Representing continuous star-shaped probability measures in spaces with suitably constructed geometries

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Abstract

The local approach to the notion of a star generalized surface measure, consisting of taking derivatives of sector volumes, is proved to be equivalent to a suitable generalization of the well known integral (or diffential geometric) approach to the common notion of surface content. For star-shaped probability laws having a density contour defining star body K, a known geometric measure representation which is based upon the local approach to the star-generalized surface measure, in consequence appears in the new light of being a representation in the space $(R^n, \hat{h}_{K^*})$ where $\hat{h}_{K^*}$ is a slight modification of the Minkowski functional of a certain generalized ball K^* which is constructed in dependence of K.

Key words: spaces with geometries, constructing a geometry, generalized metric geometry, star-shaped distributions, star-generalized surface content, star-generalized uniform distribution, geometric measure representation, disintegration, generalized balls, ball number function

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1 Introduction

It is well known from [8] and several papers mentioned there that integrating a suitably chosen functional of the normal vector along a certain surface generalizes the common differential geometric approach to the notion of surface content in a way being fundamentally useful for the purposes of probability theory and mathematical statistics. The functional of interest is the Minkowski functional of a certain generalized ball K^* and defines a geometry which is a metric one or is of another type if K^* is convex and symmetric or not, respectively.

It will be shown here that the indicated generalized differential geometric or integral approach to the notion of star-generalized surface content is equivalent to the local one which makes use of taking derivatives of sector volumes. These derivatives are w.r.t. the star radius instead of the Euclidean one.

The notion of a star-generalized surface measure is basic for the construction of a star-generalized uniform distribution on a star sphere and the broad class of all star-shaped probability laws in $\mathbb{R}^n$. We shall concentrate our attention here to distributions having a density contour defining star body, K say. The geometric measure representation (2) of such a distribution was proved in [8] using the local approach of defining the star-generalized surface measure $\mathfrak{O}_S$ and can be viewed now in the new light of being a representation in the space $(\mathbb{R}^n, \hat{h}_{K^*})$. The functional $\hat{h}_{K^*}$ which defines the geometry of this space may, in dependence on the properties of K, be a norm, an antinorm, a semi-antinorm or a homogeneous functional of another type not yet mentioned. For antinorms and semi-antinorms we refer to [6].

It follows from [4] that, under certain assumptions upon the function $\hat{h}_{K^*}$, K solves the isoperimetric problem in the space $(R^n, \hat{h}_{K^*})$. A lower bound for isoperimetric constants of product probability measures is found in [1], and an approach to isoperimetric inequalities for probability measures with logarithmically concave densities is dicussed in [2]. In [4], an application of the Brunn-Minkowski inequality to probability distributions has been presented.

Star bodies and their Minkowski functionals play quite another role in probabilistic distribution theory if they are used to represent the corresponding characteristic functions, see [5]. Yet another way of using star bodies for describing (convex) measures is presented in [3].

The paper is organized as follows. Basic facts from [8] dealing with the star-shaped distribution theory are recalled in Section 2 in a condensed way. The choice of a suitable geometry (by choosing its generalized unit ball K^*) for defining the notion of star-generalized surface content $\mathfrak{O}_S$ is presented in Section 3. Section 4 deals with an extension of the ball number function which itself reflects one of the most essential properties of the star-generalized surface measure $\mathfrak{O}_S$. The final Section 5 deals with special cases of constructing the generalized ball K^*.

2 Star-shaped distributions

Let $K \subset \mathbb{R}^n$ denote a nonempty star-shaped set that is compact and equal to the closure of its interior, and that has the origin 0_n in its interior. We denote the boundary of the star body K by S, and the Minkowski functional of K by h_K. The functional $h_K : \mathbb{R}^n \to [0, \infty)$ is defined as $h_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}, x \in \mathbb{R}^n$ and is assumed to be positive-homogeneous of degree one, $h_K(\lambda x) = \lambda h_K(x), \lambda > 0$. The sets $rK = \{(rx_1, ..., rx_n)^T : x \in K\} = K(r)$ and $rS = S(r)$ are called a star ball and a star sphere of star radius $r > 0$, respectively.

