisfy the assumptions $0 < I(g) < \infty$ where $I(g) = \int_0^\infty r^{n-1} g(r) dr$. Such function is called a density generating function (dgf). A dgf can be used modeling light or heavy distribution centers and tails. Moreover, it allows even modeling complete level distributions. Well known examples are the power-exponential dgf $g_{PE}(r) = \exp\{-\frac{r^p}{p}\}$, Kotz type dgf $g_K(r) = r^{M-1} e^{-\beta r^\gamma}, \beta, \gamma > 0, 2M + n > 2$ and Pearson-VII-type dgf $g_{PT7}(r) = (1 + \frac{r}{m})^{-M}, M > n/2, m > 0$ where $I(g_{PE}) = p^{n/p-1} \Gamma(n/p)$, $I(g_K) = \frac{\Gamma((n+M-1)/\gamma)}{\gamma \beta^{(n+M-1)/\gamma}}$ and $I(g_{PT7}) = m^n B(M - n, n)$, respectively. Student- and Cauchy-type dgf's are special cases of g_{PT7}.

If the star-shaped distributed random vector X has a probability density (p.d.), $\varphi_{g,K,\nu}$ say, then

$$\varphi_{g,K,\nu}(x) = C(g, K) g(h_K(x - \nu)), x \in \mathbb{R}^n.$$

Densities of this type have been considered in [2] and [8]. Here,

$$C(g, K) = 1/(I(g) O_S(S)), \tag{5}$$

and the p.d. of the generating variate R is

$$f(r) = I_{(0,\infty)}(r) r^{n-1} g(r).$$

The corresponding probability distribution allows the geometric measure representation

$$\Phi_{g,K,\nu}(B) = \frac{1}{I(g)} \int_0^\infty r^{n-1} g(r) \mathfrak{F}_S(B, r) dr, \ B \in \mathfrak{B}^n \tag{6}$$

where

$$\mathfrak{F}_S(B, r) = \omega_S([\frac{1}{r}(B - \nu)] \cap S), \ r > 0$$

denotes the intersection proportion function (ipf) of the set B.

If K is the Euclidean unit ball then $K^* = K$ and O_S is the Euclidean surface content measure. In this case, representation (6) has been considered already in [25, 26]. For specific non-Euclidean cases, (6) has been considered in [27, 28, 31, 32].

Surveys of different types of applications of formula (6) are given for $n = 2$ in [34] and for arbitrary n in [30, 32]. For two elementary applications, see Section 2.6.

2.5 Building subclasses and class extensions

Building subclasses of distributions by modeling R. Let us denote the probability law of the generating variate R by P^R, and let ν be a σ-finite measure on the Borel σ-field $\mathfrak{B}^{(1)}$ (or its restriction to the positive real line) such that P^R is absolutely continuous w.r.t. ν. If ν is the Lebesgue measure on $\mathfrak{B}^{(1)}$ then $\frac{dP^R}{d\nu}(r) = f(r)$ is the p.d. considered in Section 2.4. In Sections 2.2 and 2.3, P^R can be chosen as the distribution of any non-negative random variable thus $\frac{dP^R}{d\nu}$ being a mixture of an absolutely continuous, a discrete and a singular component, in general.

Extending distribution classes by restricting U. We emphasize for the rest of this section that S circumscribes the origin as in inner point. Let $\mathcal{M}$ be a measurable subset of the star sphere S. We call a random vector $U_\mathcal{M}$ star-generalized uniformly distributed on $\mathcal{M}$ (or the $\mathcal{M}$-restriction of U) if $U_\mathcal{M}$ follows the distribution law

$$P^{U_\mathcal{M}}(A) = \frac{O_S(A)}{O_S(\mathcal{M})}, \ A \in \mathcal{B}_S \cap \mathcal{M}.$$

Note that $U_S \overset{d}{=} U$ and that pos $\mathcal{M}$ is not an absorbing set unless for $\mathcal{M} = S$.

For an example of an $\mathcal{M}$-restriction of U occuring in depth-based classification for distributions with non-convex support, we refer to [13].

We call
$$\mathcal{D}_S(\mathcal{M}) = \{\mathcal{L}\{R \cdot U_\mathcal{M}\}, \ R \text{ is a non-negative random variable }\}$$

where $\mathfrak{L}(Y)$ means the distribution law of Y the class of directionally $\mathcal{M}$-out-of-S restricted distributions and
$$\mathcal{D}_S = \{\mathcal{D}_S(\mathcal{M}) : \mathcal{M} \text{ is a measurable subset of the star sphere } S \}$$

the class of directionally out-of-S restricted distributions. Finally,
$$\mathcal{D} = \{\mathcal{D}_S : S \text{ is a star sphere }\}$$

is called the class of directionally out-of a star sphere restricted distributions in $\mathbb{R}^n$.

