Circle numbers of regular convex polygons

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Abstract. The circle number function is extended here to regular convex polygons. To this end, the radius of the polygonal circle is defined as the Minkowski functional of the circumscribed polygonal disc, and the arc-length of the polygonal circle is measured in a generalized Minkowski space having the rotated polar body as the unit disc.

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1. Introduction

Regular convex polygons are studied in a broad variety of literature about convex geometry and polytopes. In this respect, we refer to [6], [4], [24], and [23]. Generalized circle numbers are introduced in [19], [21], [15] and [13] for convex and non-convex $l_{2,p}$-circles, ellipses and star discs, respectively.
To define circle numbers of regular polygons it is necessary, to introduce two basic notions. The first one generalizes what we call a disc and its radius. For symmetric balls, a closely related discussion of some positively homogeneous functions on $\mathbb{R}^n$ can be found in [11]. From this article, it turns out that the Minkowski functional h_S of a star disc S can be used for defining the radius of convex and even non-convex discs. The present paper will deal, however, with non-symmetric cases, too.
The second main notion for establishing circle numbers is that of a suitably defined non-Euclidean arc-length measure. One method to construct this measure is introduced in [13] and consists in considering a generalized Minkowski space $(\mathbb{R}^2, h_T)$ to a suitably chosen symmetric star disc T and replacing the Euclidean norm of the vector normal to the boundary of S in the defining integral of the Euclidean arc-length by h_T. In the present paper, for every symmetric polygon we introduce a non-Euclidean arc-length and prove, that our results apply to non-symmetric cases, too. This complements

the results in [13]. While this way of generalizing the notion of arc-length deals with integration, another way deals with taking derivatives of area contents of suitably defined disc sectors with respect to the generalized radius of the polygonal disc. Within a more general frame work, this local way to define a suitable non-Euclidean arc-length measure was introduced first in [17] and proved to be equivalent to the integral way in [14]. The equivalence of the local and integral approaches to a generalized notion of arc-length has been proved later for ellipses in [15].
Notice that the notion of generalized arc-length is related to the non-Euclidean generalization of the method of indivisibles and to the notion of a generalized uniform distribution on a generalized circle being the topological boundary of a generalized disc. This method was developed in several articles of the first author and surveyed to some extent in [16]. For an introduction to the topic of Cavalieri's classical method of indivisibles and the related so called Cavalieri integration, we refer to [1] and to [2] and [8], respectively.
For similar considerations on specific polyhedra in dimension three, see [22].

The paper is structured as follows. We discuss some properties of the polygonal radius and the polygonal Minkowski functional in Section 2. The polygonally generalized arc-length measure will be considered based upon both the integral and the local approaches in Section 3, and their equivalence will be shown. The notion of polygonal circle numbers will be introduced in Section 4, and a generalized uniform probability distribution on the regular convex polygons will be studied in Section 5. Section 6 deals with a polygonal disintegration formula of the Lebesgue measure which is closely connected with a non-Euclidean generalization of the method of indivisibles. Finally, we will note other possible choices of the reference radius as to consider the regular convex polygons being generalized circles in Section 7, and discuss their properties compared to our choice.

2. Radius

The first step to generalize the circle number π for generalized circles is to adapt the notion of radius. The idea is to start from a generalized reference circle and to multiply it by a positive number, this way defining the set of all points, having this number as the same distance from a given point. This change of the notion radius was introduced in [19] and it turned out that the Minkowski functional of this generalized reference circle can serve as its generalized radius functional. The Minkowski functional is for every finite dimensional convex body E with the origin in its interior defined by

$$h_E(x) = \inf\{r > 0 : x/r \in E\}. \tag{2.1}$$

It is subadditive and has moreover the property

$$h_E(\alpha x) = \alpha h_E(x), \text{ for all } \alpha > 0 \text{ and } x \in \mathbb{R}^n. \tag{2.2}$$

Originally, this definition was introduced by Hermann Minkowski in [9] and can be found e.g. in [7]. Note that there is a slightly other definition of

the Minkowski functional in the literature which assumes E to be centrally symmetric, so that (2.2) becomes $h_E(\alpha x) = |\alpha| h_E(x)$, for all $\alpha \in \mathbb{R}$. For this definition and some basic knowledge on the topic of Minkowski functionals, we refer to [7] and [23]. Moreover, we refer to [23], concerning the topic of Minkowski geometry and Minkowski spaces in general. In this paper, we follow the more general approach that is also given in [7], by calling, for every absorbing set E, h_E in (2.1) the Minkowski functional of E.
Let P_n be a regular polygon with n vertices on the Euclidean unit circle. Without loss of generality, we assume that the vertices of P_n are the points $I_{n,1}, ..., I_{n,n}$, where

$$I_{n,i}^T = \left(\cos\left(\frac{2\pi}{n}(i-1)\right), \sin\left(\frac{2\pi}{n}(i-1)\right)\right), \ i \in \{1, ..., n\}, \ n \geq 3, \quad (2.3)$$

see Figure 1. For simplicity, we shall use the same notation $I_{n,i}$ for a vector being oriented from the origin to the point $I_{n,i}$. We will consider P_n as a generalized reference circle and call it polygonal unit circle. Let K_n be the convex body circumscribed by P_n. We call K_n the unit polygonal disc. Note that $P_n = \{(x, y)^T \in \mathbb{R}^2 : h_{K_n}(x, y) = 1\}$, and, for $r > 0$,

$$r \cdot K_n = \{(rx, ry)^T \in \mathbb{R}^2 : h_{K_n}(x, y) \leq 1\} = \{(x, y)^T \in \mathbb{R}^2 : h_{K_n}(x, y) \leq r\}$$

because of (2.2). This motivates us to call $K_n(r) = r \cdot K_n$ a polygonal disc of polygonal radius r. The Minkowski functional of the unit disc takes the value r on the boundary of the disc $K_n(r)$,