Let $\mathfrak{B}_n$ be the Borel-σ-field in $\mathbb{R}^n$ and $\mathfrak{B}_S = S \cap \mathfrak{B}_n = \{S \cap B : B \in \mathfrak{B}_n\}$. Moreover, let $CPC(A) = \{x \in \mathbb{R}^n : x/h_K(x) \in A\}$ denote the central projection cone of a set $A \in \mathfrak{B}_S$ and $\text{sector}(A, r) = CPC(A) = CPC(A) \cap K(r)$ the intersection of $CPC(A)$ with the generalized ball $K(r)$. Under a mild technical condition, Assumption 1 in [5], the star-generalized surface measure may be defined on $r\mathfrak{B}_S = S(r) \cap \mathfrak{B}_n$ by $\mathfrak{O}_S(A) = f'(r)$ where $f(r) = \mu(sector(A, r))$ and μ denotes the Lebesgue measure in $\mathbb{R}^n$.

A random vector U_S defined on a probability space $(\Omega, \mathfrak{A}, P)$ is said to follow the star-generalized uniform probability distribution if $P(U_S \in A) = \mathfrak{O}_S(A)/\mathfrak{O}_S(S), A \in \mathfrak{B}_S$.

If for a random vector $Y : \Omega \to \mathbb{R}^n$ there are a vector $\nu \in \mathbb{R}^n$, a star body K with $0_n \in \text{int}K$ and a random variable $R : \Omega \to [0, \infty)$ being independent of U_S such that $Y - \nu$ allows the stochastic representation $Y - \nu \overset{d}{=} R \cdot U_S$ then Y follows a star-shaped distribution. Here, $V \overset{d}{=} W$ means that the random vectors V and W follow the same distribution law.

A function $g : [0, \infty) \to [0, \infty)$ satisfying $0 < I(g) < \infty$ with $I(g) = \int\limits_0^\infty r^{n-1} g(r) dr$ is called a density generating function (dgf). If Y has the probability density (pd)

$$\varphi_{g,K,\nu}(x) = C(g,K) g(h_K(x - \nu)), x \in \mathbb{R}^n$$

this means that R has the pd $f(r) = \frac{1}{I(g)} r^{n-1} g(r), r > 0$, K is then called the density contour defining star body and the normalizing constant allows the representation

$$C(g,K) = 1/(\mathfrak{O}_S(S) I(g)). \tag{1}$$

The corresponding probability measure allows the disintegration formula

$$\Phi_{g,K,\nu}(B) = C(g,K) \int\limits_0^\infty r^{n-1} g(r) \mathfrak{O}_S([\frac{1}{r}(B - \nu)] \cap S) dr, B \in \mathfrak{B}^n. \tag{2}$$

Numerous problems where the representation (2) successfully applies are discussed in [7, 8] and several papers mentioned there.

Geometric disintegration formulas for the Lebesgue measure underlying formula (2) are proved in [7, 8] and in [10]. In these considerations, integrating measures assigned to surfaces being intersections of a Borel set and generalized spheres yields volumes of those Borel sets. While, to this end, in [7, 8] non-Euclidean surface areas are integrated, in [10] certain volume (cone) measures assigned to surfaces are integrated. The method used in [7, 8] and several earlier papers mentioned there generalizes and extends the classical method of indivisibles of Cavalieri and Torricelli.