2.6   Two elementary types of applications of formula (6)

The ipf is the indicator function of an interval. Assume that the random vector X follows the continuous star-shaped distribution law $\Phi_{g,K,\nu}$, and let the statistic $T : \mathbb{R}^n \to [0, \infty)$ be defined by $T(x) = h_K(x - \nu)$. The ipf of the set $B(t) = \{x \in \mathbb{R}^n : T(x) < t\}$ allows then the representation
$$\mathfrak{F}_S(B(t), r) = \frac{O_S([\frac{1}{r}(B(t) - \nu)] \cap S)}{O_S(S)} = \begin{cases} 1 & r \in I(t) \\ 0 & \text{elsewhere} \end{cases}$$

where S is still the boundary of K and $I(t)$ is the particular interval $[0, t)$. It follows from (6) that

$$P(T(X) < t) = \Phi_{g,K,\nu}(\{x \in \mathbb{R}^n : h_K(x - \nu) < t\}) = \frac{1}{I(g)O_S(S)} \int\limits_0^\infty r^{n-1} g(r) I_{[0,t)} dr.$$

Now, a density formula can be derived from the representation

$$P(T(X) < t) = \frac{1}{I(g)O_S(S)} \int\limits_{I(t)} r^{n-1} g(r) dr.$$

Similar results hold for arbitrary intervals $I(t)$. For a specific application of the present type we refer to the calculation of generalized Chi-square densities.

The ipf is constant. Let $\mathfrak{F}_{B(t)}(r) = C(t)$ where $C(t) \in (0, 1)$ does not depend on r. Then it follows from (6) that $P(T < t) = C(t)$, and T is called a g-robust statistic. For specific such applications we refer to the geometric derivation of generalized Student and Fisher distributions. The notion of robustness considered here was studied already in [6] for elliptically contoured distributions.

Several other types of sets for which the ipf has been calculated already in the past can be found in some of author's papers surveyed in [30, 32].

3 Inference on scaling parameters

3.1 The power exponential sample distribution

A basic scaling parameter model. Let the i.i.d. random variables $X_1, ..., X_n$ follow the power exponential (or p-generalized Gaussian or p-generalized Laplace) density $f_p(.; \mu, \sigma)$, where $p > 0$ and $\mathbb{E}X_1 = \mu$ are known, and

$$f_p(x; \mu, \sigma) = \frac{C_p}{\sigma} \exp\{-\frac{|x - \mu|^p}{p\sigma^p}\}, \; -\infty < x < \infty. \tag{7}$$

Note that $C_p = p^{1-1/p}/(2\Gamma(1/p))$ and $V(X_1) = \sigma^2 \Gamma(\frac{3}{p})/\Gamma(\frac{1}{p})$. Let X_0 follow the p.d. of the standardized p-generalized normal distribution, $f_p(.; 0, 1)$. Then X_1 satisfies the representation $X_1 \overset{d}{=} \mu + \sigma X_0$. The quantity σ is called a scaling parameter.

Norm and antinorm contoured sample distributions. The density of the sample vector $X = (X_1, ..., X_n)$ is

$$\prod_{i=1}^{n} f_p(x_i; \mu, \sigma) = \frac{C_p^n}{\sigma^n} \exp\{-\sum_{i=1}^{n} \frac{|x_i - \mu|^p}{p\sigma^p}\}, \; x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

The density super level sets $\{x \in \mathbb{R}^n : \prod_{i=1}^{n} f_p(x_i; \mu, \sigma) \geq t\}$, $t > 0$ are generated by the functional

$$x \to |x|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

which is a norm if $p \geq 1$ and an antinorm if $0 < p \leq 1$. Here, $K = B_p = \{x \in \mathbb{R}^n : |x|_p \leq 1\}$.

For getting a more concrete idea of the shape of this density let us recall that the boundaries of the two-dimensional density super level sets are the star circles

$$\{(x_1, x_2)^T \in \mathbb{R}^2 : |x_1|^p + |x_2|^p = r^p\}, \; r > 0.$$

The star discs circumscribed by these p-circles are convex if $p \geq 1$ and radially concave if $0 < p \leq 1$. For the latter notion, we refer to [20]. Similar properties hold in the case of arbitrary finite dimension.

In the geometric measure representation (6), the star generalized surface content measure allows for every $p > 0$ and every measurable subset A of the upper (or lower) half of the star sphere S the representation

$$O_S(A) = \int_{G(A)} (1 - \sum_{1}^{n-1} |x_i|^p)^{(1-p)/p} dx \tag{8}$$

for $p \geq 1$ and $p \in (0, 1]$ being each time a special case of (3) and (4), respectively.