$$h_{K_n}(x, y) = r, \ \forall (x, y)^T \in P_n(r) = rP_n.$$

The notion of a polygonal radius will be used throughout this paper. For some basic knowledge on regular polygons, we refer to [6], [24] and [5].
Let $a_{n,i}^T = (\cos(\pi/n))^{-1} \left(\cos\left((2i-1)\frac{\pi}{n}\right), \sin\left((2i-1)\frac{\pi}{n}\right)\right), \ i \in \{1, ..., n\}$. For every point $(x, y)^T \in \mathbb{R}^2$, the Minkowski functional h_{K_n} of the polygonal unit disc K_n can be expressed by $h_{K_n}(x, y) = max\left\{a_{n,i}^T(x, y)^T, \ i = 1, ..., n\right\}$. To prove this, note that K_n can be expressed as $K_n = \{(x, y)^T \in \mathbb{R}^2 : A_n(x, y)^T \leq \mathbf{1}_n\}$, where $A_n \in \mathbb{R}^{n\times 2}$ and $\mathbf{1}_n = (1, ..., 1)^T$. Since $K_n = \{(x, y)^T \in \mathbb{R}^2 : \max\{A_n[i](x, y)^T, i = 1, ..., n\} \leq 1\}$, where $A_n[i]$ denotes the i-th row of A_n, and because of the definition of the Minkowski functional, the formula for h_{K_n} follows. Note that if n is even, then P_n is symmetric with respect to the origin, $h_{K_n}(\cdot)$ is a norm in $\mathbb{R}^2$, and its representation reduces to
$h_{K_n}(x, y) = max\left\{a_{n,i}^T(|x|, |y|)^T, \ i = 1, ..., \lceil n/4 \rceil\right\}, \ \forall (x, y)^T \in \mathbb{R}^2$, where $\lceil \cdot \rceil$ denotes the entire function.

3. Circumference

The notion of generalized arc-length measure has been dealt with in [19] and in [15] for convex $l_{2,p}$-circles and ellipses, respectively. The corresponding vocabulary of generalized surface content for the higher dimensional case was introduced in [14] and [18] for $l_{n,p}$-balls and ellipsoids, respectively. In

[14] and [18], integral and local definitions are given and their equivalence is proven. Here, we introduce first the integral approach to the polygonal arc-length and give a local go up to such notion afterwards. Finally, we show their equivalence.
For any successive and positive (anticlockwise) oriented partition
$\mathfrak{Z}_n = \{z_0, z_1, ..., z_n = z_0\}$ of $P_n(\rho)$, let us define the positive directed T-arc-length of $P_n(\rho)$ by

$$AL_{K_n,T}(\rho) = \sup_{n,\mathfrak{Z}_n} \sum_{j=1}^{n} h_T(z_j - z_{j-1}).$$

To study the properties of $AL_{K_n,T}$, we introduce generalized trigonometric functions. For this, let us consider a parameter representation of the unit polygonal disc K_n by

$$K_n = \{r(\cos\varphi, \sin\varphi)^T,\ 0 \le \varphi \le 2\pi, 0 \le r \le R_n(\varphi)\}.$$

Because of $h_{K_n}(x,y) = 1,\ \forall (x,y)^T \in P_n$, it follows that
$h_{K_n}(\cos\varphi, \sin\varphi) = 1/R_n(\varphi)$, $0 \le \varphi \le 2\pi$, which allows the representation

$$P_n = \left\{ \left(\frac{\cos\varphi}{M_n(\varphi)}, \frac{\sin\varphi}{M_n(\varphi)} \right)^T,\ 0 \le \varphi \le 2\pi \right\},$$

where $M_n(\varphi) = h_{K_n}(\cos\varphi, \sin\varphi)$. This motivates the definition of polygonally generalized sine- and cosine-functions as

$$\cos_n(\varphi) = \frac{\cos\varphi}{M_n(\varphi)} \text{ and } \sin_n(\varphi) = \frac{\sin\varphi}{M_n(\varphi)},\ \varphi \in [0, 2\pi).$$

Note that these generalized trigonometric functions can be interpreted geometrically by considering a right angled triangle with vertices $(0,0)^T$, $(x,0)^T$ and $(x,y)^T$. Then, the polygonally generalized sine and cosine of the angle $\varphi \in [0, 2\pi)$ between the positive x-axis and the vector $(x,y)^T$ are obtained by the formulae $\sin_n(\varphi) = y/h_{K_n}(x,y)$ and $\cos_n(\varphi) = x/h_{K_n}(x,y)$, respectively. Furthermore, these functions satisfy the equation
$h_{K_n}(\cos_n(\varphi), \sin_n(\varphi)) = 1$ which generalizes the well known formula $\cos^2\varphi + \sin^2\varphi = 1$.
The introduction of polygonally generalized trigonometric functions makes it possible to define a polygonal polar coordinate transformation $Pol_n : [0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$ analogously to the Euclidean one by $x = r\cos_n(\varphi)$ and $y = r\sin_n(\varphi)$, for $0 \le \varphi \le 2\pi$ and $0 < r < \infty$. The following theorem tells that, here, r is the polygonal radius and φ is the usual polar angle. Let $Q_1, ..., Q_4$ denote the anticlockwise enumerated quadrants of $\mathbb{R}^2$.

Theorem 3.1. *The map Pol_n is almost one-to-one, for $x \neq 0$, its inverse Pol_n^{-1} is given by*

$$r = h_{K_n}(x,y),\ \arctan(|y/x|) = \varphi \textit{ in } Q_1,$$
$$\pi - \varphi \textit{ in } Q_2, \varphi - \pi \textit{ in } Q_3 \textit{ and } 2\pi - \varphi \textit{ in } Q_4,$$

and its Jacobian is $J(r,\varphi) = r/M_n^2(\varphi)$, $\varphi \in [0, 2\pi)$, $r > 0$.