3 Construction of a suitably generalized metric geometry

The star-generalized surface content of a set $A \in \mathfrak{B}_S$ can according to Definition 1, Lemma 1 and Corollary 1 in [8] be represented symbolically as a single integral, being in fact a sum of integrals, as

$$\mathfrak{O}_S(A) = \int\limits_{G(A)} J^*(\vartheta) d\vartheta \text{ where } J^*(\vartheta) = |\eta(\vartheta) - \sum_{i=1}^{n-1} \vartheta_i \frac{d}{d\vartheta_i} \eta(\vartheta)|$$

and $G(A) = \{\vartheta \in \mathbb{R}^{n-1} : \exists \eta = \eta(\vartheta) \text{ with } (\vartheta^T, \eta)^T \in A\}$. If we denote the outer normal vector to the star sphere S at the point $(\vartheta^T, \eta(\vartheta))^T$ by $N(\vartheta)$ then

$$\mathfrak{O}_S(A) = \int\limits_{G(A)} (\vartheta^T, \eta(\vartheta)) N(\vartheta) d\vartheta.$$

Let the set of gradients of the functional h_K along that part of the boundary of the star body K where the gradient is defined be $\text{Grad}(S) = \{\nabla h_K(x), \, x \in S\}$ and

$$\text{Grad}^+(S) = \{\lambda x : x \in \text{Grad}(S), 0 < \lambda < 1\}. \tag{3}$$

Note that $Grad^+(S)$ is not necessarily an absorbing set, thus the notion of Minkowski functional is not always defined for this set.

Let, however, a slight modification of the Minkowski functional of this set be defined by

$$\hat{h}_{\mathrm{Grad}^+(S)}(t) = \inf\{\lambda > 0 : t \in \lambda \ \mathrm{Grad}^+(S)\}, t \in \mathrm{pos}\,\mathrm{Grad}^+(S).$$

Here, pos $x = \{\lambda x : \lambda \geq 0\}$ and pos M $= \bigcup\limits_{x \in M}$ pos x. For every $x \in S$ where $\nabla h_K(x)$ exists, all points from the ray $R_x = \{y \in \mathbb{R}^n : \frac{y}{\hat{h}_{\mathrm{Grad}^+(S)}(y)} = \nabla h_K(x)\}$ satisfy the equation $x^T y = x^T \nabla h_K(x) \hat{h}_{\mathrm{Grad}^+(S)}(y)$. Because the Minkowski functional of the star body K is homogeneous of degree one, it allows a.e. the representation

$$h_K(x) = x^T \nabla h_K(x),$$

generalizing immediately that from the case of norms which was dealt with in [11]. Thus,

$$x^T y = h_K(x) \hat{h}_{\mathrm{Grad}^+(S)}(y), \forall y \in R_x \text{ and all } x \in S \text{ where } \nabla h_K(x) \text{ exists}.$$

Since $N(\vartheta) \in R_{(\vartheta^T, \eta(\vartheta))^T}, \vartheta \in G(A)$, $(\vartheta^T, \eta(\vartheta))^T \in S$ and therefore $h_K((\vartheta^T, \eta(\vartheta))^T) = 1$, it follows that $(\vartheta^T, \eta(\vartheta))N(\vartheta) = \hat{h}_{\mathrm{Grad}^+(S)}(N(\vartheta))$. The resulting representation

$$\mathfrak{O}_S(A) = \int\limits_{G(A)} \hat{h}_{K^*}(N(x))dx \tag{4}$$

with K^* chosen as

$$K^* = \mathrm{Grad}^+(S) \tag{5}$$

may be considered as a star-generalization of the common differential geometric definition of the notion of surface content and may therefore be taken as the definition of the notion of star-generalized surface content measure itself.

Note that it does not have any problematic influence onto the measure representation in formula (4) if S is not smooth in a countable set of points. Particularly, if S is a finite union of sets S^o with constant value of $N(x)$ for all $x \in S^o$, thus K not being strongly convex, then the geometry generated by $Grad^+(S)$ will be a finite one. Finally, if h_K is a norm then $\hat{h}_{K^*}$ is a norm, too.

We recall that according to [8] the symbolic, single integral (4) has actually to be read as a sum of integrals, in general.