Extended scaling parameter model. Let now $X_1, ..., X_n$ be i.i.d. random variables following the densities

$$f_p(x; \mu_i, \sigma^2 a_i^2) = \frac{C_p}{\sigma a_i} \exp\{-\frac{|x - \mu_i|^p}{p(\sigma a_i)^p}\}, \; x \in \mathbb{R}, \; i = 1, ..., n \tag{9}$$

where $\mathbb{E}X_i = \mu_i \in \mathbb{R}, p > 0, a_i > 0$ are known, and $\sigma > 0$ is unknown. Note that

$$\mathbb{V}(X_i) = \sigma^2 a_i^2 \frac{\Gamma(3/p)}{\Gamma(1/p)}$$

and that X_i allows the representation

$$X_i \overset{d}{=} \mu_i + \sigma a_i X_0$$

where X_0 has the p.d. $f_p(.; 0, 1, 1)$ and σ is a common scaling parameter.

3.2 Inference on a single scaling parameter

Basic scaling parameter model. The maximum likelihood estimator of σ in the model (7) is the sufficient statistic

$$\hat{\sigma} = mle(\sigma) = \left(\frac{1}{n}\sum_{i=1}^{n}|X_i - \mu|^p\right)^{1/p}.$$

Note that

$$\mathbb{E}(\hat{\sigma}^p) = \sigma^p \text{ and } \mathbb{V}(\hat{\sigma}^p) = \frac{p}{n}\sigma^{2p},$$

$\hat{\sigma}^p$ thus being an unbiased and consistent estimator for σ^p. The distribution of $\hat{\sigma}$ can be derived, e.g., on using formula (6) with the help of formula (8). The result follows below from the consideration for the extended scaling parameter model (9).

In the sense of Section 2.6, it is an elementary problem to evaluate the S-generalized surface content of subsets of S according to (8) in the present case because we are dealing here with a set the ipf of which is just the indicator function of an interval. This way it is proved (within a more general framework) in [28] (and with a slight modification already in [27]) that $n(\hat{\sigma}/\sigma)^p$ follows the χ^p-(or p-generalized χ^2-) distribution with n d.f., $n(\hat{\sigma}/\sigma)^p \sim \chi^p(n)$ where $Y \sim Q$ means that the random variable Y follows the probability distribution Q. The p.d. of the $\chi^p(n)$ distribution is according to [27, 28]

$$f_{n,p}(t) = \frac{t^{n/p-1}e^{-t/p}}{p^{n/p}\Gamma(n/p)}I_{(0,\infty)}(t). \tag{10}$$

Although assigning in its derivation the number $\omega_{n,p}$ the Euclidean interpretation instead of the right non-Euclidean one, density (10) was derived later again in [17]. For a considerable generalization of this density, see (14) below and [28]. We denote the quantile of order $q \in (0,1)$ of the $\chi^p(n)$ distribution by $\chi_q^p(n)$. Thus, by definition $P(n(\hat{\sigma}/\sigma)^p < \chi_q^p(n)) = q$. Note that the $\chi^p(n)$ distribution is a particular Gamma distribution having parameters $\frac{n}{p}$ and $\frac{1}{p}$. The quantiles of this distribution can therefore be computed using the inverse Gamma function being available in standard software.

Confidence intervals for σ^p. The random intervals

$$\left[\frac{|X - \mu 1_n|_p^p}{\chi_{1-\alpha_1}^p(n)}, \frac{|X - \mu 1_n|_p^p}{\chi_{\alpha_2}^p(n)}\right], \left[\frac{|X - \mu 1_n|_p^p}{\chi_{1-\alpha}^p(n)}, \infty\right) \text{ and } \left(-\infty, \frac{|X - \mu 1_n|_p^p}{\chi_{\alpha}^p(n)}\right] \tag{11}$$

are level $1-\alpha$ two-sided, and one-sided upper and lower confidence intervals for σ^p, respectively, with $\alpha = \alpha_1 + \alpha_2$ in the two-sided case, and where $1_n = (1, ..., 1)^T \in \mathbb{R}^n$. If $p = 2$ then $\chi_q^p(n)$ is the common quantile of order q of the Chi-square distribution with n d.f., $\chi_q^2(n)$. The case $p = 1$ is dealt with in [15], formula (2.6.112), where, however, $2/\chi_{1-\alpha/2}^2(2n)$ is used instead of $1/\chi_{1-\alpha/2}^1(n)$. Tables of quantiles $\chi_q^p(n)$ are given in the Appendix for several values of n, p and q.