Proof. By (2.2), $h_{K_n}(x,y) = h_{K_n}(r\cos_n(\varphi), r\sin_n(\varphi)) = r$. The relations for $\arctan(|y/x|)$ are the same as those in the case of usual polar coordinates. For calculating the Jacobian, we use the formulae $\cos_n'(\varphi) = \frac{1}{M_n^2(\varphi)}(-\sin(\varphi)M_n(\varphi) - \cos(\varphi)M_n'(\varphi))$ and $\sin_n'(\varphi) = \frac{1}{M_n^2(\varphi)}(\cos(\varphi)M_n(\varphi) - \sin(\varphi)M_n'(\varphi))$. □

We denote the restriction of Pol_n to the case $r = 1$ by Pol_n^*, $Pol_n^*(\varphi) = Pol_n(1,\varphi)$, and its inverse by Pol_n^{*-1}, $Pol_n^{*-1} : \mathfrak{B}^2 \cap P_n \to \mathfrak{B}([0, 2\pi))$, and recall that, according to [13], $AL_{K_n,T}(\rho)$ can be rewritten as

$$AL_{K_n,T}(\rho) = \rho \int_0^{2\pi} h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T}\left(Pol_n\left(\nabla h_{K_n}(x,y)\right)(r,\varphi)\right) \frac{d\varphi}{M_n^2(\varphi)},$$

where $Pol_n\left(\nabla h_{K_n}(x,y)\right)(r,\varphi) = \nabla h_{K_n}(x,y)|_{(x,y)=Pol_n(r,\varphi)}$ denotes the plug-in version of the gradient $\nabla h_{K_n}(x,y)$ in polygonally generalized coordinates. Furthermore, if for almost every $(x,y)^T \in \mathbb{R}^2$ the rotated gradient condition

$$h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T}\left(Pol_n\left(\nabla h_{K_n}(x,y)\right)(r,\varphi)\right) = 1 \tag{3.1}$$

is satisfied, then

$$AL_{K_n,T}(\rho) = \rho \int_0^{2\pi} \frac{d\varphi}{M_n^2(\varphi)}.$$

If P_n is symmetric w.r.t. the origin, h_{K_n} denotes a norm, and it is proven then according to example 2.13. in [13], that the 90° anticlockwise rotated unit polygonal disc with respect to the corresponding dual norm satisfies condition (3.1).
In what follows, we show that the disc T in (3.1) is equal to the 90° anticlockwise rotated polar body K_n^* of K_n which is defined by $K_n^* = \{y \in \mathbb{R}^2 : y^T x \leq 1,\ \forall x \in K_n\}$. Furthermore, we show that even if P_n is not symmetric then $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)K_n^*$ still satisfies condition (3.1).
Let $C = \{\sum_{i=1}^{n} \alpha_i a_{n,i}, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0\}$ denote the convex hull of the finite number of points $a_{n,1}, ..., a_{n,n}$.

Theorem 3.2. *The polar body K_n^* of K_n satisfies the equation $K_n^* = C$.*

Proof. We recall the representation $K_n = \{x \in \mathbb{R}^2 : a_{n,i}^T x \leq 1, i = 1, ..., n\}$. Let $y \in K_n^*$. Since for all $x \in K_n$, it holds $a_{n,i}^T x \leq 1, i \in \{1, ..., n\}$, we can find $\alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$ such that $\alpha_i a_{n,i}^T x \leq \alpha_i$ and $\sum_{i=1}^{n} \alpha_i a_{n,i}^T x \leq \sum_{i=1}^{n} \alpha_i = 1$. Because $y \in K_n^*$, we can identify $\sum_{i=1}^{n} \alpha_i a_{n,i}$ with y. Hence, $y \in C$. On the other hand side, if y is an element of C, then $y^T x = \sum_{i=1}^{n} \alpha_i \underbrace{a_{n,i}^T x}_{\in K_n} \leq \sum_{i=1}^{n} \alpha_i = 1$, thus $K_n^* = C$. □

Theorem 3.2 says that K_n^* is circumscribed by the polygon with vertices $I_{n,i}^* = a_{n,i}, i \in \{1, ..., n\}$. The Minkowski functional $h_{K_n^*}$ of K_n^* can be calculated analogously to that of K_n. Thus for every point $(x,y)^T \in \mathbb{R}^2$, $h_{K_n^*}$ can be expressed by $h_{K_n^*}(x,y) = \max\left\{I_{n,i+1}^T(x,y)^T, i = 1, ..., n, I_{n,n+1} = I_{n,1}\right\}$. In the following theorem, we state that for any regular polygon P_n a star-shaped set T can be constructed, such that condition (3.1) is satisfied. To be more specific, T can be chosen as the 90° anticlockwise rotated regular polygonal polar disc.

Theorem 3.3. *The star body* $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K_n^*$ *satisfies the condition (*3.1*).*

Proof. Notice that

$$h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)T}(x,y) = h_{K_n^*}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) = h_{K_n^*}(x,y).$$

Let $C_{n,i}$ denote the cone with vertex at the origin, which is spanned up by the i-th facet $\overline{I_{n,i}I_{n,i+1}}$ of P_n. For an illustration of $I_{5,i}$ and $C_{5,i}$, we refer to Figure 5. Then h_{K_n} can be reformulated by
$h_{K_n}(x,y) = \sum_{i=1}^{n} \mathbb{1}_{C_{n,i}}(x,y)\left(\frac{x\cos((2i-1)\pi/n) + y\sin((2i-1)\pi/n)}{\cos(\pi/n)}\right)$, where

$$\mathbb{1}_{C_{n,i}}(x,y) = \begin{cases} 1, & \text{if } (x,y)^T \in C_{n,i} \\ 0, & \textit{otherwise} \end{cases} = \begin{cases} 1, & \text{if } \vartheta \in \left[\frac{2\pi}{n}(i-1), \frac{2\pi}{n}i\right) \\ 0, & \textit{otherwise} \end{cases}$$

and $\vartheta = \vartheta(x,y)$ denotes the polar angle of $(x,y)^T$, i.e. the angle between the positive x-axis and the line through the origin and the point $(x,y)^T$. Furthermore,

$$\nabla h_{K_n}(x,y) = \begin{pmatrix} \sum_{j=1}^{n} \hat{\mathbb{1}}_{C_{n,j}}(x,y)\left(\frac{\cos((2j-1)\pi/n)}{\cos(\pi/n)}\right) \\ \sum_{j=1}^{n} \hat{\mathbb{1}}_{C_{n,j}}(x,y)\left(\frac{\sin((2j-1)\pi/n)}{\cos(\pi/n)}\right) \end{pmatrix},$$

where $\hat{\mathbb{1}}_{C_{n,j}}(x,y) = \begin{cases} 1, & \text{if } \vartheta \in \left(\frac{2\pi}{n}(j-1), \frac{2\pi}{n}j\right) \\ 0, & \textit{otherwise} \end{cases}$ slightly modifies $\mathbb{1}_{C_{n,j}}(x,y)$ and does not further express dependence on $(x,y)^T$. Hence,

$$\nabla h_{K_n}(x,y) = Pol_n(\nabla h_{K_n}(x,y))(r,\varphi).$$

Here and in what follows the gradient is considered a.e. Thus, if $C_{n,i}^*$ denotes the cone with vertex at the origin, which is spanned up by the i-th facet $\overline{I_{n,i-1}^* I_{n,i}^*}$ of P_n^* with $I_{n,0}^* = I_{n,n}^*$ (we refer to Figure 2 for an illustration of $I_{5,i}^*$ and $C_{5,i}^*$),

$$h_{K_n^*}(x,y) = \sum_{i=1}^{n} \mathbb{1}_{C_{n,i}^*}(x,y)\left(x\cos\left(\frac{2(i-1)\pi}{n}\right) + y\sin\left(\frac{2(i-1)\pi}{n}\right)\right)$$