As to summarize the results of this section, a suitable geometry for measuring the surface content of a measurable subset of the star sphere S is a geometry having $K^* = \mathrm{Grad}^+(S)$ as its generalized unit ball. Thus the constructed geometry may be, e.g., a metric or a finite one. For the case of a finite geometry in this context, we refer to [9]. It is aimed by the present paper to stimulate further studies of the spaces $(\mathbb{R}^n, \hat{h}_{K^*})$.

4 Extension of the ball number function

We recall that according to (1) the star-generalized surface content $\mathfrak{O}_S(S)$ plays an important role in the representation of the normalizing constant $C(g, K)$. As it was proved in Section 2, the integral (or differential geometric) approach and the local approach to the definition of star-generalized surface content lead to the same result. As an immediate consequence, we observe that the ratios $\mathfrak{O}_S(S(r))/nr^{n-1}$ and $\mu(K(r))/r^n$ do not depend on the star radius r, $r>0$ and are equal, and their common value, $\pi(K)$, is called the ball number of the star body K (possibly w.r.t. a suitable shift),

$$\frac{\mu(K(r))}{r^n} = \pi(K) = \frac{\mathfrak{O}_S(S(r))}{nr^{n-1}},\ r>0.$$

The ball number of a star body allows, among others, the representations $\pi(K) = \mu(K)$ and

$$\pi(K) = \frac{1}{n} \int\limits_{G(S)} \hat{h}_{K^*}(N(x))dx.$$

The general problem of extending the circle and ball number function $K \mapsto \pi(K)$ from Euclidean circles and balls to $l_{n,p}$-circles and balls was dealt with in earlier papers of the author. The extension of $\pi(.)$ from the set of $l_{n,p}$-balls to a set of more general balls was given in [8]. Here, we extended this function further to the set consisting of all star balls in $\mathbb{R}^n$ the Minkowski functionals of whose suitably shifted versions are homogeneous of degree one.

5 Examples

We recall that a star-shaped distribution law having a dgf g, a density contour defining star body K and a location vector ν allows the representation (2) with $\mathfrak{O}_S$ being defined according to (4),(5) where $\mathrm{Grad}^+(S)$ is given in (3), and with a normalizing constant $C(g,K)$ satisfying (1).

Example 5.1. *If the density level set defining star body K is an $l_{n,p}$-norm unit ball, $1 \le p < \infty$, then $\hat{h}_{K^*}$ in (4) can be chosen as the common Minkowski functional of the unit ball of the $l_{n,q}$-norm being dual to the given $l_{n,p}$-norm.*

Example 5.2. *If K is an $l_{n,p}$-antinorm unit ball, $0 < p \le 1$, then $\hat{h}_{K^*}$ in (4) can be expressed as the Minkowski functional of the unit ball K^* of the corresponding semi-antinorm where*

$$K^* = \{x \in \mathbb{R}^n : (|x_1|^q + ... + |x_n|^q)^{1/q} \le 1\} \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

If $p < 1$ then q satisfies the inequality $q < 0$.

The next example generalizes Examples 5.1 and 5.2 w.r.t. an additional scaling-shape parameter a. Presenting it jointly for the cases $0 < p \le 1$ and $1 \le p$ emphasizes the circumstance that both convex and non-convex cases can be dealt with in a unified way.

Example 5.3. *If the density level set defining star body is $K = \{x \in \mathbb{R}^n : (\sum\limits_{i=1}^n |x_i/a_i|^p)^{1/p} \le 1\}$ where $p > 0, a_i \neq 0, i = 1,...,n$ then h_{K^*} can be chosen in the representation (4) as the Minkowski functional of $K^* = \{x \in \mathbb{R}^n : (\sum\limits_{i=1}^n |a_i x_i|^q)^{1/q} \le 1\}$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

The final measure representations (2) with (4) turning out in all these examples and special cases of them were considered already in [8] and earlier papers mentioned there, but by using particular methods instead of the general one used here for representing $\mathfrak{O}_S$. We let it open here as an exercise to directly derive from the present general result that, in all these cases, K^* in (5) may be chosen as it was done in Examples 5.1-5.3, respectively.

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