Significance testing. If one wants to test the hypothesis $H_0 : \sigma = \sigma_0$ versus the two-sided alternative $H_A : \sigma \neq \sigma_0$ then the decision rule of a level-α significance test is defined as to reject H_0 if

$$\sum_{i=1}^{n}\left|\frac{X_i - \mu}{\sigma_0}\right|^p \in (-\infty, \chi_{\alpha_2}^p(n)) \bigcup (\chi_{1-\alpha_1}^p(n), \infty), \tag{12}$$

where $\alpha = \alpha_1 + \alpha_2, 0 < \alpha_i < 1/2, i = 1, 2$. Tests for one-sided hypotheses are analogously constructed.

Extended scaling parameter model. In the model (9), the maximum likelihood estimator of the common scaling parameter σ is

$$mle(\sigma) = \hat{\sigma} = \frac{1}{n^{1/p}}|X - \nu|_{a,p}$$

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where $\nu = (\mu_1, ..., \mu_n)^T$ and

$$|x|_{a,p} = (\sum_{i=1}^{n} |\frac{x_i}{a_i}|^p)^{1/p}, x \in \mathbb{R}^n.$$

The p.d. of X is called a p-generalized axes aligned elliptically contoured density having the contour defining star body $K = B_{a,p} = \{x \in \mathbb{R}^n : |x|_{a,p} \leq 1\}$, see [32]. This density, as well as $\hat{\sigma}$ allow functional representations

$$f_X(x) = \frac{C_p^n}{\sigma^n a_1 \cdot ... \cdot a_n} g_{PE}(h_{\sigma B_{a,p}}(x - \nu)), x \in \mathbb{R}^n$$

and

$$\hat{\sigma} = \frac{1}{n^{1/p}} h_{B_{a,p}}(X - \nu).$$

For the derivation of the density of $n(\frac{\hat{\sigma}}{\sigma})^p$, we start from the equation

$$P(n(\frac{\hat{\sigma}}{\sigma})^p < t) = \Phi_{g_{PE},B_{a,p},0_n}(t^{1/p}B_{a,p}).$$

We recall that the intersections $B \cap S_{a,p}(r), r > 0$, where $S_{a,p}(r) = rS_{a,p}$ and $S = S_{a,p}$ is the boundary of $K = B_{a,p}$ are called sometimes the 'indivisibles' of a set B.

The geometric disintegration method behind formula (6) divides integration into first an integration on a part of a generalized sphere and second an integration w.r.t. r:

$$P(n(\frac{\hat{\sigma}}{\sigma})^p < t) = \frac{1}{I(g_{PE})} \int_0^\infty r^{n-1} g_{PE}(r) \mathfrak{F}_{S_{a,p}}(t^{1/p}B_{a,p}, r)dr$$

where according to Section 2.6

$$\mathfrak{F}_{S_{a,p}}(t^{1/p}B_{a,p}, r) = I_{[0,t^{1/p}]}(r).$$

Thus

$$P(n(\frac{\hat{\sigma}}{\sigma})^p < t) = \frac{1}{I(g_{PE})} \int_0^{t^{1/p}} r^{n-1} g_{PE}(r)dr, \ t > 0.$$

Note that this result does not depend on $a = (a_1, ..., a_n)^T$. Finally,

$$n(\frac{\hat{\sigma}}{\sigma})^p \sim \chi^p(n).$$

If one replaces the functional $|.|_p^p$ by $|.|_{a,p}^p$ in the definitions of the intervals (11) then the new intervals are confidence intervals for σ^p in the extended scaling parameter model (9).

3.3 The p-generalized Student's sample distribution and beyond

We study here the distribution of the statistic $n\left(\frac{\hat{\sigma}}{\sigma}\right)^p$ if the distribution of the sample vector $X = (X_1, ..., X_n)^T$ is not the n-dimensional power exponential one but any other element of the family of $l_{n,p}$-symmetric distributions. Before doing this, let us recall that one of the most famous non-normal spherically contoured distributions is the Student's t-distribution because of its heavy tails and the remarkable property that all its marginal distributions have a density generating function of the same known type. Moreover, note that the components of a spherically contoured vector are uncorrelated but not independent unless in the Gaussian case. All these properties hold also true for the p-generalized Student's t-distribution with f degrees of freedom

which was studied for dimension one in [27] and for arbitrary finite dimension n in [1]. In the latter case, the dgf may be chosen as

$$g_{St(n,p,f)}(r) = \left(1 + \frac{r^p}{f}\right)^{-\frac{f+n}{p}}, r > 0,$$

and the density of X is

$$t_{n,p}(x; f) = D_{n,p,f}\left(1 + \frac{|x|_p^p}{f}\right)^{-\frac{f+n}{p}}, x \in \mathbb{R}^n,$$

where $D_{n,p,f} = \frac{(\frac{p}{2})^n \Gamma(\frac{n+f}{p})}{\Gamma(\frac{f}{p})(\Gamma(\frac{1}{p}))^n f^{n/p}}$. It follows that the density of the statistic $n\left(\frac{\hat{\sigma}}{\sigma}\right)^p$ is then

$$f_{n,g_{St},p}^\chi(t) = \frac{t^{\frac{n}{p}-1}}{f^{\frac{n}{p}} B(\frac{n}{p}, \frac{f}{p})\left(1 + \frac{t^p}{f}\right)^{\frac{f+n}{p}}}, t \in \mathbb{R}. \tag{13}$$