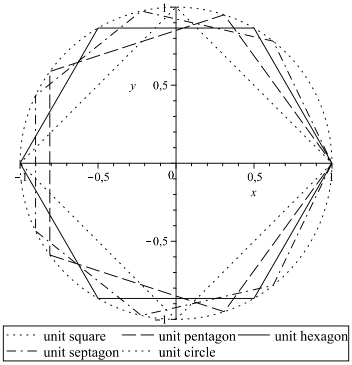


FIGURE 1. Boundaries of the unit polygons K_n for $n = 4, 5, 6, 7$ and the unit circle

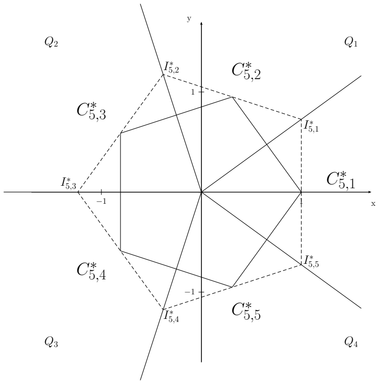


FIGURE 2. The unit pentagon (solid lines) with its corresponding polar body (dashed lines)

with $\mathbb{1}_{C^*_{n,i}}(x,y) = \begin{cases} 1, & \text{if } \vartheta \in \left[\frac{(2i-3)\pi}{n}, \frac{(2i-1)\pi}{n}\right) \\ 0, & \text{otherwise} \end{cases}$, and it follows that

$$h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^T} \left(Pol_n \left(\nabla h_{K_n}(x,y)\right)(r,\varphi)\right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{1}_{C^*_{n,i}}(x,y) \hat{\mathbb{1}}_{C_{n,j}}(x,y) \left(\cos\left(\frac{\pi}{n}\right)\right)^{-1} \cos\left(\frac{2(i-1)\pi}{n} - \frac{(2j-1)\pi}{n}\right).$$

By definition of $\mathbb{1}_{C^*_{n,i}}(x,y)$ and $\hat{\mathbb{1}}_{C_{n,j}}(x,y)$, there holds $\mathbb{1}_{C^*_{n,i}}(x,y)\hat{\mathbb{1}}_{C_{n,j}}(x,y) = 1$ if and only if $i = j$ or $i = j+1$, where $i = j+1$ for $j = n$ is defined as $i = 1$. Thus,

$$\cos\left(\frac{2(i-1)\pi}{n} - \frac{(2j-1)\pi}{n}\right) = \begin{cases} \cos(-\pi/n) = \cos(\pi/n), & \text{if } i = j \\ \cos(\pi/n), & \text{if } i = j+1 \end{cases},$$

thus

$$h_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^T} \left(Pol_n \left(\nabla h_{K_n}(x,y)\right)(r,\varphi)\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{1}_{C^*_{n,i}}(x,y) \hat{\mathbb{1}}_{C_{n,j}}(x,y) \cdot 1 = 1.$$

□

Let us denote the regular upper half-polygon of P_n by P_n^+. Furthermore, the positive directed T-arc-length of a Borel measurable set $A \subseteq P_n^+$ is defined by

$$AL_{K_n,T}(A) = -\int_{G(A)} h_T(1, y'(x))dx,$$

with $y'(x) = \frac{dy}{dx}$, $G(A) = \{x \in [-1,1] : (x, y(x))^T \in A\}$ and

$$y(x) = \sum_{j=1}^{\lceil n/2 \rceil} \left(\mathbb{1}_j(x) \frac{\cos(\pi/n) - x\cos((2j-1)\pi/n)}{\sin((2j-1)\pi/n)} \right),$$

where

$$\mathbb{1}_j(x) = \begin{cases} 1, & \text{if } x \in [\cos(2(j-1)\pi/n), \cos(2j\pi/n)) \\ 0, & \text{otherwise} \end{cases}.$$

Notice that because we consider the positive directed arc-length, and $h_T(\cdot)$ is not symmetric in general, it is essential to consider x as the integration variable decreasing from 1 to -1.
To introduce now a local definition of the polygonal arc-length measure, let us define the central projection cone generated by a Borel measurable subset of the polygonal unit circle $A \in \mathfrak{B}^2 \cap P_n$ by

$$CPC_n(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \left(\frac{1}{h_{K_n}(x,y)} \begin{pmatrix} x \\ y \end{pmatrix} \right) \in A \right\}.$$

The set $sector_n(A, r) = CPC_n(A) \cap K_n(r)$ will be called a sector of the polygonal disc $K_n(r)$ with polygonal radius $r > 0$. The derivative of the Lebesgue measure of this sector w.r.t. the polygonal radius defines a finite measure $\mathfrak{U}_n : \mathfrak{B}^2 \cap P_n(r) \to \mathbb{R}^+$ by

$$\mathfrak{U}_n(A) = \frac{d}{d\rho} \lambda(sector_n(A/r, \rho))|_{\rho=r}.$$

We call $\mathfrak{U}_n(\cdot)$ the polygonal arc-length measure on the polygonal circle $P_n(r)$. By the notation $\lambda(K_n(r)) = A_n(r)$, and the formula

$$\lambda(K_n(r)) = \int_0^r \mathfrak{U}_n(P_n(\rho))d\rho \tag{3.2}$$

which reflects a certain generalization of the method of Cavalieri and Torricelli, we obtain

$$\frac{A_n(r)}{r^2} = \frac{\mathfrak{U}_n(P_n(r))}{2r}. \tag{3.3}$$

These ratios do not only coincide but are even independent of r. Their common constant value may be considered as $A_n(1) = \frac{1}{2}\mathfrak{U}_n(P_n)$, which motivates our basic definition in Section 4.
Let $A \in \mathfrak{B}^2 \cap P_n$ and let l_n denote the Euclidean distance between the origin and the lateral faces of K_n. Since l_n equals the magnitude of the inner circle radius of P_n and the outer circle of P_n is the unit circle, it follows that $l_n = \cos(\pi/n)$. Furthermore, it is well known that $\lambda(sector_n(A, 1)) = l_n U(A)/2$ where U denotes the Euclidean arc-length measure. Hence, $\mathfrak{U}_n(A) = \frac{1}{2} \cdot l_n \cdot \mathfrak{U}_n(P_n) \cdot \frac{U(A)}{\lambda(K_n)} = l_n \cdot U(A)$.

Theorem 3.4. *The polygonal arc-length measure satisfies, for every $A \in \mathfrak{B}^2 \cap P_n$, the representation*

$$\mathfrak{U}_n(A) = \int\limits_{Pol_n^{*-1}(A)} \frac{d\varphi}{M_n^2(\varphi)} \tag{3.4}$$

and

$$\mathfrak{U}_n(A) = 2\lambda(sector_n(A,1)). \tag{3.5}$$

Proof. By changing the Cartesian with polygonally generalized polar coordinates,

$$\lambda(sector_n(A,\rho)) = \int\limits_{sector_n(A,\rho)} d(x,y) = \int\limits_0^\rho r dr \int\limits_{Pol_n^{*-1}(A)} \frac{d\varphi}{M_n^2(\varphi)}.$$

The derivation with respect to ρ yields (3.4), and (3.5) follows immediately. $\square$

Theorem 3.5. *The positive directed T-arc-length of a given Borel measurable set $A \subseteq P_n^+$ coincides with the polygonal arc-length of A, i.e. $AL_{K_n,T}(A) = \mathfrak{U}_n(A)$.*

Proof. By calculating $y'(x)$ and showing the equality $h_T(1, y'(x)) = h_{K_n^*}(y'(x), 1)$, it follows that $h_T(1, y'(x))$ reduces to $h_T(1,y'(x)) = -\sum\limits_{j=1}^{\lceil n/2 \rceil} \frac{\mathbb{1}_j(x)\cos(\pi/n)}{\sin((2j-1)\pi/n)}$. Changing variables $x = \frac{\cos\varphi}{M_n(\varphi)}$, $\varphi \in [0,\pi)$ gives

$$\mathbb{1}_j(x) = \mathbb{1}_j(\varphi) = \begin{cases} 1, & \text{if } \varphi \in [2(j-1)\pi/n, 2j\pi/n) \\ 0, & \text{otherwise} \end{cases}$$

and finally

$$\begin{aligned} -\int\limits_{G(A)} h_T(1,y'(x))dx &= \int\limits_{Pol_n^{*-1}(A)} x'(\varphi) h_T(1, y'(\varphi))d\varphi \\ &= \int\limits_{Pol_n^{*-1}(A)} \sum_{j=1}^{\lceil n/2 \rceil} \frac{\mathbb{1}_j(\varphi)\cdot 1}{M_n^2(\varphi)} d\varphi = \mathfrak{U}_n(A). \end{aligned}$$

$\square$

The reader who wants to become more acquainted with the evaluation of the numerical value of $AL_{K_n,\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)K_n^*}(A) = \mathfrak{U}_n(A)$ for a concrete Borel measurable set $A \subseteq P_n$ may find a suitable example in the Appendix. To be more specific, we measure there that part of the unit pentagon P_5 which belongs to the first quadrant Q_1, see Figure 6, and calculate its polygonally generalized arc-length as well as its T-arc-length, for $T = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)K_5^*$.

4. Polygonal circle numbers

The circumference and area content properties of Euclidean circles originally motivated the definition of the circle number π and have been discussed in [19] for $l_{2,p}$-circles and in [15] for ellipses. We follow this line and recall equation (3.3) to motivate the following definition.

Definition 4.1. The polygonal circle number π_n of the regular convex polygon P_n is defined by $\pi_n = A_n(1)$.

Because $\pi_n = \mathfrak{U}_n(P_n)/2 = \pi \cdot \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$, there holds $\pi_n \longrightarrow \pi$ as $n \to \infty$. For an illustration of this convergence, we refer to Table 1 and Figure 4, where $\epsilon(n) = 1 - \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$ denotes the relative error of approximation of π by π_n.

n	3	6	10	50	100	697
π_n	1,299	2,598	2,939	3,133	3,1395	3,1416
$\epsilon(n)$	0,587	0,173	0,0645	0,0026	0,00066	0,00001

Table 1. The convergence of π_n and its approximation error $\epsilon(n)$

Let us recall that the Euclidean isoperimetric inequality connects the circumference U and the area content A of a plane figure by

$$U^2 \geq 4\pi A.$$

Note that, corresponding to a result in [3], $\mathfrak{U}_n^2(r)$ attains the lower bound in the non-Euclidean case, i.e.

$$(\mathfrak{U}_n(r))^2 = 4\pi_n A_n(r), \forall r > 0.$$

5. Generalized uniform distribution on P_n

The notion of a polygonal arc-length measure makes it possible to consider a random vector U taking values on the polygonal unit circle P_n and to define a generalized uniform distribution on P_n by

$$P(U \in B) = \frac{\mathfrak{U}_n(B)}{\mathfrak{U}_n(P_n)}, B \in \mathfrak{B}^2 \cap P_n.$$

Note that the distribution P^U induced by a random vector U on the Borel σ-field $\mathfrak{B}^2$ has the property $P^U(A) = P(U \in A)$ for every Borel measurable set A.

Let a random vector X follow the uniform probability distribution on K_n, i.e. $P(X \in A) = \lambda(A)/\lambda(K_n)$, for $A \in \mathfrak{B}^2 \cap K_n$, and put $Y = X/h_{K_n}(X)$, where division is defined componentwise. Then the following theorem holds true, which reflects one of the main properties of the generalized uniform distribution on P_n.