If, however, the density of the sample vector X is an arbitrary $l_{n,p}$-symmetric one having dgf g,
$$f(x) = C(g,p)g(|x|_p), x \in \mathbb{R}^n,$$

then, according to [28],

$$f_{n,g,p}^\chi(t) = \frac{t^{\frac{n}{p}-1}g(t)}{pI_{n,g,p}}, t \in \mathbb{R}, \tag{14}$$

where $I_{n,g,p} = \int\limits_0^\infty r^{n-1}g(r^p)dr$.

This density can be considered as a (p,g)-generalization of the classical Chi-square density with n d.f., or as a g-generalization of the p-generalized Chi-square density $f_{n,p}$ in (10). We denote the quantile of order q of the $\chi_g^p(n)$-distribution by $\chi_{g;q}^p(n), 0 < q < 1$, that is $\chi_{g;q}^p(n)(\chi_{g;q}^p(n)) = q$. Note that $\chi_{g_{PE};q}^p(n) = \chi_q^p(n)$.

Let a random variable following density (14) be denoted by R^p. Note that tables of quantiles of the distribution of R are recently provided in [21] for $n \in \{1, 2, 3\}$, $p \in \{\frac{1}{2}, 1, 3\}$ different dgfs g of power-exponential and Student's type as well of Pearson type 2.

3.4 Inference for generalized sample distributions

If the sample vector X is distributed according to an $l_{n,p}$-symmetric distribution, see e.g. [28], with a dgf g being different from the power exponential one, $g \neq g_{PE}$, then $X_1, ..., X_n$ are not further independent. In this case of $l_{n,p}$-dependent observation variables, it is possible to study decision rules for exact inference on scaling parameters as before. Confidence and significance levels of the analogously defined estimators and tests, however, change in dependence of the actual shape of g. It is an immediate consequence from the equation

$$P(\chi_{g,\alpha_2}^p(n) < n\left(\frac{\hat{\sigma}}{\sigma}\right)^p < \chi_{g,1-\alpha_1}^p(n)) = 1 - \alpha_1 - \alpha_2$$

that the level-α confidence intervals in (11) extend to the present g-generalized case if all quantiles $\chi_q^p(n)$ are replaced with $\chi_{g;q}^p(n)$. Similarly, the same replacement of quantiles g-generalizes the level-α tests considered in (12).

3.5 The ratio of two scaling parameters

Let $X_1, ..., X_n$ and $Y_1, ..., Y_m$ be completely independent random variables with

X_i having p.d. $f_p(.; \mu_1, \sigma_1)$, $i = 1, ..., n$ and Y_i having p.d. $f_p(.; \mu_2, \sigma_2)$, $i = 1, ..., m$

where $p > 0$ and the expectations μ_1, μ_2 are known and the scaling parameters σ_1, σ_2 are unknown. With the notation $X = (X_1, ..., X_n)^T, Y = (Y_1, ..., Y_m)^T$, the $n + m$-dimensional random vector $(X^T Y^T)^T$ follows the distribution $\Phi_{g,K,\nu}$ where the dgf is the power exponential one, $g(r) = g_{PE}(r) = I_{(0,\infty)}(r) \exp\{-\frac{r^p}{p}\}$, the star body K is the p-generalized ellipsoid

$$K = B_{p,\sigma_1,\sigma_2} = \{(x^T y^T)^T \in \mathbb{R}^{n+m} : |(x^T y^T)^T|_{p,\sigma_1,\sigma_2} \leq 1\},$$

and the shift vector ν is $(\mu_1 1_n^T \mu_2 1_m^T)^T \in \mathbb{R}^{n+m}$. Here,

$$|(x^T y^T)^T|_{p,\sigma_1,\sigma_2} = (\frac{|x|_p^p}{\sigma_1^p} + \frac{|y|_p^p}{\sigma_2^p})^{1/p}, (x^T y^T)^T \in \mathbb{R}^{n+m}.$$