Theorem 5.1. *The random vector Y follows the generalized uniform distribution on P_n.*

Proof. It follows from the definition of $\mathfrak{U}_n$ that the equation (3.2) may be generalized by

$$\lambda(sector_n(A/r,r)) = \int_0^r \mathfrak{U}_n(P_n(\rho) \cap [\rho A/r])d\rho,\ A \in \mathfrak{B}^2 \cap P_n(r). \tag{5.1}$$

Thus, (3.3) extends to $\frac{\lambda(sector_n(A/r,r))}{r^2} = \lambda(sector_n(A/r,1)) = \frac{\mathfrak{U}_n(A)}{2r}$, $A \in \mathfrak{B}^2 \cap P_n(r)$. Hence, for $A \in \mathfrak{B}^2 \cap P_n$,

$$P(Y \in A) = P(X \in sector_n(A,1)) = \frac{\frac{d}{d\rho}\lambda(sector_n(A,\rho))}{\frac{d}{d\rho}\lambda(K_n(\rho))}\Bigg|_{\rho=1} = \frac{\mathfrak{U}_n(A)}{\mathfrak{U}_n(P_n)}.$$

□

The polygonal circle number π_n can be used to reformulate the definition of the generalized uniform distribution of a random Vector U on P_n. It holds

$$P(U \in B) = \frac{1}{2\pi_n} \int\limits_{Pol^{*-1}(B)} \frac{d\varphi}{M_n(\varphi)^2},\ \text{for } B \in \mathfrak{B}^2 \cap P_n.$$

6. Disintegration of the Lebesgue measure

Polygonal disintegration of the Lebesgue measure is closely related to the notion of geometric measure representation that has been dealt with in [17], [14] and [20] for $l_{n,p}$-balls as well as in [18] for ellipsoids. A corresponding survey is given in [16]. In this sense, we introduce a disintegration formula for $\lambda(B),\ B \subset \mathbb{R}^2$ Borel measurable, using the polygonally generalized arc-length measure. Further, we give an example to show how polygonal disintegration of the Lebesgue measure applies, and present a thin layers property for the Lebesgue measure in this section.

Theorem 6.1. *For every regular convex polygon P_n,*
$\lambda(B) = \int_0^\infty \mathfrak{U}_n(B \cap P_n(r))dr,\ \ B \in \mathfrak{B}^2.$

To prove this theorem, one can follow the proof of Theorem 4 in [17] in an analogous way, wherefore this is omitted here. Theorem 6.1 reflects a certain generalization of the method of indivisibles of Cavalieri and Torricelli. Note, that in general $\lambda(B) \neq \int_0^\infty U_n(B \cap P_n(r))dr$, but by using the relation between the polygonal arc-length measure $\mathfrak{U}_n(\cdot)$ and the Euclidean arc-length measure $U(\cdot)$ it follows that $\lambda(B) = l_n \int_0^\infty U(B \cap P_n(r))dr$, $B \in \mathfrak{B}^2$.

Corollary 6.2. *For every Borel measurable subset B from $\mathbb{R}^2$ it holds true that*

$$\lambda(B) = \int_0^\infty r\mathfrak{U}_n\left(\left[\frac{1}{r}B\right] \cap P_n\right)dr = l_n \int_0^\infty rU\left(\left[\frac{1}{r}B\right] \cap P_n\right)dr.$$

Proof. Let $A \in \mathfrak{B}^2 \cap P_n$. Then

$$\mathfrak{U}_n(rA) = \frac{d}{d\rho}\lambda(sector_n(A,\rho))|_{\rho=r} = 2r\lambda(sector_n(A,1)) = r\mathfrak{U}_n(A).$$

By Theorem 6.1, with $rA = B \cap P_n(r)$, the corollary follows. □

Let $B \subset \mathbb{R}^2$ be Borel measurable. The function $r \to \mathfrak{F}_n(B,r)$ defined by

$$\mathfrak{F}_n(B,r) := \frac{U\left(\left[\frac{1}{r}B\right] \cap P_n\right)}{U(P_n)}, \text{ for } r > 0$$

is called the polygonal intersection proportion function (*ipf*) of the set B. With this function, the quantity $\lambda(B)$ can be reformulated by $\lambda(B) = \omega_n \int_0^\infty r\mathfrak{F}_n(B,r)dr$, where $\omega_n = \mathfrak{U}_n(P_n) = l_n U(P_n)$. This representation will be called the regular polygonal geometric measure representation of the Lebesgue measure.

Example. We reprove the well known result for the area content of the square with the vertices $(-1,-1)^T$, $(1,-1)^T$, $(1,1)^T$, $(-1,1)^T$. For this, we use the geometric measure representation based upon the *ipf* of the hexagon. Let $B := \{(x,y)^T \in \mathbb{R}^2 : max\{|x|,|y|\} \leq 1\}$. We have $\omega_6 = 3\sqrt{3}$ and therefore $\lambda(B) = 3\sqrt{3}\int_0^\infty r\mathfrak{F}_6(B,r)dr$. To calculate $\mathfrak{F}_6(B,r)$, we have to distinguish four cases with respect to the choice of the hexagonal radius r, see Figure 3. Because of

$$\mathfrak{F}_6(B,r) = \begin{cases} 1, & 0 < r \leq 1, \\ 1 - \frac{4(r-1)}{3r}, & 1 < r \leq \frac{2}{\sqrt{3}}, \\ \frac{2}{3r}\left(\frac{2}{\sqrt{3}} - 2(r-1)\right), & \frac{2}{\sqrt{3}} < r \leq \frac{1}{3}(3+\sqrt{3}), \\ 0, & \frac{1}{3}(3+\sqrt{3}) < r, \end{cases}$$

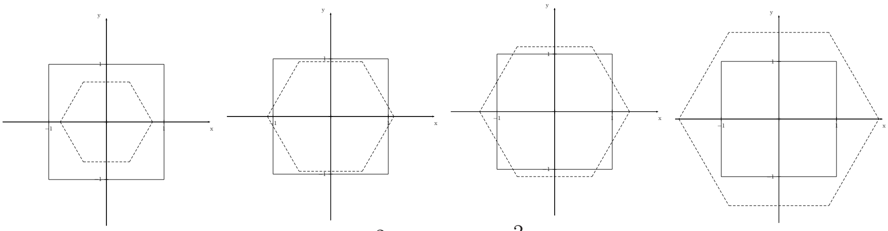
it follows

$$\lambda(B) = 3\sqrt{3}\left[\int_0^1 rdr + \int_1^{\frac{2}{\sqrt{3}}} r - \frac{4}{3}(r-1)dr + \frac{2}{3}\int_{\frac{2}{\sqrt{3}}}^{\frac{1}{3}(3+\sqrt{3})} \frac{2}{\sqrt{3}} - 2(r-1)dr\right] = 4.$$

Theorem 6.3. *Let $L(r,\epsilon) = \{(x,y)^T \in \mathbb{R}^2 : r \leq h_{K_n}(x,y) \leq r+\epsilon\}$. The Lebesgue measure $\lambda(L(r,\epsilon))$ has the polygonal thin layers property*

$$\lambda(L(r,\epsilon)) \sim 2\pi_n r\epsilon, \ \epsilon \to +0.$$

Proof. Because $\lambda(K_n(r)) = A_n(r) = \int_0^r \rho\mathfrak{U}_n(P_n)d\rho$ and $\pi_n = \mathfrak{U}_n(P_n)/2$, we obtain $\lambda(K_n(r)) = 2\pi_n \int_0^r \rho d\rho$. Thus, $\lambda(L(r,\epsilon)) = 2\pi_n \int_r^{r+\epsilon} \rho d\rho$
$= 2\pi_n\left(\frac{1}{2}(r+\epsilon)^2 - \frac{1}{2}r^2\right) = 2\pi_n(r\epsilon + \frac{1}{2}\epsilon^2).$ □



$0 < r \leq 1$ $1 < r \leq \frac{2}{\sqrt{3}}$ $\frac{2}{\sqrt{3}} < r \leq \frac{1}{3}(3+\sqrt{3})$ $\frac{1}{3}(3+\sqrt{3}) < r$

FIGURE 3. The hexagonal *ipf* takes constant values according to four regions of the hexagonal radius r.

7. Discussion

In this final section, we make somehow an excursion to the history of approximating the circle number π. The idea to use the radius $R_{o,n}$ of the outer circle of P_n as generalized reference circle and the Euclidean circumference $U_n(R_{o,n})$ of a regular polygon is rather old. Following it, it is possible to define numbers

$$\pi_{o,n,1} = U_n(R_{o,n})/(2R_{o,n}) = \frac{2R_{o,n} n \sin(\pi/n)}{2R_{o,n}} = n \sin(\pi/n)$$

and

$$\pi_{o,n,2} = A_n(R_{o,n})/R^2_{o,n} = \frac{\frac{1}{2} n R^2_{o,n} \sin(2\pi/n)}{R^2_{o,n}} = \pi_n$$

which are independent of $R_{o,n}$ but not equal. Thus, neither they do solve the isoperimetric problem nor can the circumference be written as the derivative of the area content of a regular polygon with respect to $R_{o,n}$. The use of the principle of Cavalieri and Torricelli to derive a geometric measure representation for the Lebesgue measure is not possible, wherefore there is no thin layers property as in Theorem 6.3. Hence, $\pi_{o,n,1}$ and $\pi_{o,n,2}$ appear as generalized circle numbers with weaker properties than π_n and they are impractical to be used for probabilistic purposes as those were dealt with in [13]-[22].

Using the radius $R_{i,n}$ of the inscribed circle, the Euclidean circumference $U_n(R_{i,n}) = 2nR_{i,n} \tan(\pi/n)$ and the area content $A_n(R_{i,n}) = nR^2_{i,n} \tan(\pi/n)$ inside of a regular polygon, we define

$$\pi_{i,n} = \frac{A_n(R_{i,n})}{R^2_{i,n}} = n \tan\left(\frac{\pi}{n}\right) = \frac{U_n(R_{i,n})}{2R_{i,n}}.$$

Each of the considered sequences of circle numbers has got the property to approximate π. For an illustration of all these approximations of π, see Table 2 and Figure 4. Furthermore, Figure 5 shows the inner and outer approximations of the unit circle by regular pentagons. Note that the coordinates of the center of each facet of the outer pentagon $A_{5,i},\ i = 1,..,5$ can be constructed by an anticlockwise rotation of the vertices of the inner pentagon $I_{5,i},\ i = 1,..,5$ through $36°$. Thus $A_{5,i} = \begin{pmatrix} \cos(36°) & -\sin(36°) \\ \sin(36°) & \cos(36°) \end{pmatrix} I_{5,i}$, for $i = 1,...,5$.

n	$\pi_{o,n,1}$	$\pi_{o,n,2} = \pi_n$	$\pi_{i,n}$
5	2,93	2,38	3,63
6	3	2,6	3,46
8	3,06	2,83	3,31
10	3,09	2,94	3,24

TABLE 2. Approximations of π

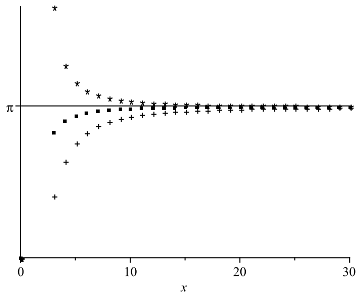


FIGURE 4. Approximation of π using the inner ($\star$), outer ($\blacksquare$) and polygonal (+) radius of P_n

Appendix

Example. To prove the Relation $AL_{K_5,\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)K_5^*}(A) = \mathfrak{U}_5(A)$ for the set $A = \{x, y > 0 : h_{K_5}(x,y) = 1\}$, which represents that part of the pentagonal

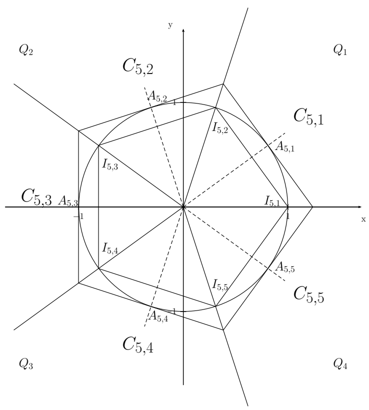


FIGURE 5. Approximation of the unit circle by an inner and outer regular pentagon

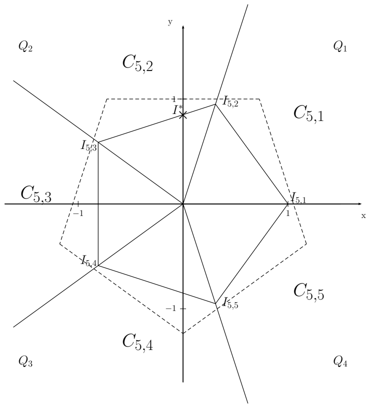


FIGURE 6. The unit polygonal circle P_5 (solid lines) and its $90°$ anticlockwise rotated unit polar pentagon (dashed lines)

circle that belongs to the first quadrant, we have $\mathfrak{U}_5(A) = \cos(\pi/5) \cdot U(A)$. To calculate $U(A)$, one has to sum up the Euclidean lengths of the facets $\overline{I_{5,1}I_{5,2}}$ and $\overline{I_{5,2}I^*}$, see Figure 6, where $I^* = (0, \cos(\pi/5)/\sin(3\pi/5))^T$. Since $\overline{I_{5,1}I_{5,2}}$ is an edge of P_5 it holds, for example according to [5] that $\overline{I_{5,1}I_{5,2}} = 10/(\sqrt{50+10\sqrt{5}})$.