Let, in accordance with Section 3.2,

$$\hat{\sigma}_1^p = \frac{1}{n}|X - \mu_1 1_n|_p^p, \quad \hat{\sigma}_2^p = \frac{1}{m}|Y - \mu_2 1_m|_p^p, \tag{15}$$

and put

$$T = \frac{(\hat{\sigma}_1/\sigma_1)^p}{(\hat{\sigma}_2/\sigma_2)^p}.$$

Note that, for $t > 0$,

$$P(T < t) = \Phi_{g_{PE}, B_{p;\sigma_1,\sigma_2},\nu}(\{(x^T y^T)^T \in \mathbb{R}^{n+m} : \frac{|\frac{x-\mu_1 1_n}{\sigma_1}|_p^p}{|\frac{y-\mu_2 1_m}{\sigma_2}|_p^p} < \frac{n}{m} t\})$$

$$= \Phi_{g_{PE}, B_{p;1,1},0}(C_{n,m;p}(t))$$

where

$$C_{n,m;p}(t) = \{(x^T y^T)^T \in \mathbb{R}^{n+m} : \frac{|x|_p^p}{|y|_p^p} < \frac{n}{m} t\}$$

denotes a cone in $\mathbb{R}^{n+m}$. The ipf of this cone is studied in [28]. It is shown there that $C_{n,m;p}(t)$ belongs to the second elementary type of sets considered in Section 2.6, thus

$$P(T < t) = \mathfrak{F}_{S_{a,\sigma_1,\sigma_2}}(C_{n,m;p}(t)), \; t > 0$$

does not depend on the dgf g. To be specific,

$$\mathfrak{F}_{S_{a,\sigma_1,\sigma_2}}(C_{n,m;p}(t)) = \frac{p}{B(\frac{n}{p}, \frac{m}{p})} \int_{\text{arccot}((\frac{n}{m}t)^{1/p})}^{\pi/2} \frac{(\cos\varphi)^{n-1}(\sin\varphi)^{m-1}}{((\cos\varphi)^p + (\sin\varphi)^p)^{n/p}} d\varphi.$$

Hence, the p.d. of T is

$$f_{n,m;p}(t) = \frac{(n/m)^{n/p} t^{n/p-1}}{B(n/p, m/p)(1 + nt/m)^{(n+m)/p}} I_{(0,\infty)}(t).$$

The distribution law of T, denoted by $F_{n,m}(p)$, is called in [28] the p-generalized Fisher distribution with (n, m) d.f.. Finally, it turns out that

$$T = \frac{m\sigma_2^p |X - \mu_1 1_n|_p^p}{n\sigma_1^p |Y - \mu_2 1_m|_p^p} \sim F_{n,m}(p).$$

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Let us denote the q'th order quantile of the p-generalized Fisher distribution having (n, m) d.f. by $F_{n,m;q}(p), q \in (0, 1)$.

Confidence intervals. Two-sided, possibly asymmetric, and one-sided upper and lower confidence intervals for $\frac{\sigma_1^p}{\sigma_2^p}$ are, correspongdingly,

$$\xi[\frac{1}{F_{n,m;1-\alpha_1}(p)}, \frac{1}{F_{n,m;\alpha_2}(p)}], \ \xi[\frac{1}{F_{n,m;1-\alpha}(p)}, \infty) \text{ and } \xi(-\infty, \frac{1}{F_{n,m;\alpha}(p)}]$$

where $\xi[e, f] = \{\xi \cdot s : e \leq s \leq f\}$ and

$$\xi = \frac{m|X - \mu_1 1_n|_p^p}{n|Y - \mu_2 1_m|_p^p}.$$

Significance test. For deciding

$$H_0 : \sigma_1 = \sigma_2 \text{ vs. } H_A : \sigma_1 \neq \sigma_2 \tag{16}$$

we use the distribution of T if H_0 is true to construct a level α significance test. In this case, $T = \frac{m}{n} \frac{|X - \mu_1 1_n|_p^p}{|Y - \mu_2 1_m|_p^p}$. Reject H_0 if $T < F_{n,m;\alpha_2}(p)$ or if $T > F_{n,m;1-\alpha_1}(p)$, respectively, where $\alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 = \alpha < 1$.

Proposition 1. *If sample sizes n and m are equal then this rule is equivalent to rejecting H_0 if the likelihood ratio statistic is smaller than a constant $c \in (0, 1)$ chosen s.t.*

$$P(\text{ reject } H_0 \text{ although } H_0 \text{ is true }) = \alpha.$$

To show this, let us consider the likelihood function

$$L(\sigma_1, \sigma_2) = (n + m) \ln C_p - n \ln \sigma_1 - m \ln \sigma_2 - \frac{1}{p\sigma_1^p} \sum_{i=1}^{n} |x_i - \mu_1|^p - \frac{1}{p\sigma_2^p} \sum_{i=1}^{m} |y_i - \mu_2|^p.$$