Further, $\overline{I_{5,2}I^*} = \sqrt{\cos^2(2\pi/5) + \left(\sin(2\pi/5) - \frac{\cos(\pi/5)}{\sin(3\pi/5)}\right)^2}$ and since according to [10] $\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$, $\cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$, $\sin(\pi/5) = \frac{\sqrt{10-2\sqrt{5}}}{4}$, and $\sin(2\pi/5) = \sin(3\pi/5) = \frac{\sqrt{10+2\sqrt{5}}}{4}$, it follows that

$$\overline{I_{5,2}I^*} = \frac{\sqrt{6-2\sqrt{5}}}{\sqrt{10+2\sqrt{5}}} \quad \text{and} \quad \mathfrak{U}_5(A) = l_5(\overline{I_{5,1}I_{5,2}} + \overline{I_{5,2}I^*}) = \frac{10+14\sqrt{5}}{4\sqrt{50+10\sqrt{5}}}.$$

By the definition of $AL_{K_n,T}(A)$ it follows that

$$AL_{K_5,\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)K_5^*}(A) = -\int_0^1 h_{\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)K_5^*}(1, y'(x))dx = \cos(\pi/5)\cdot$$

$$\cdot \int_0^1 \sum_{j=1}^{2} \frac{\mathbb{1}_j(x)}{\sin((2j-1)\pi/5)} dx = \cos(\pi/5)\left(\frac{\cos(2\pi/5)}{\sin(3\pi/5)} + \frac{1-\cos(2\pi/5)}{\sin(\pi/5)}\right)$$

which results in the same value as that of $\mathfrak{U}_5(A)$.

References

[1] Andersen, K., Cavalieri's Method of Indivisibles, Archive for History of Exact Sciences 31, 4 (1985) 291-367.

[2] Ackermann, E.R., Grobler, T.L., Kleynhans, W., Olivier, J.C., Salmon, B.P. and van Zyl, A.J., Cavalieri integration, Quaestiones Mathematicae- Journal of the South African Math. Soc. 35, 2 (2012) 265-296.

[3] Busemann, H. The isoperimatric problem in the Minkowski plane, Amer. J. Math. 69 (1947) 863-871.

[4] Böhm, J., Hertel, E., Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung, Deutscher Verlag der Wissenschaften, 1981.

[5] Bronstein, I.N., Semendjajew, K.A., Musiol, G., Mühlig, H., Taschenbuch der Mathematik, Verlag Harri Deutsch, 6., vollständig überarbeitete und ergänzte Auflage, 2005.

[6] Coxeter, H.S.M., Regular Polytopes, Mcmillian Company, 1963.

[7] Kolmogorov, A.N., Fomin, S.V., Reelle Funktionen und Funktionalanalysis, VEB Deutscher Verlag der Wissenschaften, Berlin, 1975.

[8] Malik, M.A., A note on Cavalieri integration, Mathematics Magazine 57, 3 (1984) 154-156.

[9] Minkowski, H., Geometrie der Zahlen, Teubner, 1910.

[10] Myland, J., Oldham, K. B., Spanier, J., An Atlas of Functions, Second Edition, Springer, 2008.

[11] Moszyńska, M., Richter, W.-D., Reverse triangle inequality. Antinorms and semi-antinorms, Studia Scientiarum Mathematicarum Hungarica 49(1) (2012) 120-138. doi: 10.1556/SScMath.49.2012.1.1192.

[12] Quaisser, E., Sprengel, H.-J., Geometrie in Ebene und Raum, VEB Deutscher Verlag der Wissenschaften, Berlin, 1989.

[13] Richter, W.-D., Circle numbers for star discs, ISRN Geometry, Volume 2011, Article ID 479262 (2011) 16 pages; doi:10.5402/2011/479262.

[14] Richter, W.-D., Continuous $l_{n,p}-$symmetric distributions, Lithuanian Math. J. 49(1) (2009) 93-108. doi: 10.1007/s10986-009-9030-3.

[15] Richter, W.-D., Ellipses numbers and geometric measure representations, Journal of Applied Analysis 17 (2011) 165-179. doi: 10.1515/JAA.2011.011.

[16] Richter, W.-D., Exact distributions under non-standard model assumptions, AIP Conf.Proc. **1479** 442 (2012). doi: 10.1063/1.4756160

[17] Richter, W.-D., Generalized spherical and simplicial coordinates, J. Math. Anal. Appl. 336 (2007) 1187-1202. doi: 10.1016/j.jmaa.2007.03.047.

[18] Richter, W.-D., Geometric and stochastic representations for elliptically contoured distributions, Communications in Statistics: Theory and Methods 42 (2013) 579-602. doi: 10.1080/03610926.2011.611320.

[19] Richter, W.-D., On $l_{2,p}$-circle numbers, Lithuanian Math. J. 48(2) (2008) 228-234. doi: 10.1007/s10986-008-9002-z.

[20] Richter, W.-D., On the ball number function, Lithuanian Math. J. 51(3) (2011) 440-449. doi: 10.1007/s10986-011-9138-0.

[21] Richter, W.-D., On the Pi-function for nonconvex $l_{2,p}$-circle discs. Lithuanian Math. J. 48(3) (2008) 332-338.

[22] Richter, W.-D., Schicker, K., Ball numbers of platonic bodies. J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.03.007

[23] Thompson, A.C., Minkowski Geometry, Cambridge University Press, 1996.

[24] Ziegler, G. M., Lectures on Polytopes, Springer-Verlag, New York, 1995.

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