The maximum likelihood estimators of the scaling parameters are given in (15). If $\sigma_1 = \sigma_2 = \sigma$, say, then $L^*(\sigma) = L(\sigma, \sigma)$ is the likelihood function for estimating σ, and

$$\hat{\sigma} = mle(\sigma) = \left(\frac{\sum_{i=1}^{n} |x_i - \mu_1|^p + \sum_{i=1}^{m} |y_i - \mu_2|^p}{n + m} \right)^{1/p}.$$

The likelihood ratio is

$$Q = \frac{\sup_{\sigma_1 = \sigma_2} L(\sigma_1, \sigma_2)}{\sup_{\sigma_1 > 0, \sigma_2 > 0} L(\sigma_1, \sigma_2)} = \frac{L^*(\hat{\sigma})}{L(\hat{\sigma}_1, \hat{\sigma}_2)}$$

$$= (\frac{(n + m)^{n+m}}{n^n m^m})^{1/p} \frac{|x - \mu_1 1_n|_p^n |y - \mu_2 1_m|_p^m}{(|x - \mu_1 1_n|_p^p + |y - \mu_2 1_m|_p^p)^{(n+m)/p}}.$$

Let $\xi = |x - \mu_1 1_n|_p^{p/2}$ and $\eta = |y - \mu_2 1_m|_p^{p/2}$. Because $n = m$ it follows that $Q \leq t, t \in (0, 1)$ iff $l(\xi) \geq 0$ where $l(\xi) = \xi^2 + p\xi + q$, $p = -2\eta/t^{p/(2m)}$ and $q = \eta^2$. The two positive solutions of the equation $l(\xi) = 0$ are

$$\xi_1 = -p/2 + (p^2/4 - q)^{1/2} = \frac{\eta}{t^{p/(2m)}}(1 + (1 - t^{p/m})^{1/2}),$$

and

$$\xi_2 = -p/2 - (p^2/4 - q)^{1/2} = \frac{\eta}{t^{p/(2m)}}(1 - (1 - t^{p/m})^{1/2})$$

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where $\eta > 0$. Because $l(.)$ is a convex function, the rejection area of H_o is the set of all ξ satisfying $\xi \geq \xi_1$ or $\xi \leq \xi_2$ The latter inequalities are equivalent to

$$\frac{|x - \mu_1 1_n|_p^p}{|y - \mu_2 1_m|_p^p} \geq \frac{2(1 + (1 - t^{p/m})^{1/2})}{t^{p/m}} - 1$$

and

$$\frac{|x - \mu_1 1_n|_p^p}{|y - \mu_2 1_m|_p^p} \leq \frac{2(1 - (1 - t^{p/m})^{1/2})}{t^{p/m}} - 1.$$

This proves the equivalence of the considered decision rules.

Remark 1. *As we have seen in this section, inferring on the scaling parameters ratio in $l_{n,p}$-symmetric sample distributions can be done in a g-robust way. For a discussion of this property see [28], and note that it goes back in the case $p = 2$ to [6].*

4 Conclusions

As to shortly summarize the results of this paper, we derived exact confidence intervals and significance tests for single scaling parameters and ratios of two such parameters if the n-dimensional sample vector follows a norm or antinorm contoured $l_{n,p}$-symmetric distribution. This assumption includes modeling heavy and light tails of the sample vector's distribution as well as independence and $l_{n,p}$-dependence sampling. Certain classical results are (g, p)-generalized in the present work. The additional methodological benefit coming from the present paper may be estimated having a look at [30] where a rather general frame of future work is sketched.

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5 Appendix

Table 1 gives a brief comparison of norms and antinorms at the hand of the functional $|.|_q$, $q \in R$. Note that $N_q = \{x \in \mathbb{R}^n : |x|_q = 0\}$ and $H_i = \{(x_1, ..., x_n)^T \in \mathbb{R}^n : x_i = 0\}, i = 1, ..., n$ are hyper planes in $\mathbb{R}^n$.

Table 1: Norms and antinorms

parameter	q	$\in [1, \infty)$	$\in (0, 1]$	$\in (-\infty, 0)$
functional	$|.|_q$	norm	antinorm	semi-antinorm, $z \neq 0$
unit ball	B_q	convex	radially concave	radially concave
			w.r.t. standard fan in $\mathbb{R}^n$	w.r.t. standard fan in $\mathbb{R}^n$
inverse image of 0	N_q	$\{0\}$	$\{0\}$	$\cup_1^d H_i$

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Table 2: Quantiles $\chi_q^p(n)$, $p = 3.3$

q \ n	1	2	3	4	5	10	15	20	25	30
0.005	0.00	0.00	0.01	0.05	0.12	1.14	2.92	5.17	7.73	10.51
0.01	0.00	0.00	0.02	0.08	0.20	1.47	3.51	6.00	8.77	11.76
0.025	0.00	0.01	0.06	0.18	0.37	2.08	4.54	7.39	10.49	13.78
0.05	0.00	0.02	0.12	0.32	0.60	2.75	5.58	8.75	12.15	15.71
0.1	0.00	0.06	0.26	0.58	0.99	3.70	6.98	10.55	14.29	18.16
0.9	2.95	5.19	7.07	8.78	10.39	17.70	24.42	30.86	37.12	43.25
0.95	4.56	7.17	9.30	11.20	12.98	20.93	28.13	34.96	41.57	48.02
0.975	6.31	9.21	11.53	13.60	15.52	24.00	31.61	38.79	45.70	52.42
0.99	8.75	11.96	14.50	16.75	18.82	27.91	35.98	43.55	50.81	57.85
0.995	10.67	14.07	16.75	19.11	21.29	30.78	39.16	47.00	54.49	61.74

Table 3: Quantiles $\chi_q^p(n)$, $p = 2.5$

q \ n	1	2	3	4	5	10	15	20	25	30
0.005	0.00	0.00	0.03	0.12	0.26	1.68	3.84	6.43	9.29	12.36
0.01	0.00	0.01	0.06	0.18	0.37	2.06	4.46	7.27	10.33	13.57
0.025	0.00	0.02	0.13	0.33	0.61	2.72	5.50	8.63	11.99	15.50
0.05	0.00	0.05	0.23	0.52	0.89	3.42	6.53	9.95	13.56	17.31
0.1	0.01	0.13	0.43	0.84	1.33	4.36	7.88	11.64	15.55	19.57
0.9	2.83	4.86	6.60	8.20	9.72	16.70	23.19	29.43	35.51	41.50
0.95	4.15	6.49	8.43	10.20	11.86	19.38	26.28	32.87	39.26	45.52
0.975	5.56	8.13	10.24	12.15	13.93	21.92	29.17	36.06	42.71	49.21
0.99	7.50	10.32	12.62	14.68	16.60	25.11	32.77	40.00	46.96	53.72
0.995	9.01	12.00	14.41	16.57	18.58	27.44	35.37	42.83	50.00	56.95

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Table 4: Quantiles $\chi_q^p(n)$, $p = 0.75$

q \ n	1	2	3	4	5	10	15	20	25	30
0.005	0.02	0.19	0.50	0.92	1.40	4.35	7.76	11.43	15.25	19.19
0.01	0.03	0.25	0.62	1.08	1.61	4.74	8.31	12.10	16.04	20.08
0.025	0.06	0.36	0.82	1.36	1.96	5.37	9.16	13.14	17.24	21.43
0.05	0.10	0.49	1.02	1.64	2.29	5.96	9.94	14.08	18.33	22.65
0.1	0.17	0.68	1.31	2.00	2.73	6.69	10.89	15.22	19.64	24.10
0.9	2.14	3.64	5.01	6.32	7.59	13.63	19.43	25.10	30.68	36.22
0.95	2.71	4.34	5.82	7.21	8.55	14.89	20.91	26.77	32.52	38.20
0.975	3.27	5.02	6.58	8.04	9.45	16.04	22.25	28.27	34.17	39.99
0.99	4.00	5.88	7.53	9.08	10.57	17.45	23.88	30.09	36.16	42.12
0.995	4.55	6.51	8.23	9.84	11.37	18.45	25.04	31.38	37.55	43.62

Table 5: Quantiles $\chi_q^p(n)$, $p = 0.5$

q \ n	1	2	3	4	5	10	15	20	25	30
0.005	0.05	0.34	0.77	1.29	1.86	5.18	8.88	12.79	16.83	20.96
0.01	0.07	0.41	0.89	1.45	2.07	5.54	9.37	13.39	17.52	21.73
0.025	0.12	0.54	1.10	1.73	2.40	6.11	10.12	14.29	18.56	22.89
0.05	0.18	0.68	1.31	1.99	2.71	6.63	10.80	15.10	19.48	23.93
0.1	0.27	0.87	1.58	2.33	3.11	7.26	11.61	16.07	20.59	25.16
0.9	1.94	3.34	4.64	5.89	7.10	12.95	18.60	24.14	29.62	35.06
0.95	2.37	3.88	5.26	6.57	7.85	13.94	19.77	25.47	31.09	36.64
0.975	2.79	4.38	5.83	7.21	8.54	14.84	20.82	26.66	32.39	38.05
0.99	3.32	5.02	6.55	8.0	9.39	15.92	22.09	28.08	33.95	39.74
0.995	3.72	5.49	7.07	8.57	10.0	16.69	22.99	29.08	35.04	40.91

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